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IDEALS GENERATED BY ADJACENT 2-MINORS

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ABSTRACT. Ideals generated by adjacent 2-minors are studied. First, the problem when such an ideal is a prime ideal as well as the problem when such an ideal possesses a quadratic Gröbner basis is solved. Second, we describe explicitly a primary decomposition of the radical ideal of an ideal generated by adjacent 2-minors, and challenge the question of classifying all ideals generated by adjacent 2-minors which are radical ideals.

Introduction. Let $X = (x_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ be an $m \times n$-matrix of indeterminates, and let K be an arbitrary field. The ideals of t-minors $I_t(X)$ in $K[X] = K[(x_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,m}}]$ are well understood. A standard reference for determinantal ideals are the lecture notes [**2**] by Bruns and Vetter. See also [**1**] for a short introduction to this subject. Determinantal ideals and the natural extensions of this class of ideals, including ladder determinantal ideals arise naturally in geometric contexts which partially explains the interest in them. One nice property of these ideals is that they are all Cohen-Macaulay prime ideals.

Motivated by applications to algebraic statistics, one is led to study ideals generated by an *arbitrary* set of 2-minors of X. We refer the interested reader to the article [**3**] of Diaconis, Eisenbud and Sturmfels where the encoding of the statistical problem to commutative algebra is nicely described. In this paper we concentrate on studying ideals generated by adjacent 2-minors, that is, minors of the form $x_{i,j}x_{i+1,j+1} - x_{i+1,j}x_{i,j+1}$. Hoşten and Sullivant [**9**] describe in a very explicit way all the minimal prime ideals for the ideal generated by all adjacent 2-minors of an $m \times n$-matrix. In Theorem 3.3 we succeed

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in describing the minimal prime ideals of the ideal of a configuration of adjacent 2-minors under the assumption that this configuration is convex, a concept which had first been introduced by Qureshi [**10**]. Convex configurations include the case considered by Hoşten and Sullivant but are more general. Our description is not quite as explicit as that of Hoşten and Sullivant, but explicit enough to determine in each particular case all the minimal prime ideals. Part of the result given in Theorem 3.3 can also be derived from [**8**, Corollary 2.1] of Hoşten and Shapiro, since ideals generated by 2-adjacent minors are lattice basis ideals. Though the minimal prime ideals are known, knowledge about embedded prime ideals of an ideal generated by 2-adjacent minors is very little, let alone knowledge of its primary decomposition. In [**3**] the primary decomposition of the ideal of all adjacent 2-minors of a 4×4-matrix is given and in [**8**] that of a 3×5-matrix. It is hard to see a general pattern from these results.

Ideals generated by adjacent 2-minors tend to have a nontrivial radical and are rarely prime ideals. In the first section of this paper we classify all ideals generated by adjacent 2-minors which are prime ideals. The result is described in Theorem 1.1. They are the ideals of adjacent 2-minors attached to a chessboard configuration with no 4-cycles.

One method to show that an ideal is a radical ideal is to compute its initial ideal with respect to some monomial order. If the initial ideal is squarefree, then the given ideal is a radical ideal. In Section 2 we classify all ideals generated by adjacent 2-minors which have a quadratic Gröbner basis (Theorem 2.3). It turns out that these are the ideals of adjacent 2-minors corresponding to configurations whose components are monotone paths meeting in a suitable way. In particular, those ideals of adjacent 2-minors are radical ideals. In general, the radical of an ideal of adjacent 2-minors attached to a convex configuration can be naturally written as an intersections of prime ideals of relatively simple nature, see Theorem 3.2. These prime ideals are indexed by the so-called admissible sets. These are subsets of the set $\mathcal{S}$, which defines the configuration, and can be described in a purely combinatorial way.

In Section 4 we aim at classifying configurations whose ideal of adjacent 2-minors is a radical ideal. In this section we restrict ourselves to considering only a particular class of configurations, which we call

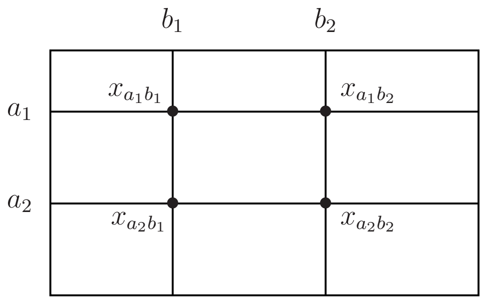

FIGURE 1. The vertices of the 2-minor $[a_1, a_2|b_1, b_2]$ in matrix X.

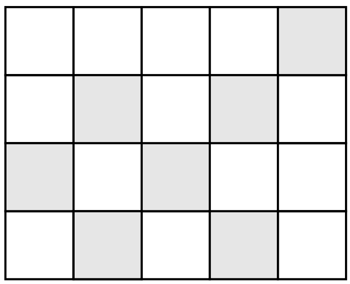

FIGURE 2. A chessboard configuration inside a 4×5-matrix.

strongly connected. It is not so hard to see (cf. Proposition 4.2) that a strongly connected configuration whose ideal of adjacent 2-minors is a radical ideal should be a path or a cycle. Computations show that the cycles should have a length at least 12. We expect that the ideal of adjacent 2-minors attached to any path is a radical ideal and prove this in Theorem 4.3 under the additional assumption that the ideal has no embedded prime ideals.

1. Prime ideals generated by adjacent 2-minors. Let $X = (x_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ be a matrix of indeterminates, and let S be the polynomial ring over a field K in the variables x_{ij}. Let $\delta = [a_1, a_2|b_1, b_2]$ be a 2-minor with rows a_1, a_2 and columns b_1, b_2. The variables x_{a_i,b_j} are called the *vertices* and the sets $\{x_{a_1,b_1}, x_{a_1,b_2}\}$, $\{x_{a_1,b_1}, x_{a_2,b_1}\}$, $\{x_{a_1,b_2}, x_{a_2,b_2}\}$ and $\{x_{a_2,b_1}, x_{a_2,b_2}\}$ the *edges* of the minor $[a_1, a_2|b_1, b_2]$, see Figure 1.

The set of vertices of δ will be denoted by $V(\delta)$. The 2-minor $\delta = [a_1, a_2 \mid b_1, b_2]$ is called *adjacent* if $a_2 = a_1 + 1$ and $b_2 = b_1 + 1$.

Let $\mathcal{C}$ be any set of adjacent 2-minors. We also call such a set a configuration of adjacent 2-minors. We denote by $I(\mathcal{C})$ the ideal generated by the elements of $\mathcal{C}$. The set of vertices of $\mathcal{C}$, denoted $V(\mathcal{C})$, is the union of the vertices of its adjacent 2-minors. Two distinct adjacent 2-minors $\delta, \gamma \in \mathcal{C}$ are called *connected*, respectively *weakly connected*, if there exist $\delta_1, \ldots, \delta_r \in \mathcal{C}$ such that $\delta = \delta_1$, $\gamma = \delta_r$, and δ_i and δ_{i+1} have a common edge, respectively a common vertex.

Any maximal subset D of $\mathcal{C}$ with the property that any two minors of D are connected, is called a *connected component of* $\mathcal{C}$. To $\mathcal{C}$ we attach a graph $G_{\mathcal{C}}$ as follows: the vertices of $G_{\mathcal{C}}$ are the connected components of $\mathcal{C}$. Let A and B be two connected components of $\mathcal{C}$. Then there is an edge between A and B if there exists a minor $\delta \in A$ and a minor $\gamma \in B$ which have exactly one vertex in common. Note that $G_{\mathcal{C}}$ may have multiple edges.

A set of adjacent 2-minors is called a *chessboard configuration*, if any two minors of this set meet in at most one vertex. An example of a chessboard configuration is given in Figure 2. An ideal $I \subset S$ is called a *chessboard ideal* if it is generated by a chessboard configuration. Note that the graph $G_{\mathcal{C}}$ of a chessboard configuration is a simple bipartite graph. Indeed, in the case of a chessboard configuration, the set of vertices V of the graph $G_{\mathcal{C}}$ corresponds to the set of 2-minors of the configuration. We define the vertex decomposition $V = V_1 \cup V_2$ of V by letting V_1 be the set of 2-minors located in the odd floors and V_2 the set of 2-minors located in the even floors.

Theorem 1.1. *Let I be an ideal generated by adjacent 2-minors. Then the following conditions are equivalent*:

(a) *I is a prime ideal.*

(b) *I is a chessboard ideal and $G_{\mathcal{C}}$ has no cycle of length 4.*

For the proof of this result, we shall need some concepts related to lattice ideals.

Let $\mathcal{L} \subset \mathbf{Z}^n$ be a lattice, in other words, a subgroup of $\mathbf{Z}^n$. Let K be a field. The *lattice ideal* attached to $\mathcal{L}$ is the binomial ideal

$I_{\mathcal{L}} \subset K[x_1, \ldots, x_n]$ generated by all binomials

$$\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \quad \text{with} \quad \mathbf{a} - \mathbf{b} \in \mathcal{L}.$$

$\mathcal{L}$ is called saturated if, for all $\mathbf{a} \in \mathbf{Z}^n$ and $c \in \mathbf{Z}$ such that $c\mathbf{a} \in \mathcal{L}$ it follows that $\mathbf{a} \in \mathcal{L}$. The lattice ideal $I_{\mathcal{L}}$ is a prime ideal if and only if $\mathcal{L}$ is saturated ([**4**]).

Recall that any subgroup of $\mathbf{Z}^n$ is a free group and finitely generated. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a basis of $\mathcal{L}$. Hoşten and Shapiro [**8**] call the ideal generated by the binomials $\mathbf{x}^{\mathbf{v}_i^+} - \mathbf{x}^{\mathbf{v}_i^-}$, $i = 1, \ldots, m$, a *lattice basis ideal* of $\mathcal{L}$. Here $\mathbf{v}^+$ denotes the vector obtained from $\mathbf{v}$ by replacing all negative components of $\mathbf{v}$ by zero, and $\mathbf{v}^- = -(\mathbf{v} - \mathbf{v}^+)$.

Fischer and Shapiro [**5**] and Eisenbud and Sturmfels [**4**] showed:

Proposition 1.2. *Let J be a lattice basis ideal of the saturated lattice $\mathcal{L} \subset \mathbf{Z}^n$. Then $J\colon (\prod_{i=1}^n x_i)^\infty = I_{\mathcal{L}}$, where $J : (\prod_{i=1}^n x_i)^\infty = \cup_{k=1}^\infty J : (\prod_{i=1}^n x_i)^k$.*

It is known from [**4**] that the ideal generated by all adjacent 2-minors of X is a lattice basis ideal and that the corresponding lattice ideal is just the ideal of all 2-minors of X. It follows that an ideal which is generated by any set of adjacent 2-minors of X is again a lattice basis ideal and that its corresponding lattice $\mathcal{L}$ is saturated. Thus, as a consequence of Proposition 1.2, we obtain:

Lemma 1.3. *Let I be an ideal generated by adjacent 2-minors. Then I is a prime ideal if and only if all variables x_{ij} are nonzerodivisors of S/I.*

Proof. As observed before, I is a lattice basis ideal and its lattice ideal $I_{\mathcal{L}}$ is a prime ideal. Assume now that all variables x_{ij} are nonzerodivisors of S/I. Then Proposition 1.2 implies that $I = I_{\mathcal{L}}$ so that I is a prime ideal. The converse implication is trivial. $\square$

For the proof of Theorem 1.1, we need the following two lemmata.

Lemma 1.4. *Let I be an ideal generated by adjacent 2-minors. For each of the minors we mark one of the monomials appearing in the minor as a potential initial monomial. Then there exists an ordering* *of the variables such that the marked monomials are indeed the initial monomials with respect to the lexicographic order induced by the given ordering of the variables.*

Proof. In general suppose that, in the set $[N] = \{1, 2, \ldots, N\}$, for each pair $(i, i+1)$ an ordering either $i < i+1$ or $i > i+1$ is given. We claim that there is a total order $<$ on $[N]$ which preserves the given ordering. Working by induction on N, we may assume that there is a total order $i_1 < \cdots < i_{N-1}$ on $[N-1]$ which preserves the given ordering for the pairs $(1,2), \ldots, (N-2, N-1)$. If $N-1 < N$, then $i_1 < \cdots < i_{N-1} < N$ is a required total order $<$ on $[N]$. If $N-1 > N$, then $N < i_1 < \cdots < i_{N-1}$ is a required total order $<$ on $[N]$.

The above fact guarantees the existence of an ordering of the variables such that the marked monomials are indeed the initial monomials with respect to the lexicographic order induced by the given ordering of the variables. □

The following examples demonstrate the construction of the monomial order given in the proof of Lemma 1.4.

Example 1.5. In Figure 3 each of the squares represents an adjacent 2-minor, and the diagonal in each of the squares indicates the marked monomial of the corresponding 2-minor. For any lexicographic order for which the marked monomials in Figure 3 are the initial monomials, the numbering of the variables in the top row must satisfy the following inequalities:

$$1 < 2 > 3 < 4 > 5 > 6.$$

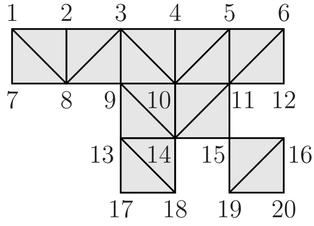

FIGURE 3.

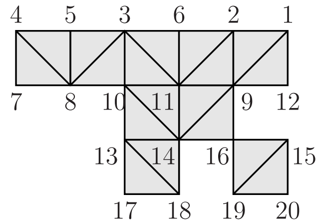

FIGURE 4. Labeling of the variables with for given initial monomials.

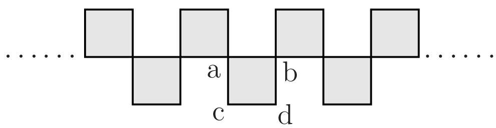

FIGURE 5. A sequence of 2-adjacent minors.

By using the general strategy given in the proof of Lemma 1.4, we relabel the top row of the vertices by the numbers 1 up to 6, and proceed in the same way in the next rows. The final result can be seen in Figure 4.

We call a vertex of a 2-minor in $\mathcal{C}$ *free*, if it does not belong to any other 2-minor of $\mathcal{C}$, and we call the 2-minor $\delta = ad - bc$ *free*, if either (i) a and d are free, or (ii) b and c are free.

Lemma 1.6. *Let $\mathcal{C}$ be a chessboard configuration with $|\mathcal{C}| \geq 2$. Suppose $G_{\mathcal{C}}$ does not contain a cycle of length 4. Then the $G_{\mathcal{C}}$ contains at least two free 2-minors.*

Proof. We may assume there is at least one nonfree 2-minor in $\mathcal{C}$, say $\delta = ad - bc$. Since we do not have a cycle of length 4, there exists a sequence of 2-minors in $\mathcal{C}$ as indicated in Figure 5. Then the left-most and the right-most 2-minor of this sequence is free. $\square$

Proof of Theorem 1.1. (a) $\Rightarrow$ (b). Let $\delta, \gamma \in I$ be two adjacent 2-minors which have an edge in common. Say $\delta = ae - bd$ and $\gamma = bf - ce$.

Then $b(af-cd) \in I$, but neither b nor $af-cd$ belongs to I. Therefore, I must be a chessboard ideal. Suppose $G_{\mathcal{C}}$ contains a cycle of length 4. Then there exist in I adjacent two minors $\delta_1 = ae-bd$, $\delta_2 = ej-fi$, $\delta_3 = hl-ik$ and $\delta_4 = ch-dg$. Then $h(bcjk-afgl) \in I$, but neither h nor $bcjk-afgl$ belongs to I.

(b) $\Rightarrow$ (a). By virtue of Lemma 1.3, what we must prove is that all variables x_{ij} are nonzerodivisors of S/I. Let $\mathcal{G}$ be the set of generating adjacent 2-minors of I. Fix an arbitrary vertex x_{ij}. We claim that for each of the minors in $\mathcal{G}$ we may mark one of the monomials in the support as a potential initial monomial such that the variable x_{ij} appears in none of the potential initial monomials and that any two potential initial monomials are relatively prime.

We are going to prove this claim by induction on $|\mathcal{G}|$. If $|\mathcal{G}| = 1$, then the assertion is obvious. Now assume that $|\mathcal{G}| \geq 2$. Then Lemma 1.6 says that there exist at least two free adjacent 2-minors in $\mathcal{G}$. Let $\delta = ad-bc$ be one of them, and assume that a and d are free vertices of δ. We may assume that $x_{ij} \neq a$ and $x_{ij} \neq d$. Let $\mathcal{G}' = \mathcal{G} \setminus \{\delta\}$. By assumption of induction, for each of the minors of $\mathcal{G}'$, we may mark one of the monomials in the support as a potential initial monomial such that the variable x_{ij} appears in none of the potential initial monomials and that any two potential initial monomials are relatively prime. Then these markings, together with the marking ad, are the desired markings of the elements of $\mathcal{G}$.

According to Lemma 1.4, there exists an ordering of the variables such that, with respect to the lexicographic order induced by this ordering, the potential initial monomials become the initial monomials. Since initial monomials are relatively prime ([**6**, Lemma 2.3.1]), it follows that $\mathcal{G}$ is a Gröbner basis of I and, since x_{ij} does not divide any initial monomial of an element in $\mathcal{G}$, it follows that x_{ij} is a nonzerodivisor of $S/\mathrm{in}(I)$, where $\mathrm{in}(I)$ is the initial ideal of I. But then x_{ij} is a nonzerodivisor of S/I as well. $\square$

2. Ideals generated by adjacent 2-minors with a quadratic Gröbner basis. A configuration $\mathcal{P}$ of adjacent 2-minors is called a *path*, if there exists an ordering $\delta_1, \ldots, \delta_r$ of the elements of $\mathcal{P}$ such that

$$\delta_j \bigcap \delta_i \subset \delta_{i-1} \bigcap \delta_i$$

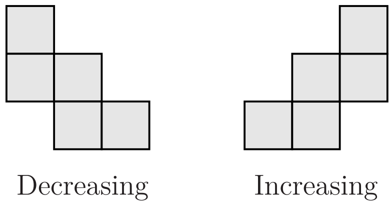

FIGURE 6. Monotone paths.

for all $j < i$, and

$$\delta_{i-1} \bigcap \delta_i \quad \text{is an edge of } \delta_i.$$

Such an ordering is called a path ordering. A path $\mathcal{P}$ with path ordering $\delta_1, \ldots, \delta_r$ where $\delta_i = [a_i, a_i + 1 | b_i, b_i + 1]$ for $i = 1, \ldots, r$ is called *monotone* if the sequences of integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are monotone sequences. The monotone path $\mathcal{P}$ is called *decreasing* if the sequences $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are both increasing or both decreasing, and the monotone path is called *increasing* if one of the sequences is increasing and the other one is decreasing, see Figure 6.

If for $\mathcal{P}$ we have $a_1 = a_2 = \cdots = a_r$, or $b_1 = b_2 = \cdots = b_r$, then we call $\mathcal{P}$ a *line path*. Notice that a line graph is both monotone increasing and monotone decreasing.

Let $\delta = ad - bc$ be an adjacent 2-minor with $a = x_{ij}$, $b = x_{ij+1}$, $c = x_{i+1j}$ and x_{i+1j+1}. Then the monomial ad is called the *diagonal* of δ.

Lemma 2.1. *Let $\mathcal{P}$ be a monotone increasing* (*decreasing*) *path of* 2-*minors. Then, for any monomial order $<$ for which $I(\mathcal{P})$ has a quadratic Gröbner basis, the initial monomials of the generators are all diagonals* (*anti-diagonals*).

Proof. Suppose first that $\mathcal{P}$ is a line path. If $I(\mathcal{P})$ has a quadratic Gröbner basis, then initial monomials of the 2-minors of $\mathcal{P}$ are all diagonals or all anti-diagonals, because otherwise there would be two 2-minors δ_1 and δ_2 in $\mathcal{P}$ connected by an edge such that $\operatorname{in}(\delta_1)$ is a diagonal and $\operatorname{in}(\delta_2)$ is an anti-diagonal. The S-polynomial of δ_1 and δ_2

FIGURE 7. Sub-paths of a monotone increasing path.

FIGURE 8.

is a binomial of degree 3 which belongs to the reduced Gröbner basis of I, a contradiction. If all initial monomials of the 2-minors in $\mathcal{P}$ are diagonals, we interpret $\mathcal{P}$ as a monotone increasing path and, if all initial monomials of the 2-minors in $\mathcal{P}$ are anti-diagonals, we interpret $\mathcal{P}$ as a monotone decreasing path.

Now assume that $\mathcal{P}$ is not a line path. We may assume that $\mathcal{P}$ is monotone increasing. (The argument for a monotone decreasing path is similar). Then, since $\mathcal{P}$ is not a line path, it contains one of the following sub-paths displayed in Figure 7.

For both sub-paths the initial monomials must be diagonals, otherwise $I(\mathcal{P})$ would not have a quadratic Gröbner basis. Then, as in the case of line paths, one sees that all the other initial monomials of $\mathcal{P}$ must be diagonals. $\square$

A configuration of adjacent 2-minors which are of the form shown in Figure 8, or are obtained by rotation from them, are called *square*, *pin* and *saddle*, respectively.

Lemma 2.2. *Let $\mathcal{A}$ be a connected configuration of adjacent 2-minors. Then $\mathcal{A}$ is a monotone path if and only if $\mathcal{A}$ contains neither a square nor a pin nor a saddle.*

Proof. Assume that $\mathcal{A} = \delta_1, \delta_2, \dots, \delta_r$ with $\delta_i = [a_i, a_i + 1|b_i, b_i + 1]$ for $i = 1, \dots, r$ a monotone path. Without loss of generality, we may assume the both sequences $a_1, \dots, a_r$ and $b_1, \dots, b_r$ are monotone increasing. We will show by induction on r that it contains no square, no pin and no saddle. For $r = 1$, the statement is obvious. Now let us assume that the assertion is true for $r - 1$. Since $\mathcal{A}' = \delta_1, \delta_2, \dots, \delta_{r-1}$ is monotone increasing, it follows that the coordinates of the minors δ_i for $i = 1, \dots, r - 1$ sit inside the rectangle R with corners $(a_1, b_1), (a_{r-1}+1, b_1), (a_{r-1}+1, b_{r-1}+1), (a_1, b_{r-1}+1)$, and $\mathcal{A}'$ has no square, no pin and no saddle. Since $\mathcal{A}$ is monotone increasing, $\delta_r = [a_{r-1}, a_{r-1} + 1 \mid b_{r-1} + 1, b_{r-1} + 2]$ or $\delta_r = [a_{r-1} + 1, a_{r-1} + 2 \mid b_{r-1}, b_{r-1} + 1]$. It follows that, if $\mathcal{A}$ would contain a square, a pin or a saddle, then the coordinates of one of the minors δ_i, $i = 1, \dots, r - 1$ would not be inside the rectangle R.

Conversely, suppose that $\mathcal{A}$ contains no square, no pin and no saddle. Then $\mathcal{A}'$ contains no square, no pin and no saddle as well. Thus, arguing by induction on r, we may assume that $\mathcal{A}'$ is a monotone path. Without loss of generality, we may even assume that $a_1 \le a_2 \le \cdots \le a_{r-1}$ and $b_1 \le b_2 \le \cdots \le b_{r-1}$. Now let δ_r be connected to δ_i (via an edge). If $i \in \{2, \dots, r - 2\}$, then $\mathcal{A}$ contains a square, a pin or a saddle which involves δ_r, a contradiction. If $i = 1$ or $i = r-1$ and $\mathcal{A}$ is not monotone, then $\mathcal{A}$ contains a square or a saddle involving δ_r. $\square$

With the notation introduced, we have

Theorem 2.3. *Let $\mathcal{C}$ be a configuration of adjacent 2-minors. Then the following conditions are equivalent*:

(a) *$I(\mathcal{C})$ has a quadratic Gröbner basis with respect to the lexicographic order induced by a suitable order of the variables.*

(b) (i) *Each connected component of $\mathcal{C}$ is a monotone path.*

(ii) *If $\mathcal{A}$ and $\mathcal{B}$ are components of $\mathcal{C}$ which meet in a vertex which is not an endpoint of $\mathcal{A}$ nor an endpoint of $\mathcal{B}$ and, if $\mathcal{A}$ is monotone increasing, then $\mathcal{B}$ must be monotone decreasing and vice versa.*

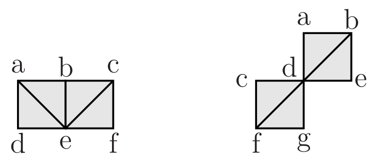

FIGURE 9.

(c) *The initial ideal of $I(\mathcal{C})$ with respect to the lexicographic order induced by a suitable order of the variables is a complete intersection.*

Proof. (a) $\Rightarrow$ (b). (i) Suppose there is component $\mathcal{A}$ of $\mathcal{C}$ which is not a monotone path. Then, according to Lemma 2.2, $\mathcal{A}$ contains a square, a pin or a saddle. In all three cases, no matter how we label the vertices of component $\mathcal{A}$, it will contain, up to a rotation or reflection, two adjacent 2-minors with leading terms as indicated in Figure 9.

In the first case the S-polynomial of the two minors is $abf - bcd$, and in the second case it is $aef - bcg$. We claim that, in both cases, these binomials belong to the reduced Gröbner basis of $I(\mathcal{C})$, which contradicts assumption (a).

Indeed, first observe that the adjacent 2-minors generating the ideal $I(\mathcal{C})$ is the unique minimal set of binomials generating $I(\mathcal{C})$. Therefore, the initial monomials of degree 2 are exactly the initial monomials of these binomials. Suppose now that $abf - bcd$ does not belong to the reduced Gröbner basis of I; then one of the monomials ab, af or bf must be the leading monomial of an adjacent 2-minor, which is impossible. In the same way, one argues in the second case.

(ii) Assume $\mathcal{A}$ and $\mathcal{B}$ have a vertex c in common. Then c must be a corner of $\mathcal{A}$ and $\mathcal{B}$, that is, a vertex which belongs to exactly one 2-minor of $\mathcal{A}$ and exactly one 2-minor of $\mathcal{B}$, see Figure 10.

If, for both components, the initial monomials are the diagonals (anti-diagonals), then the S-polynomial of the 2-minor in $\mathcal{A}$ with vertex c and the 2-minor of $\mathcal{B}$ with vertex c is a binomial of degree 3 whose initial monomial is not divisible by any initial monomial of the gener-

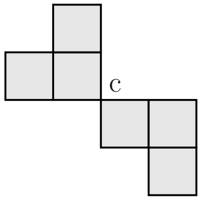

FIGURE 10. Two components of a configuration meeting at vertex c.

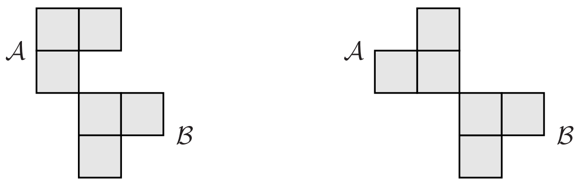

FIGURE 11. Monotone increasing paths meeting in a vertex.

ators of $\mathcal{C}$, unless c is an endpoint of both $\mathcal{A}$ and $\mathcal{B}$. Thus, the desired conclusion follows from Lemma 2.1.

(b) $\Rightarrow$ (c). Condition (b) implies that any pair of initial monomials of two distinct binomial generators of $I(\mathcal{C})$ are relatively prime. Hence, the initial ideal is a complete intersection.

(c) $\Rightarrow$ (a). Since the initial monomial of the 2-minors generating $I(\mathcal{C})$ belong to any reduced Gröbner basis of $I(\mathcal{C})$, they must form a regular sequence. This implies that S-polynomials of any two generating 2-minors of $I(\mathcal{C})$ reduce to 0. Therefore, $I(\mathcal{C})$ has a quadratic Gröbner basis. $\square$

Corollary 2.4. *Let $\mathcal{C}$ be a configuration satisfying the conditions of Theorem* 2.3 (b). *Then $I(\mathcal{C})$ is a radical ideal generated by a regular sequence.*

Proof. Let $\mathcal{C} = \delta_1, \ldots, \delta_r$. By Theorem 2.3, there exist a monomial order $<$ such that $\mathrm{in}_<(\delta_1), \ldots, \mathrm{in}_<(\delta_r)$ is a regular sequence. It follows that $\delta_1, \ldots, \delta_r$ is a regular sequence. Since the initial monomials are squarefree and form a Gröbner basis of $I(\mathcal{C})$, it follows that $I(\mathcal{C})$ is a radical ideal, see for example [**7**, Proof of Corollary 2.2]. $\square$

To demonstrate Theorem 2.3, we consider the following two examples displayed in Figure 11.

In both examples, the components $\mathcal{A}$ and $\mathcal{B}$ are monotone increasing paths. In the first example, $\mathcal{A}$ and $\mathcal{B}$ meet in a vertex which is an endpoint of $\mathcal{A}$; therefore, condition (b) (ii) of Theorem 2.3 is satisfied, and the ideal $I(\mathcal{A} \cup \mathcal{B})$ has a quadratic Gröbner basis. However, in the second example $\mathcal{A}$ and $\mathcal{B}$ meet in a vertex which is not an endpoint of $\mathcal{A}$ nor an endpoint of $\mathcal{B}$. Therefore, condition (b) (ii) of Theorem 2.3 is not satisfied, and the ideal $I(\mathcal{A} \cup \mathcal{B})$ does not have a quadratic Gröbner basis for the lexicographic order induced by any order of the variables.

3. Minimal prime ideals of convex configurations of adjacent 2-minors. Let $[a_1, a_2 \mid b_1, b_2]$ be a 2-minor. Each of the adjacent 2-minors $[a, a+1 \mid b, b+1]$ with $a_1 \leq a < a_2$ and $b_1 \leq b < b_2$ is called an adjacent 2-minor of $[a_1, a_2 \mid b_1, b_2]$.

Let $\mathcal{C}$ be a configuration of adjacent 2-minors, and let $\delta = [a_1, a_2 \mid b_1, b_2]$ be a 2-minor whose vertices belongs to $V(\mathcal{C})$. Then δ is called an *inner minor* of $\mathcal{C}$, if all adjacent 2-minors of δ belong to $\mathcal{C}$. The set of inner minors of $\mathcal{C}$ will be denoted by $\mathcal{G}(\mathcal{C})$ and the ideal they generate by $J(\mathcal{C})$.

Following Quereshi [**10**], we call a weakly connected configuration $\mathcal{C}$ of adjacent 2-minors *convex* if each minor $[a_1, a_2 \mid b_1, b_2]$ whose vertices belong to $V(\mathcal{C})$ is an inner minor of $\mathcal{C}$. An arbitrary configuration $\mathcal{C}$ of adjacent 2-minors is called convex if each of its weakly connected components is convex.

In this section we want to describe a primary decomposition of $\sqrt{I(\mathcal{C})}$. For this purpose, we have to introduce some terminology. Let $\mathcal{C} = \delta_1, \delta_2, \ldots, \delta_r$ be an arbitrary configuration of adjacent 2-minors. A subset W of the vertex set of $\mathcal{C}$ is called *admissible* if, for each index i, either $W \cap V(\delta_i) = \varnothing$ or $W \cap V(\delta_i)$ contains an edge of δ_i. For example, the admissible sets of the configuration shown in Figure 12

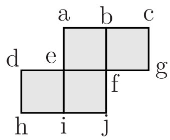

FIGURE 12.

are the following

$$\varnothing, \{c,g\}, \{d,h\}, \{a,e,i\}, \{b,f,j\}, \{a,b,c\}, \ldots, \{a,b,c,d,e,f,g,h,i,j\}.$$

Let $W \subset V(\mathcal{C})$ be an admissible set. We define an ideal $P_W(\mathcal{C})$ containing $I(\mathcal{C})$ as follows: let $\mathcal{C}' = \{\delta \in \mathcal{C} : V(\delta) \cap W = \varnothing\}$. Then the generators of $P_W(\mathcal{C})$ are the variables belonging to W and the generators of the ideal $J(\mathcal{C}')$ of inner minors of $\mathcal{C}'$. Note that $P_W(\mathcal{C}) = (W, P_\varnothing(\mathcal{C}'))$.

For example, if we take the configuration displayed in Figure 12, then

$$\begin{aligned} P_\varnothing(\mathcal{C}) &= (af-be, aj-bi, ej-fi, ag-ce, bg-cf, \\ &\qquad\qquad di-eh, dj-fh), \\ P_{\{d,h\}}(\mathcal{C}) &= (d, h, af-be, aj-bi, ej-fi, ag-ce, bg-cf). \end{aligned}$$

Lemma 3.1. *Let $\mathcal{C}$ be a convex configuration of adjacent 2-minors, and let $P_W(\mathcal{C}) = (W, P_\varnothing(\mathcal{C}'))$, where $\mathcal{C}' = \{\delta \in \mathcal{C} : V(\delta) \cap W = \varnothing\}$ and $W \subset V(\mathcal{C})$ is an admissible set. Then $\mathcal{C}'$ is again a convex configuration of 2-adjacent minors. In particular, for any admissible set $W \subset V(\mathcal{P})$, the ideal $P_W(\mathcal{C})$ is a prime ideal.*

Proof. Let $\mathcal{C}''$ be one of the weakly connected components of $\mathcal{C}'$, and let $[a_1, a_2 \mid b_1, b_2]$ be a minor whose vertices belong to $V(\mathcal{C}'')$. We want to show that $[a_1, a_2 \mid b_1, b_2]$ is an inner minor of $\mathcal{C}''$, in other words, that all adjacent 2-minors $\delta = [a, a+1 \mid b, b+1]$ of $[a_1, a_2 \mid b_1, b_2]$ belong to $\mathcal{C}''$. Suppose one of these adjacent 2-minors, say $\delta = [i, i+1 \mid j, j+1]$, does not belong to $\mathcal{C}'$. Then one of the edges of δ belongs to W, say $\{x_{i+1,j}, x_{i+1,j+1}\}$. If δ does not meet the vertices on the border lines connecting the corners x_{a_1,b_2} and x_{a_2,b_2}, and x_{a_2,b_1} and x_{a_2,b_2}, then $\delta' = [i+1, i+2 \mid j+1, j+2]$ belongs to $[a_1, a_2 \mid b_1, b_2]$, and hence it belongs to $\mathcal{C}$ since $\mathcal{C}$ is convex. Since $V(\delta') \cap W \neq \varnothing$ and W is an admissible set of $\mathcal{C}$, we see that either $x_{i+1,j+2} \in W$ or $x_{i+2,j+1} \in W$. Proceeding in this way, we see that W meets a border line of $[a_1, a_2 \mid b_1, b_2]$. We may assume that $x_{j,b_2} \in W$ for some j with $a_1 + 1 < j < a_2 - 1$.

Now, if the adjacent 2-minor $[j, j+1 \mid b_2, b_2+1] \in \mathcal{C}$, then either $x_{j+1,b_2} \in W$ or $x_{j,b_2+1} \in W$. Proceeding in this way, we find a sequence of elements $x_{i_1,j_1}, \ldots, x_{i_r,j_r}$ which belongs to W with the properties that:

(i) $(i_1, j_1) = (j, b_2)$,

(ii) for all k with $1 \leq k < r$, we have $(i_{k+1}, j_{k+1}) = (i_k + 1, j_k)$ or $(i_{k+1}, j_{k+1}) = (i_k, j_k + 1)$, and

(iii) the adjacent 2-minor $[i_r, i_r + 1 \mid j_r, j_r + 1]$ does not belong to $\mathcal{C}$ (otherwise the sequence could be extended).

Moreover, for $1 \leq k < r$, we have that $\delta_k = [i_k, i_k + 1 \mid j_k, j_k + 1] \in \mathcal{C}$ and $\delta_k \cap W \neq \varnothing$ for all k. By construction, $\delta_{r-1} = [i_r - 1, i_r \mid j_r, j_r + 1]$ or $\delta_{r-1} = [i_r, i_r + 1 \mid j_r - 1, j_r]$ belong to $\mathcal{C}$. We may assume that $\delta_{r-1} = [i_r - 1, i_r \mid j_r, j_r + 1]$. Then it follows that all the adjacent 2-minors $\gamma_k = [k, k+1 \mid j_r, j_r + 1]$ for $k = i_r, \ldots, m-1$ do not belong to $\mathcal{C}$. Indeed, if $\gamma_k \in \mathcal{C}$ for some k, then since $\delta_{r-1} = [i_r - 1, i_r \mid j_r, j_r + 1]$ belongs to $\mathcal{C}$ and since $\mathcal{C}$ is convex, it would follow that $[i_r, i_r + 1 \mid j_r, j_r + 1]$ belongs to $\mathcal{C}$, a contradiction. Similarly, there exists $x_{k_1,l_1}, \ldots, x_{k_s,l_s}$ which belongs to W with the properties that

(i) $(k_1, l_1) = (j, b_2)$,

(ii) for all t with $1 \leq t < s$, we have $(i_{t+1}, j_{t+1}) = (i_t - 1, j_t)$ or $(i_{t+1}, j_{t+1}) = (i_t, j_t - 1)$, and either the adjacent 2-minors $[i_s - 1, i_s \mid k-1, k]$ do not belong to $\mathcal{C}$ for $k = 1, \ldots, j_s$, or the adjacent 2-minors $[k-1, k \mid j_s - 1, j_s]$ do not belong to $\mathcal{C}$ for $k = 1, \ldots, i_s$.

Since vertices x_{a_1,b_2} and x_{a_2,b_2} belong to the weakly connected component $\mathcal{C}''$ of $\mathcal{C}'$, there exists a chain $\sigma_1, \ldots, \sigma_v$ of adjacent 2-minors in $\mathcal{C}'$ with $V(\sigma_i) \cap V(\sigma_{i+1}) \neq \varnothing$ for all i and such that $x_{a_1,b_2} \in \sigma_1$ and $x_{a_2,b_2} \in \sigma_v$. It follows that $\{x_{i_1,j_1}, \ldots, x_{i_r,j_r}, x_{k_1,l_1}, \ldots, x_{k_s,l_s}\} \cap \sigma_i \neq \varnothing$ for some i. Therefore, $V(\sigma_i) \cap W \neq \varnothing$, a contradiction since $\sigma_i \in \mathcal{C}'$.

Now, since $\mathcal{C}'$ is a convex configuration, it follows by a result of Qureshi [**10**, Theorem 2.2] that $P_{\varnothing}(\mathcal{C}')$ is a prime ideal. Therefore, $P_W(\mathcal{C})$ is a prime ideal as well. $\square$

Theorem 3.2. *Let $\mathcal{C}$ be a convex configuration of adjacent 2-minors. Let P be a minimal prime ideal of $I(\mathcal{C})$. Then there exists an admissible set $W \subset V(\mathcal{C})$ such that $P = P_W(\mathcal{C})$. In particular.*

$$\sqrt{I(\mathcal{C})} = \bigcap_W P_W(\mathcal{C}),$$

where the intersection is taken over all admissible sets $W \subset V(\mathcal{C})$.

Proof. Let P be any minimal prime ideal of $I(\mathcal{C})$, and let W be the set of variables among the generators of P. We claim that W is admissible. Indeed, suppose that $W \cap V(\delta) \neq \varnothing$ for some adjacent 2-minor of $\mathcal{C}$. Say, $\delta = ad - bc$ and $a \in W$. Then $bc \in P$. Hence, since P is a prime ideal, it follows that $b \in P$ or $c \in P$. Thus, W contains the edge $\{a, c\}$ or the edge $\{a, b\}$ of δ.

Since $I(\mathcal{C}) \subset P$, it follows that $(W, I(\mathcal{C})) \subset P$. Observe that $(W, I(\mathcal{C})) = (W, I(\mathcal{C}'))$, where $W \cap V(\mathcal{C}') = \varnothing$ and $\mathcal{C}'$ is again a convex configuration, see the proof of Lemma 3.1. Modulo W, we obtain a minimal prime ideal $\overline{P}$, which contains no variables of the ideal $I(\mathcal{C}')$.

By [**10**, Theorem 2.2] the ideal $P_\varnothing(\mathcal{C}')$ is a prime ideal containing $I(\mathcal{C}')$. Thus, the assertion of the theorem follows once we have shown that $P_\varnothing(\mathcal{C}') \subset \overline{P}$.

Since $P_\varnothing(\mathcal{C}')$ is generated by the union of the set of 2-minors of certain $r \times s$-matrices, it suffices to show that if P is a prime ideal having no variables among its generators and containing all adjacent 2-minors of the $r \times s$-matrix X, then it contains all 2-minors of X. In order to prove this, let $\delta = [a_1, a_2 \mid b_1, b_2]$ be an arbitrary 2-minor of X. We prove that $\delta \in P$ by induction on $(a_2 - a_1) + (b_2 - b_1)$. For $(a_2 - a_1) + (b_2 - b_1) = 2$, this is the case by assumption. Now let $(a_2 - a_1) + (b_2 - b_1) > 2$. We may assume that $a_2 - a_1 > 1$. Let $\delta_1 = [a_1, a_2 - 1 | b_1, b_2]$ and $\delta_2 = [a_2 - 1, a_2 | b_1, b_2]$. Then $x_{a_2-1,b_1}\delta = x_{a_2,b_1}\delta_1 + x_{a_1,b_1}\delta_2$. Therefore, by induction hypothesis, $x_{a_2-1,b_1}\delta \in P$. Since P is a prime ideal, and $x_{a+k-1,1} \notin P$, it follows that $\delta \in P$, as desired. $\square$

In general, it seems to be pretty hard to find the primary decomposition for ideals generated by adjacent 2-minors. This seems to be even difficult for ideals described in Theorem 2.3. For example, the primary decomposition (computed with the help of *Singular*) of the ideal $I(\mathcal{C})$

of adjacent 2-minors shown in Figure 13 is the following:

$$\begin{aligned} I(\mathcal{C}) &= (ae-bd, ch-dg, ej-fi, hl-ik) \\ &= (ik-hl, fi-ej, dg-ch, bd-ae, bcjk-afgl) \\ &\quad \cap (d,e,h,i). \end{aligned}$$

It turns out that $I(\mathcal{C})$ is a radical ideal. On the other hand, if we add the minor $di-eh$, we get a connected configuration $\mathcal{C}'$ of adjacent 2-minors. The ideal $I(\mathcal{C}')$ is not radical, because it contains a pin, see Proposition 4.2. Indeed, one has

$$\begin{aligned} \sqrt{I(\mathcal{C}')} = (&ae-bd, ch-dg, ej-fi, hl-ik, di-eh, \\ &fghl-chjl, bfhl-aejl, bchk-achl, bcfh-acej) \end{aligned}$$

Applying Theorem 3.3, we get

$$\begin{aligned} \sqrt{I(\mathcal{C}')} &= (ae-bd, ch-dg, ej-fi, hl-ik, di-eh, fghl-chjl, \\ &\qquad bfhl-aejl, bchk-achl, bcfh-acej) \\ &= (-ik+hl, -fi+ej, -ek+dl, -fh+dj, -eh+di, -fg+cj, \\ &\qquad -eg+ci, -dg+ch, -bk+al, -bh+ai, -bd+ae) \\ &\quad \cap (d,e,h,i) \cap (a,d,h,i,j) \cap (d,e,f,h,k) \cap (c,d,e,i,l) \cap (b,e,g,h,i) \\ &\quad \cap (a,d,h,k,ej-fi) \cap (c,d,e,f,hl-ik) \cap (b,e,i,l,ch-dg) \\ &\quad \cap (g,h,i,j,ae-bd). \end{aligned}$$

The presentation of $\sqrt{I(\mathcal{C})}$ as an intersection of prime ideals as given in Theorem 3.2 is usually not irredundant. In order to obtain an irredundant intersection, we have to identify the minimal prime ideals of $I(\mathcal{C})$ among the prime ideals $P_W(\mathcal{C})$.

For any configuration $\mathcal{C}$, we denote by $\mathcal{G}(\mathcal{C})$ the set of adjacent 2-minors generating $P_\varnothing(\mathcal{C})$.

Theorem 3.3. *Let $\mathcal{C}$ be a convex configuration of adjacent 2-minors, let $V, W \subset V(\mathcal{C})$ be admissible sets of $\mathcal{C}$, and let $P_V(\mathcal{C}) = (V, \mathcal{G}(\mathcal{C}'))$ and $P_W(\mathcal{C}) = (W, \mathcal{G}(\mathcal{C}''))$ where $\mathcal{C}' = \{\delta \in \mathcal{C} : V(\delta) \cap V = \varnothing\}$ and $\mathcal{C}'' = \{\delta \in \mathcal{C} : V(\delta) \cap W = \varnothing\}$. Then*

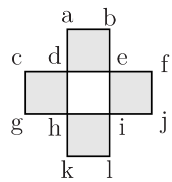

FIGURE 13.

(a) $P_V(\mathcal{C}) \subset P_W(\mathcal{C})$ *if and only if* $V \subset W$*, and for all elements*

$$\delta \in \mathcal{G}(\mathcal{C}') \setminus \mathcal{G}(\mathcal{C}''),$$

one has that $W \cap V(\delta)$ *contains an edge of* δ*.*

(b) $P_W(\mathcal{C}) = (W, \mathcal{G}(\mathcal{C}''))$ *is a minimal prime ideal of* $I(\mathcal{C})$ *if and only if, for all admissible subsets* $V \subset W$ *with* $P_V(\mathcal{C}) = (V, \mathcal{G}(\mathcal{C}'))$*, there exists a*

$$\delta \in \mathcal{G}(\mathcal{C}') \setminus \mathcal{G}(\mathcal{C}'')$$

such that the set $W \cap V(\delta)$ *does not contain an edge of* δ*.*

Proof. (a) Suppose that $P_V(\mathcal{C}) \subset P_W(\mathcal{C})$. The only variables in $P_W(\mathcal{C})$ are those belonging to W. This shows that $V \subset W$. The inclusion $P_V(\mathcal{C}) \subset P_W(\mathcal{C})$ implies that $\delta \in (W, \mathcal{G}(\mathcal{C}''))$ for all $\delta \in \mathcal{G}(\mathcal{C}')$. Suppose $W \cap V(\delta) = \varnothing$. Then δ belongs to $P_\varnothing(\mathcal{C}'') = (\mathcal{G}(\mathcal{C}''))$. Let $f = u - v \in \mathcal{G}(\mathcal{C}'')$. Neither u nor v appears in another element of $\mathcal{G}(\mathcal{C}'')$. Therefore, any binomial of degree 2 in $P_\varnothing(\mathcal{C}'')$ belongs to $\mathcal{G}(\mathcal{C}'')$. In particular, $\delta \in \mathcal{G}(\mathcal{C}'')$, a contradiction. Therefore, $W \cap V(\delta) \neq \varnothing$.

Suppose that $W \cap V(\delta)$ does not contain an edge of $\delta = ad - bc$. We may assume that $a \in W \cap V(\delta)$. Then, since $\delta \in P_W(\mathcal{C})$, it follows that $bc \in P_W(\mathcal{C})$. Since $P_W(\mathcal{C})$ is a prime ideal, we conclude that $b \in P_W(\mathcal{C})$ or $c \in P_W(\mathcal{C})$. Then $b \in W$ or $c \in W$, and hence either the edge $\{a, b\}$ or the edge $\{a, c\}$ belongs to $W \cap V(\delta)$.

The 'if' part of statement (a) is obvious.

(b) is a simple consequence of Theorem 3.2 and statement (a). $\square$

In Figure 14 we display all the minimal prime ideals $I(\mathcal{P})$ for the path $\mathcal{P}$ shown in Figure 12. The fat dots mark the admissible sets, and the

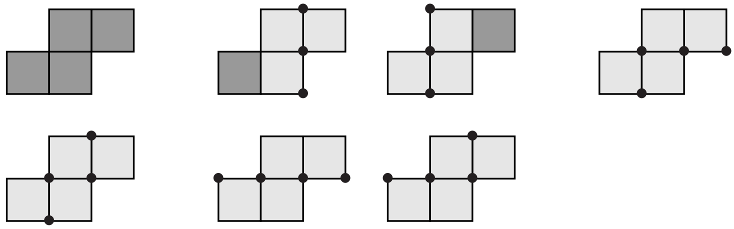

FIGURE 14. The admissible sets of the configuration shown in Figure 12.

dark shadowed areas mark the regions where the inner 2-minors have to be taken.

4. Strongly connected configurations which are radical. We call a connected configuration of adjacent 2-minors *strongly connected* if the following condition is satisfied: for any two adjacent 2-minors $\delta_1, \delta_2 \in \mathcal{C}$ which have exactly one vertex in common, there exists a $\delta \in \mathcal{C}$ which has a common edge with δ_1 and a common edge with δ_2.

This section is devoted to studying the strongly connected configuration of adjacent 2-minors $\mathcal{C}$ for which $I(\mathcal{C})$ is a radical ideal.

We call a configuration $\mathcal{C}$ of adjacent 2-minors a *cycle* if, for each $\delta \in \mathcal{C}$, there exist exactly two $\delta_1, \delta_2 \in \mathcal{C}$ such that δ and δ_1 have a common edge and δ and δ_2 have a common edge.

Lemma 4.1. *Let $\mathcal{C}$ be a strongly connected configuration which does not contain a pin. Then $\mathcal{C}$ is a path or a cycle.*

Proof. If $\mathcal{C}$ does not contain a pin, then for each adjacent 2-minor $\delta \in \mathcal{C}$ there exists at most two adjacent 2-minors in $\mathcal{C}$ which have a common edge with δ. Thus, if $\mathcal{C}$ is not a cycle but connected, there exist $\delta_1, \delta_2 \in \mathcal{C}$ such that δ_1 has a common edge only with δ_1. Now in the configuration $\mathcal{C}' = \mathcal{C} \setminus \{\delta_1\}$, the element δ_2 has at most one edge in common with another element of $\mathcal{C}'$. If δ_2 has no edge in common with another element of $\mathcal{C}'$, then $\mathcal{C} = \{\delta_1, \delta_2\}$. Otherwise, continuing this argument, a simple induction argument yields the desired conclusion. $\square$

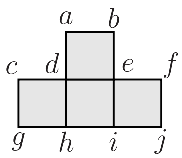

FIGURE 15.

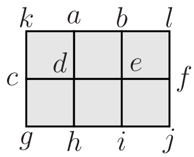

FIGURE 16.

Proposition 4.2. *Let $\mathcal{C}$ be a strongly connected configuration of adjacent 2-minors. If $I(\mathcal{C})$ is a radical ideal, then $\mathcal{C}$ is a path or a cycle.*

Proof. By Lemma 4.1, it is enough to prove that $\mathcal{C}$ does not contain a pin. Suppose $\mathcal{C}$ contains the pin $\mathcal{C}'$ as shown in Figure 15.

Then $q = acej - bcfh \notin I(\mathcal{C}')$ but $q^2 \in I(\mathcal{C}') \subset I(\mathcal{C})$. We consider two cases. In the first case, suppose that the adjacent 2-minors $kd - ac$ and $bf - le$ do not belong to $\mathcal{C}$, see Figure 16.

Then $q \notin (I(\mathcal{C}), W)$ where W is the set of vertices which do not belong to $\mathcal{C}'$. It follows that $q \notin I(\mathcal{C})$. In the second case we may assume that $ac - kd \in \mathcal{C}$. Let $\mathcal{C}''$ be the configuration with the adjacent 2-minors $kd - ac, ae - bd, ch - dg, di - eh$. Then $r = kdi - aeg \notin I(\mathcal{C}'')$ but $r^2 \in I(\mathcal{C}'') \subset I(\mathcal{C})$. Then $r \notin (I(\mathcal{C}), V)$ where V is the set of vertices in $\mathcal{C}$ which do not belong to $\mathcal{C}'$. It follows that $r \notin I(\mathcal{C})$. Thus, in both cases, we see that $I(\mathcal{C})$ is not a radical ideal. $\square$

For the cycle $\mathcal{C}$ displayed in Figure 17, the ideal $I(\mathcal{C})$ is not radical. Indeed we have $f = b^2hino - abhjno \notin I(\mathcal{C})$, but $f^2 \in I(\mathcal{C})$. By computational evidence, we expect that the ideal of adjacent 2-minors of a cycle is a radical ideal if and only if the length of the cycle is

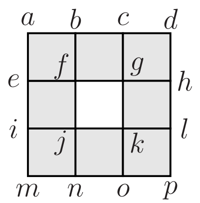

FIGURE 17.

≥ 12. On the other hand, if $\mathcal{P}$ is a monotone path, we know from Theorem 2.3 that $I(\mathcal{P})$ has a squarefree initial ideal. This implies that $I(\mathcal{P})$ is a radical ideal. More generally, we expect that ideal of adjacent 2-minors of a path $\mathcal{P}$ is always a radical ideal, and prove this under the assumption that the ideal $I(\mathcal{P})$ has no embedded prime ideals. In [**3**, Theorem 4.2], the primary decomposition of the ideal of adjacent 2-minors of a 4×4-matrix is given, from which it can be seen that in general the ideal of adjacent 2-minors of a strongly connected configuration may have embedded prime ideals.

Theorem 4.3. *Let $\mathcal{P}$ be path, and suppose that $I(\mathcal{P})$ has no embedded prime ideals. Then $I(\mathcal{P})$ is a radical ideal.*

The proof will require several steps.

Lemma 4.4. *Let $I \subset S$ be an ideal, and let $a, b \in S$ be such that a is a nonzerodivisor modulo (b, I). Then*

$$(ab, I) = (a, I) \cap (b, I).$$

Proof. Obviously one has $(ab, I) \subset (a, I) \cap (b, I)$. Conversely, let $f \in (a, I) \cap (b, I)$. Then

$$f = ag_1 + h_1 = bg_2 + h_2 \quad \text{with} \quad g_1, g_2 \in S \text{ and } h_1, h_2 \in I.$$

Therefore, $ag_1 \in (b, I)$. Since a is a nonzerodivisor modulo (b, I), it follows that $g_1 = cb + h$ for some $c \in S$ and $h \in I$. Hence, we get that $f = ag_1 + h_1 = a(cb + h) + h_1 = abc + (ah + h_1)$. Thus, $f \in (ab, I)$. $\square$

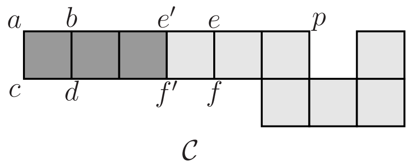

FIGURE 18.

Lemma 4.5. *Let $\mathcal{P} = \delta_1, \delta_2, \ldots, \delta_r$ be a path which is not a line, and let $i > 1$ be the smallest index for which δ_i has a free vertex c (which we call a corner of the path). Let a be the free vertex of δ_1 whose first or second coordinate coincides with that of c, see Figure 18. Then a does not belong to any minimal prime ideal of $I(\mathcal{P})$.*

In particular, if $I(\mathcal{P})$ has no embedded prime ideals, element a is a nonzerodivisor of $S/I(\mathcal{P})$.

Proof. We may assume that, like in Figure 18, the first 2-adjacent minors up to the first corner form a horizontal path. Let W be an admissible set with $a \in W$. In our discussion, we refer to the notation given in Figure 18. Then, since W is admissible, we have $b \in W$ or $c \in W$.

First suppose that $c \in W$. If $W = \{a, c\}$, then $P_\varnothing(\mathcal{P})$ is a proper subset of $P_W(\mathcal{P})$, and so $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$. Hence, we may assume that $\{a, c\}$ is a proper subset of W. In case of $d \in W$, it follows that $V = W \setminus \{a\}$ is an admissible set with $\mathcal{G}(V) = \mathcal{G}(W)$. In case of $b \in W$, it follows that $V = W \setminus \{c\}$ is an admissible set with $\mathcal{G}(V) = \mathcal{G}(W)$. Hence, in both cases it follows from Theorem 3.3 that $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$. On the other hand, in case of $d \notin W$ and $b \notin W$, it follows that $V = W \setminus \{a, b\}$ is an admissible set with either $\mathcal{G}(V) = \mathcal{G}(W)$ or $\mathcal{G}(W) = \mathcal{G}(V) \cup \{ad - bc\}$. Hence by Theorem 3.3, $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$.

In the second case, suppose that $c \notin W$. Then $b \in W$. Let $a = (i, j)$ and $p = (k, j)$ with $k > i + 1$, and let $[a, p] = \{(l, j) : i \leq l \leq k\}$. Then $b \in [a, c]$. If $W = [a, p]$, then $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$ because, in that case, $P_\varnothing(\mathcal{P})$ is a proper subset of $P_W(\mathcal{P})$. On the other hand, if W is a proper subset of $[a, p]$, then W is not admissible. Hence, there exists an $e \in [a, p]$ such that $[a, e] \subset W$ and,

FIGURE 19.

moreover, f, as indicated in Figure 18, belongs to W. We may assume that $[b, f'] \cap W = \varnothing$. Let $V = W \setminus [a, e']$. Then V is admissible. Since $\mathcal{G}(V) \setminus \mathcal{G}(W)$ consists of those adjacent 2-minors which are indicated in Figure 18 as the dark shadowed area, it follows from Theorem 3.3 that $P_W(\mathcal{P})$ is not a minimal prime ideal of $I(\mathcal{C})$. ◻

The vertex c in Figure 18 is the first corner of path $\mathcal{C}$. Therefore, according to Lemma 4.5, element a is not contained in any minimal prime ideal of $I(\mathcal{C})$.

Proof of Theorem 4.3. Let $\mathcal{P} = \delta_1, \delta_2, \ldots, \delta_r$ be a path, and choose the vertex $a \in \delta_1$ as described in Proposition 4.5. Then our hypothesis implies that a is a nonzerodivisor modulo $I(\mathcal{P})$. The graded version of Lemma 4.4.9 in [**1**] implies then that $I(\mathcal{P})$ is a radical ideal if and only if $(a, I(\mathcal{P}))$ is a radical ideal. Thus, it suffices to show that $(a, I(\mathcal{P}))$ is a radical ideal.

Let $\delta_1 = ad - bc$ and $\mathcal{P}' = \delta_2, \ldots, \delta_r$ be the path which is obtained from $\mathcal{P}$ by removing δ_1. Then $(a, I(\mathcal{P})) = (a, bc, I(\mathcal{P}'))$. Thus, $(a, I(\mathcal{P}))$ is a radical ideal if $(bc, I(\mathcal{P}'))$ is a radical ideal, because a is a variable which does not appear in $(bc, I(\mathcal{P}'))$. Since c is regular modulo $(b, I(\mathcal{P}'))$, we may apply Lemma 4.4 and get that $(bc, I(\mathcal{P}')) = (b, I(\mathcal{P}')) \cap (c, I(\mathcal{P}'))$. By using induction of the length of the path, we may assume that $I(\mathcal{P}')$ is a radical ideal. Since c does not appear $I(\mathcal{P}')$ it follows that $(c, I(\mathcal{P}))$ is a radical ideal. Thus, it remains to be shown that $(b, I(\mathcal{P}'))$ is a radical ideal. Observe that b is one of the vertices of δ_2. If it is a free vertex, we can argue as before. So we may assume that b is not free. The following Figure 19 (i) and Figure 19 (ii) indicate (up to rotation and reflection) the possible positions of b in $\mathcal{P}'$.

In the case of Figure 19 (i), we have $(b, I(\mathcal{P}')) = (b, df, gf, I(\mathcal{P}''))$ where $\mathcal{P}' = \delta_4, \ldots, \delta_r$. Since the variable b does not appear in $(df, gf, I(\mathcal{P}''))$, it follows that $(b, I(\mathcal{P}'))$ is a radical ideal if and only if $(df, gf, I(\mathcal{P}''))$ is a radical ideal. Applying Lemma 4.4, we see that $(df, gf, I(\mathcal{P}'')) = (d, gf, I(\mathcal{P}'')) \cap (f, I(\mathcal{P}''))$. By induction hypothesis we may assume that $(f, I(\mathcal{P}''))$ is a radical ideal. Thus, it remains to be shown that $(d, gf, I(\mathcal{P}''))$ is a radical ideal which is the case if $(gf, I(\mathcal{P}''))$ is a radical ideal. Once again we apply Lemma 4.4 and get $(gf, I(\mathcal{P}'')) = (g, I(\mathcal{P}'')) \cap (f, I(\mathcal{P}''))$. By assumption of induction, we deduce as before that both ideals $(g, I(\mathcal{P}''))$ and $(f, I(\mathcal{P}''))$ are radical ideals. Therefore, $(gf, I(\mathcal{P}''))$ is a radical ideal.

In the case of Figure 19 (ii) a similar argument works. $\square$

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