J. Math. Kyoto Univ. (JMKYAZ)
49-3 (2009), 475–490

A variational representation for random functionals on abstract Wiener spaces

By

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Abstract

We extend to abstract Wiener spaces the variational representation

$$\mathbb{E}[e^F] = \exp\left(\sup_{v\in\mathcal{H}^a} \mathbf{E}\left[F(\cdot+v) - \frac{1}{2}\|v\|_{\mathbb{H}}^2\right]\right),$$

proved by Boué and Dupuis [1] on the classical Wiener space. Here F is any bounded measurable function on the abstract Wiener space $(\mathbb{W}, \mathbb{H}, \mu)$, and $\mathcal{H}^a$ denotes the space of $\mathcal{F}_t$-adapted $\mathbb{H}$-valued random fields in the sense of Üstünel and Zakai [11]. In particular, we simplify the proof of the lower bound given in [1, 3] by using the Clark-Ocone formula. As an application, a uniform Laplace principle is established.

1. Introduction

Let W be a standard d-dimensional Brownian motion. The following elegant formula for the Laplace transform of a bounded and measurable functional F of Brownian motion was first established by Boué and Dupuis [1]:

$$\mathbb{E}[e^F] = \exp\left(\sup_v \mathbb{E}\left[F\left(\cdot + \int_0^\cdot v_s \mathrm{d}s\right) - \frac{1}{2}\int_0^1 |v_s|^2 \mathrm{d}s\right]\right),$$

where the supremum is taken for all processes v that are progressively measurable with respect to the augmented filtration generated by Brownian motion. This result has been later extended to Hilbert space-valued Brownian motion by Budhiraja and Dupuis [3].

This formula has proved to be useful in deriving various asymptotic results in large deviations, cf. [1, 3, 2, 8, 9].

The aim of this paper is to further extend their results to the framework of abstract Wiener spaces. The notion of filtration introduced by Üstünel and Zakai in [11] will play a crucial role. Although the argument is the same as in Boué and Dupuis's original paper, some new technical difficulty needs to be overcome. In particular, the new point is that we simplify the original proof

2000 *Mathematics Subject Classification*(*s*). 60H07, 60F10.
Received May 21, 2008

of the lower bound given in [1, 3] by using the Clark-Ocone formula. In their original proof, Boué and Dupuis [1] used the complicated measurable selection principle. As an application we shall also prove a uniform Laplace principle following the method of Budhiraja and Dupuis in [3], that might be used to obtain various large deviation estimates for diffusions.

This paper is arranged as follows: in Section 2 we prepare some lemmas and recall some notions related to the construction of filtration on abstract Wiener spaces, cf. [11, 12]. In Section 3 the variational representation formula is proved. Finally, in Section 4 an abstract criterion is established for the uniform Laplace principle. In particular, when the rate function is good, the Laplace principle is equivalent to the large deviation principle (cf. [5, Theorem 1.2.3]).

2. Preliminaries

Definition 2.1. Let $(\mathbb{X}, \mathcal{F})$ be a measurable space, and let $\mathcal{P}(\mathbb{X})$ denote the set of all the probability measures defined on $\mathbb{X}$. For $\mu \in \mathcal{P}(\mathbb{X})$, the relative entropy function $R(\cdot\|\mu)$ is a mapping from $\mathcal{P}(\mathbb{X})$ into $\mathbb{R} \cup \infty$ given by

$$R(\nu\|\mu) := \mathbb{E}^{\nu}\left(\log \frac{d\nu}{\mathrm{d}\mu}\right),$$

whenever $\nu \in \mathcal{P}(\mathbb{X})$ is absolutely continuous with respect to μ such that the above integral is finite. In all other cases, $R(\nu\|\mu) := \infty$.

The following lemma is taken from Boué-Dupuis [1, Lemma 2.8]:

Lemma 2.1. *Let $\mathbb{X}$ be a Polish space, and $f : \mathbb{X} \to \mathbb{R}$ a bounded Borel measurable function. For fixed $\mu \in \mathcal{P}(\mathbb{X})$, consider a sequence of probability measures $\{\nu_n, n \in \mathbb{N}\}$ in $\mathcal{P}(\mathbb{X})$ satisfying*

$$\sup_{n\in\mathbb{N}} R(\nu_n\|\mu) < +\infty.$$

Then, the following conclusions hold:

(i) *if $\{f_n, n \in \mathbb{N}\}$ is a sequence of uniformly bounded measurable functions converges μ-a.s. to f, then*

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \mathbb{E}^{\nu_k}|f_n - f| = 0;$$

(ii) *if ν_n converges weakly to a probability measure ν, then*

$$\lim_{n\to\infty} \mathbb{E}^{\nu_n} f = \mathbb{E}^{\nu} f.$$

We need the following variational formula involving the entropy function (cf. [5, Proposition 1.4.2]).

Proposition 2.1. *Let $\mathbb{X}$ be a Polish space, and $f : \mathbb{X} \to \mathbb{R}$ a bounded Borel measurable function. Let $\mu \in \mathcal{P}(\mathbb{X})$. The following conclusions hold:*

(i) *we have the following variational formula:*

$$-\log \mathbb{E}^{\mu}(e^{-f}) = \inf_{\gamma \in \mathcal{P}(\mathbb{X})} \Big[R(\gamma \| \mu) + \mathbb{E}^{\gamma}(f) \Big];$$

(ii) *the infimum in the above formula is uniquely attained at the probability measure γ_0 defined by*

$$\mathrm{d}\gamma_0(w) = e^{-f(w)}/\mathbb{E}^{\mu}(e^{-f}) \cdot \mathrm{d}\mu(w). \tag{2.1}$$

Let $(\mathbb{W}, \mathbb{H}, \mu)$ be an abstract Wiener space. Namely, $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ is a separable Banach space, $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ is a separable Hilbert space densely and continuously embedded in $\mathbb{W}$, and μ is the Gaussian measure over $\mathbb{W}$. If we identify the dual space $\mathbb{H}^*$ with itself, then $\mathbb{W}^*$ may be viewed as a dense linear subspace of $\mathbb{H}$ so that $\ell(w) = \langle \ell, w \rangle_{\mathbb{H}}$ whenever $\ell \in \mathbb{W}^*$ and $w \in \mathbb{H}$, where $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ denotes the inner product in $\mathbb{H}$.

We now recall some notions from [11, 12] about the filtration in abstract Wiener space. In what follows, we fix a continuous and strictly monotonic resolution $\pi = \{\pi_t, t \in [0,1]\}$ of the identity in $\mathbb{H}$, i.e.:

(i) for each $t \in [0,1]$, π_t is an orthogonal projection;

(ii) $\pi_0 = 0, \pi_1 = I$;

(iii) for $0 \leqslant s < t \leqslant 1$, $\pi_s \mathbb{H} \subsetneq \pi_t \mathbb{H}$;

(iv) for any $h \in \mathbb{H}$ and $t \in [0,1]$, $\lim_{s \to t} \pi_s h = \pi_t h$.

For any $h \in \mathbb{H}$, there exists a sequence $h_n \in \mathbb{W}^*$ such that $\lim_{n\to\infty} \|h_n - h\|_{\mathbb{H}} = 0$. So,

$$\lim_{n,k\to\infty} \mathbb{E}^{\mu} |h_n(\cdot) - h_k(\cdot)|^2 = \lim_{n,k\to\infty} \|h_n - h_k\|^2_{\mathbb{H}} = 0.$$

Thus, there exists a $\delta(h) \in L^2(\mathbb{W}, \mathscr{B}(\mathbb{W}), \mu)$ such that

$$\lim_{n\to\infty} \mathbb{E}|h_n(\cdot) - \delta(h)|^2 = 0.$$

The $\delta(h)(w)$ is also written as $\langle h, w \rangle$, called the Skorohod integral of h.

Definition 2.2. The continuous filtration on $(\mathbb{W}, \mu)$ is defined by

$$\mathcal{F}_t := \sigma\{\delta(\pi_t h), h \in \mathbb{H}\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the collection of all the null sets in $\mathbb{W}$ with respect to μ. We write $\mathcal{F}_1$ as $\mathcal{F}$, and remark that $\mathscr{B}(\mathbb{W}) \subset \mathcal{F}$.

Below, we shall consider the filtered probability space $(\mathbb{W}, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mu)$. If there is no special declaration, the expectation $\mathbb{E}$ and the term "a.s." are always taken with respect to the Wiener measure μ.

Definition 2.3. For every $t \in [0,1]$, let $\mathcal{C}_t$ be the collection of all cylindrical functions with the form:

$$F(w) = g(\langle \pi_t h_1, w\rangle, \cdots, \langle \pi_t h_n, w\rangle); \quad g \in C_b^\infty(\mathbb{R}^n), \quad h_1 \cdots h_n \in \mathbb{W}^*. \tag{2.2}$$

In particular, the elements in $\mathcal{C}_t$ are measurable with respect to $\mathcal{F}_t$. We write $\mathcal{C}_1$ as $\mathcal{C}$.

We have the following simple approximation result (cf. [7]).

Lemma 2.2. *For a fixed $t \in [0,1]$, let F be an $\mathcal{F}_t$-measurable and bounded function on $\mathbb{W}$ with bound N. Then there exists a sequence $F_k \in \mathcal{C}_t$ such that*

$$\|F_k\|_\infty := \sup_{w\in\mathbb{W}} |F_k(w)| \leqslant N \text{ and } \quad F_k \to F, \quad a.s..$$

In particular, $\mathcal{C}_t$ is dense in $L^2(\mathbb{W}, \mathcal{F}_t, \mu)$.

Proof. We fix an orthogonal basis $\mathscr{E} := \{\ell_k, k \in \mathbb{N}\} \subset \mathbb{W}^*$ of $\mathbb{H}$, and define

$$\mathcal{F}_t^n := \sigma\{\delta(\pi_t \ell_i), i = 1, \cdots, n\}$$

and

$$F_n := \mathbb{E}(F|\mathcal{F}_t^n).$$

Then, clearly $\mathcal{F}_t^n \vee \mathcal{N} \uparrow \mathcal{F}_t$, and F_n is bounded by N. Thus, by the martingale convergence theorem we obtain

$$\lim_{n\to\infty} \mathbb{E}|F_n - F|^2 = 0.$$

On the other hand, for each $n \in \mathbb{N}$ there is a measurable function g_n on $\mathbb{R}^n$ bounded by N such that

$$F_n(w) = g_n(\langle \pi_t \ell_1, w\rangle, \cdots, \langle \pi_t \ell_n, w\rangle).$$

For each $n \in \mathbb{N}$, we can approximate g_n by $g_{nm} \in C_b^\infty(\mathbb{R}^n)$. This can be done by the usual mollifying technique. Finally, by extracting a suitable subsequence, we may find the desired approximation F_k. $\square$

Definition 2.4. An $\mathbb{H}$-valued random variable v is called adapted to $\mathcal{F}_t$ if for every $t \in [0,1]$ and $h \in \mathbb{H}$, $\langle \pi_t h, v\rangle_\mathbb{H} \in \mathcal{F}_t$. All the adapted $\mathbb{H}$-valued random variables in $L^2(\mathbb{W}, \mathcal{F}, \mu; \mathbb{H})$ is denoted by $\mathcal{H}^a$. The set of all bounded elements in $\mathcal{H}^a$ is denoted by $\mathcal{H}_b^a$, i.e.

$$\mathcal{H}_b^a := \{v \in \mathcal{H}^a : \|v(w)\|_\mathbb{H} \leqslant N \text{ a.s.-}w \text{ for some } N > 0\}.$$

A $v \in \mathcal{H}^a$ is called simple if it has the following form

$$v(w) = \sum_{i=0}^{n-1} \xi_i(w)(\pi_{t_{i+1}} - \pi_{t_i})h_i; \quad \xi_i \in \mathcal{C}_{t_i}, \quad h_i \in \mathbb{H}, \tag{2.3}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$. The set of all simple elements in $\mathcal{H}^a$ is denoted by $\mathcal{S}^a$. We write $\mathcal{S}^a_b := \mathcal{S}^a \cap \mathcal{H}^a_b$.

We have the following proposition (cf. [11, 12]).

Proposition 2.2. *$\mathcal{H}^a$ is a closed subspace of $L^2(\mathbb{W}, \mathcal{F}, \mu; \mathbb{H})$, and $\mathcal{S}^a_b$ is dense in $\mathcal{H}^a$.*

Basing on this proposition, for any $v \in \mathcal{H}^a$, we can define Itô's integral $\delta(v)$ such that (c.f. [12, p. 43])

$$\mathbb{E}|\delta(v)|^2 = \mathbb{E}\|v\|^2_{\mathbb{H}}.$$

On the other hand, for any bounded real Borel measurable function f on $[0, 1]$ and $h \in \mathbb{H}$, we may also define the following integral with respect to the vector valued measure (cf. [12, p. 42]):

$$\int_0^1 f(s)\mathrm{d}\pi_s h$$

such that

$$\left\| \int_0^1 f(s)\mathrm{d}\pi_s h \right\|^2_{\mathbb{H}} = \int_0^1 |f(s)|^2 \mathrm{d}\langle \pi_s h, h\rangle_{\mathbb{H}}. \tag{2.4}$$

It is standard to prove the following result.

Lemma 2.3. *Let f be a left-continuous $\mathcal{F}_t$-adapted process bounded by N. Then for any $h \in \mathbb{H}$, there exists a sequence $v^h_k \in \mathcal{S}^a_b$ such that*

$$\|v^h_k(w)\|_{\mathbb{H}} \leqslant N \cdot \|h\|_{\mathbb{H}}, \quad a.s. - w,$$

and

$$\lim_{k\to\infty} \mathbb{E} \left\| v^h_k - \int_0^1 f(s)\mathrm{d}\pi_s h \right\|^2_{\mathbb{H}} = 0.$$

Proof. First of all, we define for every $n \in \mathbb{N}$

$$f_n(s) := \sum_{j=0}^{2^n-1} f(j2^{-n}) 1_{[j2^{-n}, (j+1)2^{-n})}(s).$$

Then, by (2.4) and the dominated convergence theorem we have

$$\lim_{n\to\infty} \mathbb{E} \left\| \int_0^1 (f_n(s) - f(s))\mathrm{d}\pi_s h \right\|^2_{\mathbb{H}} = 0.$$

For each $n \in \mathbb{N}$ and $j = 0, \cdots, 2^n - 1$, by Lemma 2.2 one can find $F^j_{n,k} \in \mathcal{C}_{j2^{-n}}$ such that

$$\|F^j_{n,k}\|_\infty \leqslant N, \quad F^j_{n,k} \to f(j2^{-n}), \quad a.s. \quad k \to \infty.$$

Lastly, we define

$$v^h_{n,k}(w) := \sum_{j=0}^{2^n-1} F^j_{n,k}(w) \cdot (\pi_{(j+1)2^{-n}} - \pi_{j2^{-n}})h.$$

By the diagonalization method, we may find the desired sequence v^h_k. $\square$

We need the following simple result.

Proposition 2.3. *Let $0 < c \leqslant F \leqslant C$ be a Borel measurable function on $\mathbb{W}$. Then there exists a $v \in \mathcal{H}^a$ such that*

$$\mathbb{E}(F|\mathcal{F}_t) = \mathbb{E}F \cdot \exp\left\{\delta(\pi_t v) - \frac{1}{2}\|\pi_t v\|^2_{\mathbb{H}}\right\}, \quad \forall t \in [0,1].$$

Proof. Set $M_t := \mathbb{E}(F|\mathcal{F}_t)$. Then $\{M_t, \mathcal{F}_t\}$ is a martingale bounded from above by C and from below by c. By the representation formula of martingales (cf. [12, p. 45, Theorem 2.6.4]), there is a $u \in \mathcal{H}^a$ such that

$$M_t = \mathbb{E}F + \delta(\pi_t u).$$

Now define

$$v := \int_0^1 \frac{\mathrm{d}\pi_t u}{M_t}, \quad m_t := \delta(\pi_t v).$$

Then, clearly $v \in \mathcal{H}^a$ and $\{m_t, \mathcal{F}_t\}$ is a martingale with square variation process $t \mapsto \|\pi_t v\|_{\mathbb{H}}$. Thus, by [12, p. 44, Lemma 2.6.1] we have

$$M_t = \mathbb{E}F + \int_0^t M_s \mathrm{d}m_s.$$

The desired formula follows. $\square$

We also need the following Clark-Ocone formula (cf. [7]).

Proposition 2.4. *For any $F \in \mathcal{C}$ with the form* (2.2)*, it then holds*

$$\mathbb{E}(F|\mathcal{F}_t) = \mathbb{E}F + \delta(\pi_t v), \quad \forall t \in [0,1],$$

where

$$v := \sum_{i=1}^n \int_0^1 \mathbb{E}\Big[(\partial_i g)(\langle h_1, \cdot\rangle, \cdots, \langle h_n, \cdot\rangle)|\mathcal{F}_t\Big] \mathrm{d}\pi_t h_i \in \mathcal{H}^a_b.$$

Proof. First of all, by the representation formula of martingales (c.f. [12, p. 45]), there exists a $u \in \mathcal{H}^a$ such that

$$\mathbb{E}(F|\mathcal{F}_t) = \mathbb{E}F + \delta(\pi_t u), \quad \forall t \in [0,1].$$

Let DF be the Malliavin derivative of F defined by

$$DF(w) := \sum_{i=1}^{n} (\partial_i g)(\langle h_1, w\rangle, \cdots, \langle h_n, w\rangle) h_i =: \sum_{i=1}^{n} G_i(w) h_i.$$

Then by the integration by parts formula (cf. [7]), we have for any $z \in \mathcal{S}_b^a$

$$\mathbb{E}(\langle DF, z\rangle_{\mathbb{H}}) = \mathbb{E}(F \cdot \delta(z)) = \mathbb{E}(\delta(u)\delta(z)) = \mathbb{E}\langle u, z\rangle_{\mathbb{H}}, \tag{2.5}$$

where the last step is due to the isometry property of Itô's integral.

We now assume that $z \in \mathcal{S}_b^a$ takes the following form:

$$z(w) := \sum_{i=0}^{m-1} \xi_i(w)(\pi_{t_{i+1}} - \pi_{t_i}) h_i'; \quad \xi_i \in \mathcal{C}_{t_i}, \quad h_i' \in \mathbb{H}.$$

Then we have

$$\begin{aligned}
\mathbb{E}\langle DF, z\rangle_{\mathbb{H}} &= \mathbb{E}\left(\int_0^1 \mathrm{d}\langle \pi_t(DF), z\rangle_{\mathbb{H}}\right) \\
&= \sum_{i=1}^{n}\sum_{j=0}^{m-1} \int_0^1 \mathbb{E}(\xi_j \cdot G_i) \mathrm{d}\langle \pi_t h_i, (\pi_{t_{j+1}} - \pi_{t_j}) h_j'\rangle_{\mathbb{H}} \\
&= \sum_{i=1}^{n}\sum_{j=0}^{m-1} \int_0^1 \mathbb{E}\Big[1_{\{t \geqslant t_j\}} \xi_j \cdot \mathbb{E}(G_i|\mathcal{F}_t)\Big] \mathrm{d}\langle \pi_t h_i, (\pi_{t_{j+1}} - \pi_{t_j}) h_j'\rangle_{\mathbb{H}} \\
&= \mathbb{E}\left(\sum_{i=1}^{n} \int_0^1 \mathbb{E}(G_i|\mathcal{F}_t) \mathrm{d}\langle \pi_t h_i, z\rangle_{\mathbb{H}}\right) = \mathbb{E}\langle v, z\rangle_{\mathbb{H}},
\end{aligned}$$

which yields the desired formula by (2.5) and Proposition 2.2. □

3. Variational representation formula

Let $\mathcal{H}_c^a$ be the set of all $v \in \mathcal{H}^a$ satisfying

$$\mathbb{E}\left[\exp\left\{\delta(v) - \frac{1}{2}\|v\|_H^2\right\}\right] = 1.$$

For $v \in \mathcal{H}_c^a$, we define

$$T_v(w) := w - v(w)$$

and

$$\mathrm{d}\mu_v = \exp\left\{\delta(v) - \frac{1}{2}\|v\|_{\mathbb{H}}^2\right\} \mathrm{d}\mu. \tag{3.1}$$

Then by the Girsanov theorem in [12, p. 45, Theorem 2.6.3], we have for any $A \in \mathscr{B}(\mathbb{W})$

$$\mu_v(w : T_v w \in A) = \mu(A). \tag{3.2}$$

We first prepare the following lemma for later use.

Lemma 3.1. *For $v \in \mathcal{H}_c^a$, let μ_v be defined by* (3.1)*. Then*

$$R(\mu_v \| \mu) = \frac{1}{2}\mathbb{E}^{\mu_v}\|v\|_{\mathbb{H}}^2.$$

Proof. By [12, p. 44, Corollary 2.6.1], we know that $t \mapsto \delta(\pi_t v)$ is a continuous square integrable martingale with respect to $\mathcal{F}_t$, whose square variation process is given by $t \mapsto \|\pi_t v\|_{\mathbb{H}}^2$.

Thus, by Girsanov's theorem (cf. [10, p. 329, Theorem 1.7]), the process

$$M_t := \delta(\pi_t v) - \|\pi_t v\|_{\mathbb{H}}^2$$

is a martingale with respect to $\mathcal{F}_t$ under the new probability measure μ_v. Thus,

$$\begin{aligned} R(\mu_v \| \mu) &= \mathbb{E}^{\mu_v}\left(\log \frac{\mathrm{d}\mu_v}{\mathrm{d}\mu}\right) = \mathbb{E}^{\mu_v}\left(\delta(v) - \frac{1}{2}\|v\|_{\mathbb{H}}^2\right) \\ &= \mathbb{E}^{\mu_v}\left(M_1 + \frac{1}{2}\|v\|_{\mathbb{H}}^2\right) = \frac{1}{2}\mathbb{E}^{\mu_v}\|v\|_{\mathbb{H}}^2. \end{aligned}$$

The result follows. □

We now prove the following result.

Theorem 3.1. *Let F be any bounded Borel measurable function on $\mathbb{W}$. Then*

$$-\log \mathbb{E}(e^{-F}) = \inf_{v \in \mathcal{H}_c^a} \mathbb{E}^{\mu_v}\left(F + \frac{1}{2}\|v\|_{\mathbb{H}}^2\right),$$

where μ_v is defined by (3.1)*. Moreover, the infimum is uniquely attained at some $v_0 \in \mathcal{H}_c^a$.*

Proof. For any $v \in \mathcal{H}_c^a$, by Jensen's inequality and Lemma 3.1 we have

$$\begin{aligned} -\log \mathbb{E}(e^{-F}) &= -\log \mathbb{E}^{\mu_v}\left(e^{-F - \log \frac{\mathrm{d}\mu_v}{\mathrm{d}\mu}}\right) \\ &\leqslant \mathbb{E}^{\mu_v}(F) + R(\mu_v \| \mu) \\ &= \mathbb{E}^{\mu_v}\left(F + \frac{1}{2}\|v\|_{\mathbb{H}}^2\right), \end{aligned}$$

which gives the upper bound.

Let γ_0 be the probability measure defined by (2.1). By Proposition 2.3, there is a $v_0 \in \mathcal{H}_c^a$ such that for all $t \in [0,1]$

$$\mathbb{E}(\mathrm{d}\gamma_0/\mathrm{d}\mu | \mathcal{F}_t) = \exp\left\{\delta(\pi_t v_0) - \frac{1}{2}\|\pi_t v_0\|_{\mathbb{H}}^2\right\}.$$

Thus by Lemma 3.1 again, we have

$$-\log \mathbb{E}(e^{-F}) = \mathbb{E}^{\gamma_0}(F) + R(\gamma_0\|\mu) = \mathbb{E}^{\gamma_0}\left(F + \frac{1}{2}\|v_0\|^2_{\mathbb{H}}\right).$$

The uniqueness of infimum follows from (ii) of Proposition 2.1. □

The following proposition will be crucial for the proof of Theorem 3.2 below.

Proposition 3.1. *Let F be any bounded Borel measurable function on $\mathbb{W}$. For any $v \in \mathcal{S}^a_b$, there are two $\tilde{v}, \hat{v} \in \mathcal{S}^a_b$ such that*

$$\mathbb{E}^{\mu_{\tilde{v}}}\left(F + \frac{1}{2}\|\tilde{v}\|^2_{\mathbb{H}}\right) = \mathbb{E}\left(F(\cdot + v) + \frac{1}{2}\|v\|^2_{\mathbb{H}}\right) \tag{3.3}$$

$$\mathbb{E}^{\mu_{v}}\left(F + \frac{1}{2}\|v\|^2_{\mathbb{H}}\right) = \mathbb{E}\left(F(\cdot + \hat{v}) + \frac{1}{2}\|\hat{v}\|^2_{\mathbb{H}}\right). \tag{3.4}$$

Moreover,

$$R(\mathcal{L}_\mu(\cdot + v)\|\mu) = \frac{1}{2}\mathbb{E}\|v\|^2_{\mathbb{H}}, \tag{3.5}$$

where $\mathcal{L}_\mu(\cdot + v)$ denotes the law of $w \mapsto w + v(w)$ in $(\mathbb{W}, \mathcal{F})$ under μ.

Proof. Let v have the following form:

$$v(w) = \sum_{i=0}^{n-1} \xi_i(w)(\pi_{t_{i+1}} - \pi_{t_i})h_i,$$

where $\xi_i \in \mathcal{C}_{t_i}$ takes the form:

$$\xi_i(w) := g_i(\langle \pi_{t_i} h_{i1}, w\rangle, \cdots, \langle \pi_{t_i} h_{in_i}, w\rangle); g_i \in C^\infty_b(\mathbb{R}^{n_i}), h_{i1} \cdots h_{in_i} \in \mathbb{H}$$

We define two families of random variables $\{\tilde{\xi}_i(w), \hat{\xi}_i(w); i = 0, \cdots, n-1\}$ as follows:

$$\tilde{\xi}_0(w) := \xi_0(w), \qquad \hat{\xi}_0(w) := \xi_0(w),$$

and for $i = 1, \cdots, n-1$,

$$\tilde{\xi}_i(w) := g_i(\langle \pi_{t_i} h_{i1}, w - \tilde{u}_i(w)\rangle, \cdots, \langle \pi_{t_i} h_{in_i}, w - \tilde{u}_i(w)\rangle),$$
$$\hat{\xi}_i(w) := g_i(\langle \pi_{t_i} h_{i1}, w + \hat{u}_i(w)\rangle, \cdots, \langle \pi_{t_i} h_{in_i}, w + \hat{u}_i(w)\rangle),$$

where

$$\tilde{u}_i(w) := \sum_{j=0}^{i-1} \tilde{\xi}_j(w)(\pi_{t_{j+1}} - \pi_{t_j})h_j, \quad \hat{u}_i(w) := \sum_{j=0}^{i-1} \hat{\xi}_j(w)(\pi_{t_{j+1}} - \pi_{t_j})h_j.$$

It is clear that $\tilde{\xi}_i, \hat{\xi}_i \in \mathcal{C}_{t_i}$. If we define

$$\tilde{v}(w) = \sum_{i=0}^{n-1} \tilde{\xi}_i(w)(\pi_{t_{i+1}} - \pi_{t_i})h_i, \qquad \hat{v}(w) = \sum_{i=0}^{n-1} \hat{\xi}_i(w)(\pi_{t_{i+1}} - \pi_{t_i})h_i,$$

then $\tilde{v}, \hat{v} \in \mathcal{S}_b^a$.

We now show that $\tilde{v}$ and $\hat{v}$ have the desired properties. In the following proof, it should be noticed that the measures $\mu_{\tilde{v}}, \mu_v$ and μ have the same null sets.

Note that

$$\xi_0(w) = \hat{\xi}_0(T_v(w)), \quad \tilde{\xi}_0(w) = \xi_0(T_{\tilde{v}}(w)), \quad a.s. - w,$$

and

$$\begin{aligned}\langle v(w), \pi_{t_i} h_{ij} \rangle_{\mathbb{H}} &= \sum_{j=0}^{i-1} \langle \xi_j(w)(\pi_{t_{i+1}} - \pi_{t_i})h_i, \pi_{t_i} h_{ij} \rangle_{\mathbb{H}}, \\ \langle \tilde{v}(w), \pi_{t_i} h_{ij} \rangle_{\mathbb{H}} &= \langle \tilde{u}_i(w), \pi_{t_i} h_{ij} \rangle_{\mathbb{H}}, \quad i = 0, \cdots, n-1, \ j = 1, \cdots, n_i,\end{aligned}$$

By induction, it is easy to see that

$$\xi_i(w) = \hat{\xi}_i(T_v(w)), \qquad v(w) = \hat{v}(T_v(w)), \quad a.s.$$

and

$$\tilde{\xi}_i(w) = \xi_i(T_{\tilde{v}}(w)), \qquad \tilde{v}(w) = v(T_{\tilde{v}}(w)), \quad a.s.,$$

where $T_v(w) := w - v(w)$ and $T_{\tilde{v}}(w) := w - \tilde{v}(w)$.

Thus, for any $A \in \mathscr{B}(\mathbb{W})$ and $B \in \mathscr{B}(\mathbb{H})$, we have by (3.2)

$$\begin{aligned}\mu_{\tilde{v}}(w : T_{\tilde{v}}(w) \in A, \tilde{v}(w) \in B) &= \mu_{\tilde{v}}(w : T_{\tilde{v}}(w) \in A, v(T_{\tilde{v}}(w)) \in B) \\ &= \mu(w : w \in A, v(w) \in B),\end{aligned} \tag{3.6}$$

which means that $(T_{\tilde{v}}(w), \tilde{v}(w))$ and (w, v) have the same laws under $\mu_{\tilde{v}}$ and μ. Similarly,

$$\begin{aligned}\mu_v(w : w \in A, v \in B) &= \mu_v(w : T_v(w) + \hat{v}(T_v(w)) \in A, \hat{v}(T_v(w)) \in B) \\ &= \mu(w : w + \hat{v} \in A, \hat{v}(w) \in B),\end{aligned}$$

which means that (w, v) and $(w + \hat{v}(w), \hat{v}(w))$ have the same laws under μ_v and μ. The identities (3.3) (3.4) now follow.

Moreover, by (3.6) we also have $\mathcal{L}_\mu(\cdot + v) = \mu_{\tilde{v}}$, and so

$$R(\mathcal{L}_\mu(\cdot + v) \| \mu) = R(\mu_{\tilde{v}} \| \mu) = \frac{1}{2} \mathbb{E}^{\mu_{\tilde{v}}} \|\tilde{v}\|_{\mathbb{H}}^2 = \frac{1}{2} \mathbb{E}^{\mu_{\tilde{v}}} \|v(T_{\tilde{v}}(\cdot))\|_{\mathbb{H}}^2 = \frac{1}{2} \mathbb{E} \|v\|_{\mathbb{H}}^2.$$

The proof is complete. $\square$

We are ready to prove that

Theorem 3.2. *Let F be a bounded Borel measurable function on $\mathbb{W}$. Then we have*

$$-\log \mathbb{E}(e^{-F}) = \inf_{v\in\mathcal{H}^a} \mathbb{E}\left(F(\cdot+v)+\frac{1}{2}\|v\|_{\mathbb{H}}^2\right)$$
$$= \inf_{v\in\mathcal{S}_b^a} \mathbb{E}\left(F(\cdot+v)+\frac{1}{2}\|v\|_{\mathbb{H}}^2\right).$$

Proof. **(Upper bound):** Let $v \in \mathcal{H}^a$. By Proposition 2.2 we may choose a sequence of $v_n \in \mathcal{S}_b^a$ such that

$$\lim_{n\to\infty} \mathbb{E}\|v_n - v\|_{\mathbb{H}}^2 = 0.$$

So, (w, v_n) converges in distribution to (w, v) in $\mathbb{W}\times\mathbb{H}$, and $\mathcal{L}_\mu(\cdot+v_n)$ converges weakly to $\mathcal{L}_\mu(\cdot+v)$. Noting that by (3.5)

$$\sup_n R(\mathcal{L}_\mu(\cdot+v_n)\|\mu) = \frac{1}{2}\sup_n \mathbb{E}\|v_n\|_{\mathbb{H}}^2 < +\infty,$$

we have by Lemma 2.1 (ii)

$$\lim_{n\to\infty} \mathbb{E}(F(\cdot+v_n)) = \mathbb{E}(F(\cdot+v)).$$

Therefore, by Theorem 3.1 and (3.3), we get the upper bound

$$-\log \mathbb{E}(e^{-F}) \leqslant \lim_{n\to\infty} \mathbb{E}\left(F(\cdot+v_n)+\frac{1}{2}\|v_n\|_{\mathbb{H}}^2\right) = \mathbb{E}\left(F(\cdot+v)+\frac{1}{2}\|v\|_{\mathbb{H}}^2\right).$$

(Lower bound): We divide the proof into two steps.

(*Step* 1): We first assume that $F \in \mathcal{C}$ with the form

$$F(w) := g(\langle h_1, w\rangle, \cdots, \langle h_n, w\rangle), \quad g \in C_b^\infty(\mathbb{R}^n), \quad h_1\cdots h_n \in \mathbb{W}^*.$$

By Proposition 2.3, there is a $v \in \mathcal{H}^a$ such that for all $t \in [0,1]$

$$\mathbb{E}(e^{-F}|\mathcal{F}_t) = \mathbb{E}(e^{-F})\cdot\exp\left\{\delta(\pi_t v) - \frac{1}{2}\|\pi_t v\|_{\mathbb{H}}^2\right\}. \tag{3.7}$$

By Proposition 2.4, we in fact find from the proof of Proposition 2.3 that

$$v := \sum_{i=1}^n \int_0^1 \frac{1}{\mathbb{E}(e^{-F}|\mathcal{F}_t)}\cdot\mathbb{E}\Big[e^{-F}\cdot(\partial_i g)(\langle h_1,\cdot\rangle,\cdots,\langle h_n,\cdot\rangle)|\mathcal{F}_t\Big]\mathrm{d}\pi_t h_i.$$

Thus, $v \in \mathcal{H}_b^a$, and by Lemma 2.3 there exists a sequence $v_k \in \mathcal{S}_b^a$ satisfying for some $C_F > 0$ and any $k \in \mathbb{N}$

$$\|v_k\|_{\mathbb{H}} \leqslant C_F \quad a.s.$$

such that

$$\lim_{k\to\infty} \mathbb{E}\|v_k - v\|_{\mathbb{H}}^2 = 0.$$

By extracting a subsequence if necessary, we may further assume that

$$\|v_k - v\|_{\mathbb{H}}^2 \to 0 \ a.s., \quad \delta(v_k) \to \delta(v) \ a.s.$$

Hence, recalling (3.1) we have by the dominated convergence theorem

$$\mathbb{E}^{\mu_{v_k}}\left(F + \frac{1}{2}\|v_k\|_{\mathbb{H}}^2\right) \to \mathbb{E}^{\mu_v}\left(F + \frac{1}{2}\|v\|_{\mathbb{H}}^2\right) \quad \text{as } k \to \infty. \tag{3.8}$$

Here we have used the uniform integrability of $\{e^{\delta(v_k)}, k \in \mathbb{N}\}$.

Moreover, by (3.7) and Lemma 3.1 we have

$$-\log \mathbb{E}(e^{-F}) = \mathbb{E}^{\mu_v}(F) + R(\mu_v\|\mu) = \mathbb{E}^{\mu_v}\left(F + \frac{1}{2}\|v\|_{\mathbb{H}}^2\right).$$

The lower bound now follows from (3.4) and the limit (3.8).

(*Step* 2): Let F be a bounded measurable function on $(\mathbb{W}, \mathcal{F})$. By Lemma 2.2 we can choose a sequence $F_n \in \mathcal{C}$ such that $\|F_n\|_\infty \leqslant \|F\|_\infty < \infty$, and $\lim_{n\to\infty} F_n = F$ μ-a.s.. For any $\epsilon > 0$ and F_n, by Step 1 there exists a $v_n \in \mathcal{S}_b^a$ such that

$$-\log \mathbb{E}(e^{-F_n}) \geqslant \mathbb{E}\left(F_n(\cdot + v_n) + \frac{1}{2}\|v_n\|_{\mathbb{H}}^2\right) - \epsilon. \tag{3.9}$$

In view of (3.5) and (3.9), we have

$$\sup_n R(\mathcal{L}_\mu(\cdot + v_n)\|\mu) \leqslant \frac{1}{2}\sup_n \mathbb{E}\|v_n\|_{\mathbb{H}}^2 \leqslant 2\|F\|_\infty + \epsilon.$$

So there is a subsequence n_k such that $\mathbb{E}\|v_{n_k}\|_{\mathbb{H}}^2$ converges. By Lemma 2.1 (i), we have

$$\lim_{k\to\infty} \mathbb{E}|F_{n_k}(\cdot + v_{n_k}) - F(\cdot + v_{n_k})| = 0.$$

Dominated convergence theorem gives that for sufficiently large k,

$$-\log \mathbb{E}(e^{-F}) \geqslant \mathbb{E}\left(F(\cdot + v_{n_k}) + \frac{1}{2}\|v_{n_k}\|_{\mathbb{H}}^2\right) - 2\epsilon.$$

Since $v_{n_k} \in \mathcal{S}_b^a$, we thus complete the proof of the lower bound. □

4. A uniform Laplace principle

Let $\mathbb{X}, \mathbb{Y}$ be two Polish spaces, and $\{Z_y^\epsilon, \epsilon > 0, y \in \mathbb{Y}\}$ a family of measurable mappings from $\mathbb{W}$ to $\mathbb{X}$.

Definition 4.1. A function I mapping $\mathbb{X}$ to $[0,\infty]$ is called a rate function if it is lower semi-continuous on $\mathbb{X}$. It is called a good rate function if the level set $\{f \in \mathbb{X} : I(f) \leqslant a\}$ is compact in $\mathbb{X}$. A family of rate functions $\{I_y, y \in \mathbb{Y}\}$ is said to have compact level sets on compacts if for all compact subsets K of $\mathbb{Y}$ and each $a < \infty$, $\cup_{y\in K}\{f \in \mathbb{X} : I_y(f) \leqslant a\}$ is a compact subset of $\mathbb{X}$.

Definition 4.2. Let $\{I_y, y \in \mathbb{Y}\}$ be a family of rate functions on $\mathbb{X}$. We call $\{Z_y^\epsilon, \epsilon > 0, y \in \mathbb{Y}\}$ satisfy the uniform Laplace principle on $\mathbb{X}$ with rate function I_y uniformly on compacts if for all compact subsets K of $\mathbb{Y}$ and all real bounded continuous functions g on $\mathbb{X}$:

$$\lim_{\epsilon \to 0} \sup_{y \in K} \left| \epsilon \log \mathbb{E}\left(\exp\left[-\frac{g(Z_y^\epsilon)}{\epsilon} \right] \right) - F(y, g) \right| = 0,$$

where $F(y, g) := -\inf_{f \in \mathbb{X}} \{g(f) + I_y(f)\}$.

The following lemma is easy.

Lemma 4.1. *Let $I : \mathbb{X} \to [0, \infty]$ be a measurable mapping, and I_* the lower semi-continuous regularization of I, i.e.*

$$I_*(f) := \lim_{\epsilon \downarrow 0} \inf_{f' \in B_\epsilon(f)} I(f'),$$

where $B_\epsilon(f)$ is the ball in $\mathbb{X}$ with center f and radius ϵ. Then for any real bounded continuous function g on $\mathbb{X}$

$$\inf_{f \in \mathbb{X}} \{g(f) + I(f)\} = \inf_{f \in \mathbb{X}} \{g(f) + I_*(f)\}.$$

For $N > 0$, let B_N denote the ball in $\mathbb{H}$ with radius N. Then by Theorem III.1 [6]:

Lemma 4.2. *B_N is metrizable as a compact Polish space with respect to the weak topology in $\mathbb{H}$.*

For $N > 0$, set

$$\mathcal{H}_N^a := \{v \in \mathcal{H}^a : \|v(w)\|_{\mathbb{H}} \leqslant N \ a.s. - w\}.$$

We make the following assumption:

(H) There exists a family of measurable mappings $Z_y^0 : \mathbb{H} \to \mathbb{X}$, $y \in \mathbb{Y}$ such that for any $N > 0$, if a family $\{v^\epsilon\} \subset \mathcal{H}_N^a$(as random variables in B_N) converges in distribution to some $v \in \mathcal{H}_N^a$, and y_ϵ converges to y in $\mathbb{Y}$, then for some subsequence ϵ_k

$$Z_{y_{\epsilon_k}}^{\epsilon_k}\left(\cdot + \frac{v^{\epsilon_k}(\cdot)}{\sqrt{\epsilon_k}} \right) \to Z_y^0(v) \text{ in distribution as } k \to \infty.$$

We shall follow the method of [3, 4] to prove the following uniform Laplace principle.

Theorem 4.1. *Assume* **(H)**. *Then $\{Z_y^\epsilon, \epsilon > 0, y \in \mathbb{Y}\}$ satisfies the uniform Laplace principle on $\mathbb{X}$ with rate function $(I_y)_*$ on compacts*:

$$I_y(f) := \frac{1}{2} \inf_{\{h \in \mathbb{H}:\ f = Z_y^0(h)\}} \|h\|_{\mathbb{H}}^2.$$

Proof. By Lemma 4.1 and [5, Proposition 1.2.7], it only needs to prove that for any $y_\epsilon \in \mathbb{Y}$ converging to y in $\mathbb{Y}$

$$\lim_{\epsilon\to 0} -\epsilon \log \mathbb{E}\left(\exp\left[-\frac{g(Z^\epsilon_{y_\epsilon})}{\epsilon}\right]\right) = \inf_{f\in\mathbb{X}}\{g(f)+I_y(f)\},$$

which is equivalent to prove that for any subsequence ϵ_n

$$\overline{\lim_{\epsilon_n\to 0}} -\epsilon_n \log \mathbb{E}\left(\exp\left[-\frac{g(Z^{\epsilon_n}_{y_{\epsilon_n}})}{\epsilon_n}\right]\right) \geqslant \inf_{f\in\mathbb{X}}\{g(f)+I_y(f)\} \tag{4.1}$$

and

$$\underline{\lim_{\epsilon_n\to 0}} -\epsilon_n \log \mathbb{E}\left(\exp\left[-\frac{g(Z^{\epsilon_n}_{y_{\epsilon_n}})}{\epsilon_n}\right]\right) \leqslant \inf_{f\in\mathbb{X}}\{g(f)+I_y(f)\}. \tag{4.2}$$

In the following, we omit the subscript 'n', and prove these two limits separately for sequence ϵ.

(Lower bound): By Theorem 3.2, we have for any $\epsilon > 0$

$$\begin{aligned}-\epsilon \log \mathbb{E}\left(\exp\left[-\frac{g(Z^\epsilon_{y_\epsilon})}{\epsilon}\right]\right) &= \inf_{v\in\mathcal{H}^a} \mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}(w+v) + \frac{\epsilon}{2}\|v\|^2_{\mathbb{H}}\right)\\ &= \inf_{v\in\mathcal{H}^a} \mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot + \frac{v}{\sqrt{\epsilon}}\right) + \frac{1}{2}\|v\|^2_{\mathbb{H}}\right).\end{aligned} \tag{4.3}$$

Fix a $\delta > 0$. For every $\epsilon > 0$ there is a $v^\epsilon \in \mathcal{H}^a$ such that

$$\inf_{v\in\mathcal{H}^a} \mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot + \frac{v}{\sqrt{\epsilon}}\right) + \frac{1}{2}\|v\|^2_{\mathbb{H}}\right) \geqslant \mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot + \frac{v^\epsilon}{\sqrt{\epsilon}}\right) + \frac{1}{2}\|v^\epsilon\|^2_{\mathbb{H}}\right) - \delta.$$

Since g is bounded, we have

$$\frac{1}{2}\sup_{\epsilon>0} \mathbb{E}\|v^\epsilon\|^2_{\mathbb{H}} \leqslant 2\|g\|_\infty + \delta.$$

Define for $N > 0$

$$\tau^\epsilon_N := \inf\left\{t\in[0,1] : \|\pi_t v^\epsilon\|^2_{\mathbb{H}} \geqslant N\right\}$$

and

$$v^\epsilon_N := \pi_{\tau^\epsilon_N} v^\epsilon,$$

then $v^\epsilon_N \in \mathcal{H}^a_N$ and by Chebyshev's inequality

$$\mu(w : \|v^\epsilon_N(w) - v^\epsilon(w)\|_{\mathbb{H}} \neq 0) = \mu(w : \tau^\epsilon_N(w) < 1) \leqslant \frac{2(2\|g\|_\infty + \delta)}{N}.$$

Hence,

$$\begin{aligned}&\mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot + \frac{v^\epsilon}{\sqrt{\epsilon}}\right) + \frac{1}{2}\|v^\epsilon\|^2_{\mathbb{H}}\right) - \delta\\ \geqslant\, &\mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot + \frac{v^\epsilon_N}{\sqrt{\epsilon}}\right) + \frac{1}{2}\|v^\epsilon_N\|^2_{\mathbb{H}}\right) - \frac{2\|g\|_\infty(2\|g\|_\infty + \delta)}{N} - \delta.\end{aligned}$$

By Lemma 4.2, we may pick a subsequence ϵ_k(still denoted by ϵ) such that v_N^ϵ converges in distribution to some v as B_N-valued random variables. Now, by **(H)** there exists another subsequence ϵ_k such that

$$\begin{aligned}\lim_{\epsilon_k\downarrow 0}\mathbb{E}\left(g\circ Z^{\epsilon_k}_{y_{\epsilon_k}}\left(\cdot+\frac{v_N^{\epsilon_k}}{\sqrt{\epsilon_k}}\right)+\frac{1}{2}\|v_N^{\epsilon_k}\|^2_{\mathbb{H}}\right)&\geqslant\mathbb{E}\left(g\circ Z^0_y(v)+\frac{1}{2}\|v\|^2_{\mathbb{H}}\right)\\&\geqslant\inf_{\{(f,h)\in\mathbb{X}\times\mathbb{H}:f=Z^0_y(h)\}}\left(g(f)+\frac{1}{2}\|h\|^2_{\mathbb{H}}\right)\\&\geqslant\inf_{f\in\mathbb{X}}(g(f)+I_y(f)).\end{aligned}$$

Combining the above calculations yields

$$\overline{\lim_{\epsilon\to 0}}-\epsilon\log\mathbb{E}\left(\exp\left[-\frac{g(Z^\epsilon_{y_\epsilon})}{\epsilon}\right]\right)\geqslant\inf_{f\in\mathbb{X}}(g(f)+I_y(f))-\frac{2\|g\|_\infty(2\|g\|_\infty+\delta)}{N}-\delta.$$

Finally, letting $N\to\infty$ and $\delta\to 0$ yields the lower bound (4.1).

(Upper bound): Fix a $\delta>0$. Since g is bounded, there is an $f_0\in\mathbb{X}$ such that

$$g(f_0)+I_y(f_0)\leqslant\inf_{f\in\mathbb{X}}(g(f)+I_y(f))+\delta.$$

Choose an $h_0\in\mathbb{H}$ such that

$$\frac{1}{2}\|h_0\|^2_{\mathbb{H}}\leqslant I_y(f_0)+\delta\text{ and }f_0=Z^0_y(h_0).$$

By (4.3) and **(H)**, there exists a subsequence ϵ_k such that

$$\begin{aligned}\underline{\lim}_{\epsilon\to 0}-\epsilon\log\mathbb{E}\left(\exp\left[-\frac{g(Z^\epsilon_{y_\epsilon})}{\epsilon}\right]\right)&=\underline{\lim}_{\epsilon\to 0}\inf_{v\in\mathcal{H}^a}\mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot+\frac{v}{\sqrt{\epsilon}}\right)+\frac{1}{2}\|v\|^2_{\mathbb{H}}\right)\\&\leqslant\underline{\lim}_{\epsilon\to 0}\mathbb{E}\left(g\circ Z^\epsilon_{y_\epsilon}\left(\cdot+\frac{h_0}{\sqrt{\epsilon}}\right)+\frac{1}{2}\|h_0\|^2_{\mathbb{H}}\right)\\&\leqslant\underline{\lim}_{\epsilon_k\to 0}\mathbb{E}\left(g\circ Z^{\epsilon_k}_{y_{\epsilon_k}}\left(\cdot+\frac{h_0}{\sqrt{\epsilon_k}}\right)+\frac{1}{2}\|h_0\|^2_{\mathbb{H}}\right)\\&=g\circ Z^0_y(h_0)+\frac{1}{2}\|h_0\|^2_{\mathbb{H}}\\&\leqslant g(f_0)+I_y(f_0)+\delta\\&\leqslant\inf_{f\in\mathbb{X}}(g(f)+I_y(f))+2\delta.\end{aligned}$$

Letting $\delta\to 0$, we obtain the upper bound (4.2). □

Remark 1. For the large deviation principle, we need that the rate function is good. If we assume that for every $N>0$, the set $\{Z^0_y(h):h\in B_N\}$ is a compact subset of $\mathbb{X}$, then it is easy to see that $I_y(f)$ is good.

Acknowledgements. The author would like to thank Professor Nicolas Privault for his very useful suggestions.

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