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SOME CHARACTERIZATIONS OF FIRST NEIGHBORHOOD COMPLETE IDEALS IN DIMENSION TWO

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ABSTRACT. In this paper we study first neighborhood complete ideals in a two-dimensional normal Noetherian local domain $(R, \mathfrak{M})$ with algebraically closed residue field and the associated graded ring an integrally closed domain. It is shown that a complete quasi-one-fibered $\mathfrak{M}$-primary ideal I of R is a first neighborhood complete ideal if and only if $e(I) = e(\mathfrak{M}) + 1$. This implies that $(R, \mathfrak{M})$ is a rational singularity if and only if a (every) first neighborhood complete ideal has minimal multiplicity. Moreover, if $(R, \mathfrak{M})$ has minimal multiplicity, then a complete quasi-one-fibered $\mathfrak{M}$-primary ideal I of order one is a first neighborhood complete ideal if and only if certain numerical data associated with I are minimal. This yields a simple proof of the fact that first neighborhood complete ideals in such a local ring R are projectively full.

1. Introduction. Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain, that is, a two-dimensional normal Noetherian local domain with algebraically closed residue field and its associated graded ring an integrally closed domain. It follows that the $\mathfrak{M}$-adic order function ord_R is a valuation (denoted by $v_{\mathfrak{M}}$), and the blowup $\mathrm{Bl}_{\mathfrak{M}}R$ of R at $\mathfrak{M}$ is a desingularization of R.

In [**2**] complete $\mathfrak{M}$-primary ideals I adjacent from below to $\mathfrak{M}$ (that is, length $(\mathfrak{M}/I)$ equals 1) have been studied. It has been proved in [**2**, Theorem 3.2, page 1144] that these ideals are precisely the inverse transforms in R of the maximal ideals $\mathfrak{M}'$ of the immediate quadratic transforms $(R', \mathfrak{M}')$ of $(R, \mathfrak{M})$. Therefore, these ideals have been called *first neighborhood complete ideals*.

In [**2**, Lemma 3.1, page 1144], it has been shown that, if the first neighborhood complete ideal I is the inverse transform of $\mathfrak{M}'$, then the set $T(I)$ of Rees valuations satisfies $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$, where w denotes the $\mathrm{ord}_{R'}$-valuation (w is called an immediate prime

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divisor of R). Note that $v_{\mathfrak{M}}$ may or may not belong to $T(I)$ and, if $v_{\mathfrak{M}} \notin T(I)$, then $(R, \mathfrak{M})$ is regular (cf. [**2**, Theorem 3.3]).

Complete $\mathfrak{M}$-primary ideals I in a two-dimensional Muhly local domain whose set $T(I)$ of Rees valuations satisfies the conditions $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$ with $w \neq v_{\mathfrak{M}}$ a prime divisor of R, have been called quasi-one-fibered complete $\mathfrak{M}$-primary ideals in [**4**].

In this paper first neighborhood complete ideals of a two-dimensional Muhly local domain $(R, \mathfrak{M})$ will be further investigated. It will be shown in Proposition 3.2 that a complete quasi-one-fibered $\mathfrak{M}$-primary ideal I in a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is a first neighborhood complete ideal if and only if $e(I) = e(\mathfrak{M}) + 1$. Here $e(I)$ (respectively, $e(\mathfrak{M})$) denotes the multiplicity of I (respectively, $\mathfrak{M}$).

As a consequence, the following criterion for $(R, \mathfrak{M})$ to be a rational singularity is obtained in Corollary 3.4: $(R, \mathfrak{M})$ is a rational singularity if and only if a (every) first neighborhood complete ideal has minimal multiplicity. It implies that, for any first neighborhood complete ideal I in a two-dimensional Muhly local domain $(R, \mathfrak{M})$ with minimal multiplicity, we have that $\mathfrak{M}I \subseteq \operatorname{core}(I)$ (see Corollary 3.5).

Associated with a complete quasi-one-fibered $\mathfrak{M}$-primary ideal I of order one in a two-dimensional Muhly local domain R there are certain numerical data: $\operatorname{rank}(I)$, $\mu(I)$, $w(\mathfrak{M})$ and $d(I, w)$ (for an explanation, see Section 3). In Proposition 3.6 we prove that, if $(R, \mathfrak{M})$ has minimal multiplicity, then I is a first neighborhood complete ideal if and only if these numerical data are minimal. As a result, we can give a simple proof of the fact that, in a two-dimensional Muhly local domain $(R, \mathfrak{M})$ with minimal multiplicity, any first neighborhood complete ideal is projectively full (see Proposition 3.7).

2. Background. In this paper we consider two-dimensional normal Noetherian local domains $(R, \mathfrak{M})$ with algebraically closed residue field and the associated graded ring $\operatorname{gr}_{\mathfrak{M}} R$ an integrally closed domain. These rings (in fact with one more condition) were studied in the early 1960s by Muhly (jointly with Sakuma).

For ease of reference, we therefore have called them *two-dimensional Muhly local domains* (see [**2, 3**]). We recall that all powers of $\mathfrak{M}$ are integrally closed (i.e., $\mathfrak{M}$ is a normal ideal) and the $\mathfrak{M}$-adic order function $v_{\mathfrak{M}}$ is a valuation of the quotient field K of R. Moreover, the blowup $\mathrm{Bl}_{\mathfrak{M}}R$ of R at $\mathfrak{M}$ is a desingularization of R (see [**3**, page 1650]).

Any two-dimensional local ring $(R', \mathfrak{M}')$ of the blowup $\mathrm{Bl}_{\mathfrak{M}}R$ is *regular* and it has the form

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_M,$$

where $M = (x_1, (x_2/x_1), \dots, (x_d/x_1))$ is a maximal ideal of $R[\mathfrak{M}/x_1]$ lying over $\mathfrak{M}$ and $\mathfrak{M}' = (x_1, (x_2/x_1))R'$. Here $x_1, x_2, \dots, x_d$ denotes a suitably chosen minimal ideal basis of $\mathfrak{M}$ (see [**3**, page 1651]). The local ring $(R', \mathfrak{M}')$ is called an *immediate* (or a *first*) *quadratic transform* of $(R, \mathfrak{M})$.

Let I be a complete $\mathfrak{M}$-primary ideal of R, and let $(R', \mathfrak{M}')$ be an immediate quadratic transform of $(R, \mathfrak{M})$, with R' and $\mathfrak{M}'$ given as above. Then in R', we have

$$IR' = x_1^r I' \quad \text{with} \quad r := \mathrm{ord}_R(I),$$

and I' is an ideal in R' which is called the *transform* of I in R'. It mostly will be denoted by $I^{R'}$. If $I' \neq R'$ (equivalently IR' is not a principal ideal), then R' is said to be an *immediate base point* of I.

In [**4**], it has been shown that every immediate base point of I is a local ring of $\mathrm{Bl}_{\mathfrak{M}}R$ that is dominated by a Rees valuation ring of I. The converse holds in each of the following cases:

• $T(I) = \{v_{\mathfrak{M}}, w\}$ or $T(I) = \{w\}$, where w denotes a prime divisor of R with $w \neq v_{\mathfrak{M}}$.

• $(R, \mathfrak{M})$ satisfies condition (N), that is, for every prime divisor v of R there exists an $\mathfrak{M}$-primary ideal J of R such that $T(J) = \{v\}$.

(see [**4**, Proposition 1.1]).

In Section 3 of this paper we will use at several places some results from the theory of degree functions. We therefore now give a brief review in case $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain (although this theory has been developed in a more general type of local ring).

In [**12**], Rees has associated with an $\mathfrak{M}$-primary ideal I of R an integer-valued function d_I on $\mathfrak{M} \setminus \{0\}$ as follows:

$$d_I(x) = e\left(\frac{I + xR}{xR}\right)$$

where $x \in \mathfrak{M} \setminus \{0\}$ and $e(I + xR)/xR$ denotes the multiplicity of $(I + xR)/xR$. This function d_I is called the *degree function* defined by I. With every prime divisor v of R there is associated a non-negative integer $d(I, v)$, with $d(I, v) = 0$ for all except finitely many v, such that

$$d_I(x) = \sum_v d(I, v)v(x) \quad \text{for all} \quad 0 \neq x \in \mathfrak{M}$$

where the sum is over all prime divisors v of R ([**12**, Theorem 3.2]).

Since a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is analytically unramified, $d(I, v) \neq 0$ for all prime divisors v of R that are Rees valuations of I and $d(I, v') = 0$ for all other prime divisors v' of R. In [**13**] Rees and Sharp have proved that the integers $d(I, v)$ are uniquely determined, and we have called them the *degree function coefficients of* I. From this uniqueness, it follows that for $\mathfrak{M}$-primary ideals I and J in R, one has that

$$d(IJ, v) = d(I, v) + d(J, v)$$

for every prime divisor v of R (see [**13**, Lemma (5.1)]). Since R is quasi-unmixed and normal, this implies that

$$T(IJ) = T(I) \cup T(J).$$

For any $\mathfrak{M}$-primary ideal I in R, the multiplicity $e(I)$ of I is given by

$$e(I) = \sum_{v \in T(I)} d(I, v)v(I)$$

(see [**13**, Theorem (4.3)]). For $\mathfrak{M}$-primary ideals I and J in R, Rees and Sharp define

$$d_I(J) = \min\{d_I(x) \mid 0 \neq x \in J\}$$

and they prove (see [**13**, Theorem (5.2)]) that

$$d_I(J) = \sum_{v \in T(I)} d(I, v)v(J)$$

and

$$d_I(J) = d_J(I) = e_1(I \mid J)$$

where $e_1(I \mid J)$ denotes the mixed multiplicity of I and J, which is defined by $e(IJ) = e(I) + 2e_1(I|J) + e(J)$. Moreover, in [**13**, Corollary (5.3)], Rees and Sharp have shown that the following three statements are equivalent:

(i) $\overline{I} = \overline{J}$ where "$^{-}$" denotes the integral closure.

(ii) $d_I(x) = d_J(x)$ for all $x \in \mathfrak{M} \setminus \{0\}$

(iii) $d(I, v) = d(J, v)$ for every prime divisor v of R.

3. Main results. As we have already observed in the introduction, first neighborhood complete $\mathfrak{M}$-primary ideals of a two-dimensional Muhly local domain $(R, \mathfrak{M})$ are a special kind of complete quasi-one-fibered $\mathfrak{M}$-primary ideals. We therefore begin this section by recalling some background information concerning these types of ideals.

By definition, the *quasi-one-fibered* $\mathfrak{M}$-primary ideals of a two-dimensional Muhly local domain $(R, \mathfrak{M})$ are the $\mathfrak{M}$-primary ideals I such that $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$, with w a prime divisor $\neq v_{\mathfrak{M}}$ of R, together with all $\mathfrak{M}$-primary ideals I with $T(I) = \{v_{\mathfrak{M}}\}$. In this paper we will consider only quasi-one-fibered $\mathfrak{M}$-primary ideals I with $T(I) \neq \{v_{\mathfrak{M}}\}$.

Note that both possibilities, $v_{\mathfrak{M}} \in T(I)$ and $v_{\mathfrak{M}} \notin T(I)$, may occur. It follows that all degree function coefficients $d(I, v)$ of I are zero, except $d(I, w)$ and (possibly) $d(I, v_{\mathfrak{M}})$. In what follows, we therefore will consider only the degree function coefficients $d(I, w)$ and $d(I, v_{\mathfrak{M}})$ of I.

In [**4**] we have proved the following facts concerning a quasi-one-fibered $\mathfrak{M}$-primary ideal I in R:

• I has a unique immediate base point $(R', \mathfrak{M}')$, namely, the unique local ring $(R', \mathfrak{M}')$ of $\mathrm{Bl}_{\mathfrak{M}} R$ that is dominated by the valuation ring $(W, \mathfrak{M}_W)$ of w.

• There always exists a minimal ideal basis $x_1, x_2, \dots, x_d$ of $\mathfrak{M}$ such that

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_M \quad \text{with}$$
$$M = \left(x_1, \frac{x_2}{x_1}, \dots, \frac{x_d}{x_1}\right)$$

and

$$\mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right) R'.$$

• The integral closure of the transform of I in its unique immediate base point $(R', \mathfrak{M}')$ is some power of a simple complete $\mathfrak{M}'$-primary ideal I' of R'. Thus,

$$\overline{I^{R'}} = {I'}^t \text{ for some } t \in \mathbf{N}.$$

• Corresponding to I there is a unique finite quadratic sequence starting from $(R, \mathfrak{M})$ in $(W, \mathfrak{M}_W)$:

$$(R, \mathfrak{M}) < (R_1, \mathfrak{M}_1) < \cdots < (R_s, \mathfrak{M}_s) < (W, \mathfrak{M}_W)$$

such that $\overline{I^{R_s}} = \mathfrak{M}_s^n$ for some $n \in \mathbf{N}$. Further, $(R_1, \mathfrak{M}_1)$ is the unique immediate base point $(R', \mathfrak{M}')$ of I and the ord_{R_s}-valuation is the unique Rees valuation of $I^{R'}$. The length s of this sequence will be called the *rank* of I.

In the last part of this section the following numerical data corresponding to I will be considered:

• $\operatorname{ord}_R(I)$, i.e., the largest integer n such that $I \subseteq \mathfrak{M}^n$.

• $\mu(I)$, the number of elements in a minimal ideal basis of I.

• $e(I)$, the multiplicity of I.

• $\operatorname{rank}(I)$, the length of the quadratic sequence associated with I as explained above.

• $d(I, v_{\mathfrak{M}})$ and $d(I, w)$, the degree function coefficients of I.

- $w(\mathfrak{M})$, with w the ord_{R_s}-valuation.

In order to prove the first result of this section (see Proposition 3.2), we need the following lemma.

Lemma 3.1. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain, and let $(R', \mathfrak{M}')$ be immediate quadratic transform of $(R, \mathfrak{M})$. Let $x_1, x_2, \ldots, x_d$ be a minimal ideal basis of $\mathfrak{M}$ such that $R' = R[\mathfrak{M}/x_1]_M$ with $M = (x_1, (x_2/x_1), \ldots, (x_d/x_1))$ and $\mathfrak{M}' = (x_1, (x_2/x_1))R'$. Then*:

(i) *$R[\mathfrak{M}/x_1]$ is an integrally closed domain.*

(ii) *All powers of $M = (x_1, (x_2/x_1), \ldots, (x_d/x_1))$ are integrally closed.*

(iii) *$x_1^n M^n$ is integrally closed for every $n \in \mathbf{N}$.*

Proof. (i) Since R is an integrally closed domain, we have that

$$\overline{R\left[\frac{\mathfrak{M}}{x_1}\right]} = \bigcup_{n>0} \frac{\overline{\mathfrak{M}^n}}{x_1^n}.$$

Further, all powers of $\mathfrak{M}$ being integrally closed, it follows that

$$\overline{R\left[\frac{\mathfrak{M}}{x_1}\right]} = \bigcup_{n>0} \frac{\mathfrak{M}^n}{x_1^n} = R\left[\frac{\mathfrak{M}}{x_1}\right].$$

(ii) Since $(R', \mathfrak{M}')$ is a two-dimensional regular local ring, all powers of $\mathfrak{M}'$ are integrally closed. As $\mathfrak{M}' = MR[\mathfrak{M}/x_1]_M$, this implies that

$$\left(MR\left[\frac{\mathfrak{M}}{x_1}\right]_M\right)^n = M^n R\left[\frac{\mathfrak{M}}{x_1}\right]_M = \overline{M^n R\left[\frac{\mathfrak{M}}{x_1}\right]_M}.$$

Further,

$$M^n R\left[\frac{\mathfrak{M}}{x_1}\right]_M \subseteq \overline{M^n} R\left[\frac{\mathfrak{M}}{x_1}\right]_M \subseteq \overline{M^n R\left[\frac{\mathfrak{M}}{x_1}\right]_M}$$

(the last inclusion holds because of [**14**, Remark 1.1.3 (7)]. Hence,

$$M^n R\left[\frac{\mathfrak{M}}{x_1}\right]_M = \overline{M^n} R\left[\frac{\mathfrak{M}}{x_1}\right]_M.$$

Contraction to $R[\mathfrak{M}/x_1]$ yields

$$M^n = \overline{M^n}$$

since M^n and $\overline{M^n}$ are M-primary ideals.

(iii) Since $R[\mathfrak{M}/x_1]$ is an integrally closed domain and M^n is an integrally closed ideal, we have that $x_1^n M^n$ is also integrally closed ([**14**, Proposition 1.5.2 page 14]). $\square$

We are now ready to prove our first result in this section.

Proposition 3.2. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain, and suppose that I is a complete quasi-one-fibered $\mathfrak{M}$-primary ideal of R. Then I is a first neighborhood complete ideal if and only if*

$$e(I) = e(\mathfrak{M}) + 1.$$

Proof. Suppose I is a first neighborhood complete ideal of R. Then it is clear that $e(I) \geq e(\mathfrak{M}) + 1$. So we have only to prove that $e(I) \leq e(\mathfrak{M}) + 1$. In order to do so, let $(R', \mathfrak{M}')$ be the unique immediate base point of I (cf. [**2**, Theorem 3.2]). Then $R' = R[\mathfrak{M}/x_1]_M$ with $M = (x_1, (x_2/x_1), \dots, (x_d/x_1))$ and $\mathfrak{M}' = (x_1, (x_2/x_1))R'$ for a suitable minimal ideal basis $x_1, x_2, \dots, x_d$ of $\mathfrak{M}$. Since $I = x_1\mathfrak{M}' \cap R$ (cf. [**2**, Theorem 3.2]), it follows that

$$I = (x_1^2, x_2, \dots, x_d)$$

by [**2**, Lemma 3.1 (i)]. Thus, in $R[\mathfrak{M}/x_1]$, we have that

$$IR\left[\frac{\mathfrak{M}}{x_1}\right] = x_1\left(x_1, \frac{x_2}{x_1}, \dots, \frac{x_d}{x_1}\right) = x_1 M.$$

Next, we observe that

$$e(I) = \lim_{n\to\infty} \frac{2!}{n^2}\ell\left(\frac{R}{\overline{I^n}}\right)$$

and

$$\ell\left(\frac{R}{\overline{\overline{I^n}}}\right) = \ell\left(\frac{R}{\mathfrak{M}^n}\right) + \ell\left(\frac{\mathfrak{M}^n}{\overline{I^n}}\right) \quad \text{for all } n \in \mathbf{N}.$$

Here $\ell(\ \ )$ denotes the length as an R-module.

For each $n \in \mathbf{N}$, the natural morphism

$$\varphi\colon \frac{\mathfrak{M}^n}{\overline{I^n}} \to \frac{\mathfrak{M}^n R\,[\mathfrak{M}/x_1]}{\overline{I^n} R\,[\mathfrak{M}/x_1]}$$

is injective. To see this, note that $\overline{I^n}$ is complete and $R[\mathfrak{M}/x_1]$ is contained in the unique Rees valuation ring of $\overline{I^n}$, hence $\overline{I^n}$ is contracted from $R[\mathfrak{M}/x_1]$. Thus, φ is injective.

It follows that

$$\ell\left(\frac{R}{\overline{\overline{I^n}}}\right) \leq \ell\left(\frac{R}{\mathfrak{M}^n}\right) + \ell\left(\frac{\mathfrak{M}^n R\,[\mathfrak{M}/x_1]}{\overline{I^n} R\,[\mathfrak{M}/x_1]}\right)$$

for all $n \in \mathbf{N}$. Now, for any $n \in \mathbf{N}$, we have the following inclusions

$$I^n R\left[\frac{\mathfrak{M}}{x_1}\right] \subseteq \overline{I^n} R\left[\frac{\mathfrak{M}}{x_1}\right] \subseteq \overline{I^n R\left[\frac{\mathfrak{M}}{x_1}\right]}.$$

Since $I^n R[\mathfrak{M}/x_1] = x_1^n M^n$, it follows from Lemma 3.1 that $I^n R[\mathfrak{M}/x_1]$ is integrally closed. Hence,

$$\overline{I^n} R\left[\frac{\mathfrak{M}}{x_1}\right] = I^n R\left[\frac{\mathfrak{M}}{x_1}\right].$$

So we have that

$$\ell\left(\frac{R}{\overline{\overline{I^n}}}\right) \leq \ell\left(\frac{R}{\mathfrak{M}^n}\right) + \ell\left(\frac{\mathfrak{M}^n R\,[\mathfrak{M}/x_1]}{I^n R\,[\mathfrak{M}/x_1]}\right),$$

and thus

$$\ell\left(\frac{R}{\overline{\overline{I^n}}}\right) \leq \ell\left(\frac{R}{\mathfrak{M}^n}\right) + \ell\left(\frac{R'}{\mathfrak{M}'^n}\right)$$

for all $n \in \mathbf{N}$.

This implies that

$$e(I) \leq e(\mathfrak{M}) + e(\mathfrak{M}')$$

and hence $e(I) \leq e(\mathfrak{M}) + 1$ (since $e(\mathfrak{M}') = 1$). This proves $e(I) = e(\mathfrak{M}) + 1$.

Conversely, suppose $e(I) = e(\mathfrak{M}) + 1$. Then we have to prove that I is a first neighborhood complete ideal of R. To this end, we observe that there exists a first neighborhood complete ideal J in R such that $I \subseteq J$. Indeed, let J be the first neighborhood complete ideal corresponding to the unique immediate base point $(R', \mathfrak{M}')$ of I. It has been shown in [**2**] that there exists a minimal ideal basis $x_1, x_2, \dots, x_d$ of $\mathfrak{M}$ such that $R' = R[\mathfrak{M}/x_1]_M$ with $M = (x_1, (x_2/x_1), \dots, (x_d/x_1))$ and $\mathfrak{M}' = (x_1, (x_2/x_1))R'$. By [**2**, Theorem 3.2], we know that J is the inverse transform of $\mathfrak{M}'$ in R, i.e., $J = x_1\mathfrak{M}' \cap R$. In R', we have the following inclusion

$$IR' = x_1^r I' \subseteq x_1\mathfrak{M}',$$

with $r := \mathrm{ord}_R(I)$ and I' the transform of I in R'.

Contraction to R yields

$$I \subseteq x_1^r I' \cap R \subseteq x_1\mathfrak{M}' \cap R = J.$$

Since J is a first neighborhood complete ideal of $(R, \mathfrak{M})$, we have that $e(J) = e(\mathfrak{M}) + 1$ (by the first part of this proof) and $e(I) = e(\mathfrak{M}) + 1$ (by assumption). Hence, $I \subseteq J$ and $e(I) = e(J)$; thus,

$$\overline{I} = \overline{J}$$

by Proposition (4.2) in [**13**]. It follows that $I = J$, since both the ideals I and J are complete. This completes our proof. □

The "only if" part of the preceding result generalizes Proposition 3.5 in [**6**] (where the two-dimensional Muhly local domain $(R, \mathfrak{M})$ was supposed to be a rational singularity). In Example 3.3 below, a concrete illustration of the "only if" part is given in case $(R, \mathfrak{M})$ is *not* a rational singularity.

Example 3.3. Let

$$R = \frac{k[X,Y,Z]_{(X,Y,Z)}}{(X^3 - Y^3 - Z^3)_{(X,Y,Z)}}$$

with k an algebraically closed field of characteristic 0. Then

$$R = k[x,y,z]_{(x,y,z)} \quad \text{with} \quad z^3 = x^3 - y^3,$$

where x, y and z denote the natural images of X, Y and Z. The local ring $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain that is not a rational singularity (since R has not minimal multiplicity).

Let us consider the following $\mathfrak{M}$-primary ideal in R:

$$I := (x^2, x - y, z).$$

It is clear that I has no base points on $R[\mathfrak{M}/x - y]$ and $R[\mathfrak{M}/z]$. Hence, all immediate base points of I are lying on the chart $R[\mathfrak{M}/x]$. In $R[\mathfrak{M}/x]$ we have that

$$IR\left[\frac{\mathfrak{M}}{x}\right] = x\left(x, \frac{x-y}{x}, \frac{z}{x}\right).$$

Since $x \notin (x - y, z)$, one sees that $(x, (x - y/x), (z/x))$ is a maximal ideal M of $R[\mathfrak{M}/x]$ lying over $\mathfrak{M}$. Thus,

$$R' := R\left[\frac{\mathfrak{M}}{x}\right]_M$$

is the unique immediate base point of I (and $\mathfrak{M}' = (x, (z/x))R'$ is the maximal ideal of R'). Since $I \subset \mathfrak{M}$ are adjacent and $IR' \cap R \neq \mathfrak{M}$, we have that

$$I = x\mathfrak{M}' \cap R.$$

This shows that I is complete and that it is the inverse transform of $\mathfrak{M}'$ in R. By [**2**, Theorem 3.2], this means that I is a first neighborhood complete $\mathfrak{M}$-primary ideal of R. Since $(R, \mathfrak{M})$ is not regular, we have that $T(I) = \{v_{\mathfrak{M}}, w\}$ where w is the $\mathrm{ord}_{R'}$-valuation (see [**2**, Corollary 3.4]). In [**5**, Example 4.1], it is shown that I^2 is not contracted from $R[\mathfrak{M}/x]$. This implies that I^2 is not complete, hence I is *not* normal.

As I is a first neighborhood complete $\mathfrak{M}$-primary ideal in the two-dimensional Muhly local domain $(R, \mathfrak{M})$, we have

$$e(I) = e(\mathfrak{M}) + 1$$

by the "only if" part of Proposition 3.2.

In order to present a first application of Proposition 3.2, we recall the notion of an $\mathfrak{M}$-primary ideal with minimal multiplicity in a two-dimensional Muhly local domain. Since a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is Cohen-Macaulay (with infinite residue field), we have for any $\mathfrak{M}$-primary ideal I of R that

$$e(I) \geq \mu(I) + \ell\left(\frac{R}{I}\right) - \dim R$$

(see for example [**7**]). If equality holds, then I is said to have *minimal multiplicity.*

Corollary 3.4. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain. The following assertions are equivalent*:

(i) $(R, \mathfrak{M})$ *has minimal multiplicity* (*equivalently,* $(R, \mathfrak{M})$ *is a rational singularity*).

(ii) *There exists a first neighborhood complete ideal in R with minimal multiplicity.*

(iii) *Every first neighborhood complete ideal of R has minimal multiplicity.*

Proof. (i) $\Rightarrow$(iii). Suppose $(R, \mathfrak{M})$ has minimal multiplicity, i.e., $\operatorname{embdim} R = e(\mathfrak{M}) + \dim R - 1 = e(\mathfrak{M}) + 1$. (In [**5**, Theorem 3.1] it has been shown that this is equivalent to $(R, \mathfrak{M})$ being a rational singularity). Let I be a first neighborhood complete ideal of R. Then $e(I) = e(\mathfrak{M}) + 1$, by Proposition 3.2. Since $I \subset \mathfrak{M}$ are adjacent, we have that $\ell(R/I) = 2$.

Moreover, $I = (x_1^2, x_2, \ldots, x_d)$ for a suitable minimal ideal basis $x_1, x_2, \ldots, x_d$ of $\mathfrak{M}$ (see [**2**, Theorem 3.2 and Lemma 3.1]). Thus, $\mu(I) = \operatorname{embdim} R$. It follows that $e(I) = \mu(I) + \ell(R/I) - \dim R$ holds, which means that I has minimal multiplicity.

(ii) $\Rightarrow$(i). Suppose I is a first neighborhood complete ideal of R having minimal multiplicity. Since $e(I) = e(\mathfrak{M}) + 1$ (cf. Proposition 3.2), this implies that

$$e(\mathfrak{M}) + 1 = \operatorname{embdim} R + 2 - \dim R,$$

and hence $\operatorname{embdim} R = e(\mathfrak{M}) + \dim R - 1$, i.e., $(R, \mathfrak{M})$ has minimal multiplicity. $\square$

To state our second application, we need to recall the notion of the core of an ideal. The *core* of an ideal I of $(R, \mathfrak{M})$, denoted by $\mathrm{core}\,(I)$, is defined to be the intersection of all (minimal) reductions of I.

Corollary 3.5. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain having minimal multiplicity. Then, for any first neighborhood complete ideal I of R, we have*

$$\mathfrak{M}I \subseteq \mathrm{core}\,(I).$$

Proof. Because of Corollary 3.4, a first neighborhood complete ideal I of R has minimal multiplicity. By [**7**, Lemma (2.1)], this implies that

$$\mathfrak{M}I \subseteq \mathfrak{q}$$

for any minimal reduction $\mathfrak{q}$ of I, hence the assertion holds. $\square$

The last part of this section is devoted to some characterizations of the first neighborhood complete ideals among complete quasi-one-fibered $\mathfrak{M}$-primary ideals of order one in terms of certain numerical data associated with these ideals (see the beginning of this section).

We begin by giving some background information about these numerical data in case $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain having *minimal multiplicity.* Let I be a complete quasi-one-fibered $\mathfrak{M}$-primary ideal of order one in R with $T(I) \neq \{v_{\mathfrak{M}}\}$. As we have already mentioned at the beginning of this section, the ideal I has a unique immediate base point $(R', \mathfrak{M}')$, and the transform $I^{R'}$ is some power of a *simple* complete $\mathfrak{M}'$-primary ideal in R'. Hence $T(I^{R'}) = \{w\}$ for some prime divisor $w \neq v_{\mathfrak{M}}$ of R. It follows that $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$. So all degree function coefficients of I are zero, except $d(I, w)$ and (possibly) $d(I, v_{\mathfrak{M}})$. If $\mu(I)$ denotes the number of elements in a minimal ideal basis of I, then we have

$$\mu(I) \geq \dim_k \left(\frac{\mathfrak{M}}{\mathfrak{M}^2} \right)$$

where k is the residue field $R/\mathfrak{M}$ (cf. [**4**, beginning of Section 3]). The ideal I is said to be *minimally generated* if

$$\mu(I) = \dim_k \left(\frac{\mathfrak{M}}{\mathfrak{M}^2} \right).$$

From the theory of degree functions, we will need the following results:

$$e(I) = \sum_{v} d(I, v)v(I)$$

(cf. [**13**, Theorem (4.3)]) and

$$d_I(\mathfrak{M}) = d_{\mathfrak{M}}(I)$$

(cf. [**13**, Theorem (5.2)]). Since all degree function coefficients of I are zero except $d(I, w)$ and (possibly) $d(I, v_{\mathfrak{M}})$, and using the fact that $e(\mathfrak{M}) = d(\mathfrak{M}, v_{\mathfrak{M}})$ (cf. [**13**, Theorem (4.3)]), it follows that

$$w(I)\, d(I, w) + d(I, v_{\mathfrak{M}}) = e(I) \tag{1}$$

and

$$w(\mathfrak{M})d(I, w) + d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}). \tag{2}$$

Proposition 3.6. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain with minimal multiplicity, and let I be a complete quasi-one-fibered $\mathfrak{M}$-primary ideal of order one with $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$ (cf. the beginning of this section). Then the following assertions are equivalent*:

(i) *I is a first neighborhood complete ideal.*

(ii) *The numerical data $\mu(I)$, $\operatorname{rank}(I)$, $w(\mathfrak{M})$ and $d(I, w)$ are minimal.*

(iii) *$\mu(I)$ and $\operatorname{rank}(I)$ are minimal and $d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$.*

Proof. (i) $\Rightarrow$(ii). Suppose I is a first neighborhood complete ideal of R, and let $(R', \mathfrak{M}')$ denote its unique immediate base point. As we have already observed in the proof of Proposition 3.2, there exists a minimal ideal basis $x_1, x_2, \ldots, x_d$ of $\mathfrak{M}$ such that

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_M \quad \text{with} \quad M = \left(x_1, \frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right)$$

and

$$\mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right) R'.$$

It then follows that

$$I = (x_1^2, x_2, \ldots, x_d)$$

by [**2**, Theorem 3.2 and Lemma 3.1].

Thus, $\mu(I) = \dim_k(\mathfrak{M}/\mathfrak{M}^2)$, which means that $\mu(I)$ is minimal. Further, in R', we have that

$$IR' = x_1 \left(x_1, \frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right) = x_1 \mathfrak{M}'.$$

Thus, $I^{R'} = \mathfrak{M}'$, which implies that

$$(R, \mathfrak{M}) < (R', \mathfrak{M}')$$

is the unique quadratic sequence corresponding to I and w is the $\mathrm{ord}_{R'}$-valuation (cf. [**4**, Proposition 1.6]). This shows that $\mathrm{rank}\,(I) = 1$, hence $\mathrm{rank}\,(I)$ is minimal. From (1) and (2) (just before Proposition 3.6) it follows that

$$(w(I) - w(\mathfrak{M}))d(I, w) = e(I) - e(\mathfrak{M}).$$

Since, by Proposition 3.2, $e(I) - e(\mathfrak{M}) = 1$, it follows that $d(I, w) = 1$, and hence $d(I, w)$ is minimal.

Finally, $w(\mathfrak{M}) = \mathrm{ord}_{R'}(\mathfrak{M}) = \mathrm{ord}_{R'}(x_1) = 1$ because $\mathfrak{M}' = (x_1, (x_2/x_1))R'$. Thus, $w(\mathfrak{M})$ is minimal, and this completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Since $w(\mathfrak{M}) = 1$ and $d(I, w) = 1$, (iii) follows from $w(\mathfrak{M})d(I, w) + d(I, v_{\mathfrak{M}}) = e(\mathfrak{M})$ (see formula (2) before Proposition 3.6).

(iii) $\Rightarrow$ (i). Let I be a complete quasi-one-fibered $\mathfrak{M}$-primary ideal of order one satisfying (iii). In order to prove that I is a first neighborhood complete ideal, it will suffice to show that $e(I) = e(\mathfrak{M}) + 1$ (see Proposition 3.2).

To do so, we observe that (1) and (2) (just before Proposition 3.6) imply that

$$(w(I) - w(\mathfrak{M}))\, d(I, w) = e(I) - e(\mathfrak{M}).$$

It therefore suffices to show that $(w(I) - w(\mathfrak{M}))\,d(I,w) = 1$. To this end, we now prove that $w(\mathfrak{M}) = 1$, $d(I,w) = 1$ and $w(I) = 2$.

First, observe that $w(\mathfrak{M}) = 1$ and $d(I,w) = 1$, since it follows from

$$w(\mathfrak{M})\,d(I,w) + d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) \quad \text{(i.e., formula (2))}$$

and the assumption

$$d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1,$$

that

$$w(\mathfrak{M})d(I,w) = 1.$$

So, it remains to prove that $w(I) = 2$. Let $(R', \mathfrak{M}')$ denote the unique immediate base point of the quasi-one-fibered $\mathfrak{M}$-primary ideal I. As $\operatorname{rank}(I)$ is minimal (i.e., $\operatorname{rank}(I) = 1$), we have that

$$(R, \mathfrak{M}) < (R', \mathfrak{M}')$$

is the unique quadratic sequence corresponding to I and w is the $\mathrm{ord}_{R'}$-valuation (see the beginning of this section). Since Lemma 1.11 in [**9**] remains valid in this situation, we get

$$w(I) = w(I^{R'}) + \mathrm{ord}_R(I)w(\mathfrak{M}).$$

We already know that $\mathrm{ord}_R(I) = 1$ (by assumption) and $w(\mathfrak{M}) = 1$ (see above), hence we have only to determine $w(I^{R'}) = \mathrm{ord}_{R'}(I^{R'})$ in order to get $w(I)$.

To do so, we observe that, in the two-dimensional regular local ring $(R', \mathfrak{M}')$, the following "reciprocity relation" holds:

$$d_{\mathfrak{M}'}(I^{R'}) = d_{I^{R'}}(\mathfrak{M}') \quad \text{(cf. [\textbf{13}])}.$$

Using the fact that $T(\mathfrak{M}') = T(I^{R'}) = \{\mathrm{ord}_{R'}\}$, this means that

$$(\star) \qquad d(\mathfrak{M}', \mathrm{ord}_{R'})\mathrm{ord}_{R'}(I^{R'}) = d(I^{R'}, \mathrm{ord}_{R'})\mathrm{ord}_{R'}(\mathfrak{M}').$$

It will now be shown that the right-hand side of $(\star)$ is equal to 1, i.e., that $d(I^{R'}, \mathrm{ord}_{R'})$ is equal to 1. This will follow from the fact that I is normal (because $(R, \mathfrak{M})$ is a two-dimensional rational singularity) and minimally generated (since, by assumption, $\mu(I)$ is minimal). Indeed, by Proposition 3.3 in [**4**], we then have that $d(I,w)$ is invariant under the quadratic transformation $R \to R'$, i.e.,

$$d(I,w) = d(I^{R'},w).$$

As we have already shown that $d(I,w)=1$, it follows that $d(I^{R'},\mathrm{ord}_{R'}) = 1$ since $w = \mathrm{ord}_{R'}$. Hence,

$$w(I^{R'}) = \mathrm{ord}_{R'}(I^{R'}) = 1 \quad (\text{cf. } (\star)).$$

Using this in the formula $w(I) = w(I^{R'}) + \mathrm{ord}_R(I)w(\mathfrak{M})$, we get that $w(I) = 2$. This completes the proof of (iii) $\Rightarrow$ (i). $\square$

We end this paper with an application of the preceding result. Let I be a first neighborhood complete ideal in a two-dimensional Muhly local domain $(R,\mathfrak{M})$ with minimal multiplicity. In what follows, $(R,\mathfrak{M})$ is supposed to be *not regular* because the regular case is well known. Then we have

$$T(I) = \{v_{\mathfrak{M}}, w\}$$

where w denotes the $\mathrm{ord}_{R'}$-valuation and $(R',\mathfrak{M}')$ is the unique immediate base point of I (see [**2**, Theorem 3.3]). It follows from the preceding proposition that

$$d(I,w) = 1.$$

This will enable us to give a rather simple proof of the fact that I is *projectively full* (i.e., $\overline{I^n}$ with $n \in \mathbf{N}_+$, are the only integrally closed ideals that are projectively equivalent to I). We recall that an ideal J is said to be projectively equivalent to I if there exist positive integers n, m such that $\overline{I^n} = \overline{J^m}$ (see [**1**]). More precisely, the following result will be proved.

Proposition 3.7. *Let $(R,\mathfrak{M})$ be a two-dimensional Muhly local domain (not regular) with minimal multiplicity, and let I be a first neighborhood complete ideal of R. If a complete $\mathfrak{M}$-primary ideal J of R is projectively equivalent to I, then*

$$J = I^{d(J,w)}.$$

Proof. Let J be a complete $\mathfrak{M}$-primary ideal that is projectively equivalent to I, i.e.,

$$\overline{I^n} = \overline{J^m}$$

for some positive integers n and m. In fact, we have

$$I^n = J^m$$

since R is a two-dimensional rational singularity (cf. [**5**, Theorem 3.1]), and hence complete ideals are normal. If follows that

$$T(J) = T(I) = \{v_{\mathfrak{M}}, w\}.$$

Thus, the ideal J, just like I, has $(R', \mathfrak{M}')$ as its unique immediate base point.

Moreover, $I^n = J^m$ implies that

$$d(I^n, w) = d(J^m, w),$$

and hence

$$nd(I, w) = md(J, w) \quad \text{(cf. [\textbf{13}, Lemma (5.1)].)}$$

Since $d(I, w) = 1$, it follows that

$$n = md(J, w).$$

This will enable us to prove that $J = I^{d(J,w)}$.

To do this, we first observe that $I^n = J^m$ becomes

$$I^{md(J,w)} = J^m,$$

and thus

$$(I^{d(J,w)})^m = J^m.$$

Hence,

$$d((I^{d(J,w)})^m, v) = d(J^m, v)$$

for every prime divisor v of R.

By [**13**, Lemma (5.1)], this yields

$$md(I^{d(J,w)}, v) = md(J, v),$$

and hence

$$d(I^{d(J,w)}, v) = d(J, v)$$

for every prime divisor v of R. It follows that

$$\overline{I^{d(J,w)}} = \overline{J}$$

because of Corollary (5.3) in [**13**] (cf. Section 2, Background). Since $(R, \mathfrak{M})$ is a two-dimensional rational singularity and I and J are complete, this implies that

$$J = I^{d(J,w)}. \qquad \square$$

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