Flips and variation of moduli scheme
of sheaves on a surface

By
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Abstract
Let \( H \) be an ample line bundle on a non-singular projective surface \( X \), and \( M(H) \) the coarse moduli scheme of rank-two \( H \)-semistable sheaves with fixed Chern classes on \( X \). We show that if \( H \) changes and passes through walls to get closer to \( K_X \), then \( M(H) \) undergoes natural flips with respect to canonical divisors. When \( X \) is minimal and \( \kappa(X) \geq 1 \), this sequence of flips terminates in \( M(H_X) \); \( H_X \) is an ample line bundle lying so closely to \( K_X \) that the canonical divisor of \( M(H_X) \) is nef. Remark that so-called Thaddeus-type flips somewhat differ from flips with respect to canonical divisors.

1. Introduction
Let \( X \) be a non-singular projective surface over \( \mathbb{C} \), and \( H \) an ample line bundle on \( X \). Denote by \( M(H) \) (resp. \( \overline{M}(H) \)) the coarse moduli scheme of rank-two \( H \)-stable (resp. \( H \)-semistable) sheaves on \( X \) with Chern class \( (c_1, c_2) \in \text{Pic}(X) \times \mathbb{Z} \). We shall consider birational aspects of the problem how \( \overline{M}(H) \) changes as \( H \) varies.

Let \( H_- \) and \( H_+ \) be generic ample line bundles separated by just one \((c_1, c_2)\)-wall \( W \). For \( a \in (0, 1) \) one can define \( a \)-semistability of sheaves and the coarse moduli scheme \( M(a) \) (resp. \( \overline{M}(a) \)) of rank-two \( a \)-stable (resp. \( a \)-semistable) sheaves with Chern classes \((c_1, c_2)\) in such a way that \( \overline{M}(\epsilon) = M(H_+) \) and \( M(1 - \epsilon) = M(H_-) \) if \( \epsilon > 0 \) is sufficiently small. Let \( a_- < a_+ \) be generic parameters separated by only one miniwall \( a_0 \). Denote \( M_{\pm} = M(a_{\pm}) \) and \( M_0 = M(a_0) \). There are natural morphisms \( f_- : M_- \to M_0 \) and \( f_+ : M_+ \to M_0 \). After [4], let \( f : X \to Y \) be a birational proper morphism such that \( K_X \) is \( \mathbb{Q} \)-Cartier and \( -K_X \) is \( f \)-ample, and that the codimension of the exceptional set \( \text{Ex}(f) \) of \( f \) is more than 1. We say a birational proper morphism \( f_+ : X_+ \to Y \) is a flip of \( f \) if (1) \( K_{X_+} \) is \( \mathbb{Q} \)-Cartier, (2) \( K_{X_+} \) is \( f_+ \)-ample and (3) the codimension of the exceptional set \( \text{Ex}(f_+) \) is more than 1. The main result is the following.

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**Theorem 1.1.** Assume $c_2$ is so large that $\bar{M}_-$ and $\bar{M}_+$ are normal and that the codimensions of

$$\bar{M}_\pm \supset P_\pm = \{ [E] \mid E \text{ is not } a_-\text{-semistable} \}$$

and

$$\bar{M}_\pm \supset \text{Sing}(\bar{M}_\pm) = \{ [E] \mid \dim \text{Ext}^2(E, E)^0 \neq 0 \}$$

are more than 1. Suppose $K_X$ does not lie in the wall $W$ separating $H_-$ and $H_+$, and that $K_X$ and $H_-$ lie in the same connected components of $\text{NS}(X)_{\mathbb{R}} \setminus W$. (See the left figure below.) Then the birational map

$$M_+ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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of $M(H)$, although it is unknown whether $M(H_X)$ admits only terminal singularities. Note that $M(H)$ is of general type if $X$ is of general type, $H^0(K_X)$ contains a reduced curve, and $\chi(O_X) + c_1^2(E) \equiv 0(2)$ by [5]. Corollary 2.1 states that any sequence of flips occurring from the variation of moduli schemes of sheaves on a surface always stops, in relation to termination of flips.

We mention some characteristics of this paper compared with Thaddeus’ work [8], which considered the variation of GIT quotients and linearizations. By [6] the rational map $M_1 \cdots > M_2$ is a Thaddeus-flip, that is, a rational map which is an isomorphism in codimension 1 and comes from the variation of GIT quotient and linearizations. Relations about a flip with respect to the canonical divisor are not mentioned there. So-called Thaddeus-flip is weaker than a flip defined above. Moreover the birational map (1.2) is described in a moduli-theoretic way. Moduli schemes $M_-$ and $M_+$ are connected by a natural blow-up and a blow-down described in moduli theory; see [9, Prop. 4.9].

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2. Proof of Theorem

There is a union of hyperplanes $W \subset \text{Amp}(X)$ called $(c_1, c_2)$-walls in the ample cone $\text{Amp}(X)$ such that $M(H)$ changes only when $H$ passes through walls ([7]). Let $H_-$ and $H_+$ be ample line bundles separated by just one wall $W$, and $H_0 = \lambda H_- + (1 - \lambda)H_+$ an ample line bundle contained in $W$. If $c_2$ is sufficiently large with respect to a compact subset $S \subset \text{Amp}(X)$ containing $H_{\pm}$, then $M_{\pm}$ are normal and the codimension of $P_{\pm} \subset M_{\pm}$, which is defined at (1.1), are greater than 2 from [5] and [3, Thm. 4.C.7]. By [1, Sect. 3], for a number $a \in [0, 1]$ one can define the $a$-stability of a torsion-free sheaf $E$ using

$$P_a(E(n)) = \{(1 - a)\chi(E(H^-)((nH_0)) + a\chi(E(H^+(nH_0)))) / \text{rk}(E)\}.$$

There is the coarse moduli scheme $\bar{M}(a)$ of rank-two $a$-semistable sheaves on $X$ with Chern classes $(c_1, c_2)$. Denote by $M(a)$ its open subscheme of $a$-stable sheaves. There is a finite numbers $a_\pm$ called miniwall such that, as $a$ varies, $M(a)$ changes only when $a$ passes through miniwalls. Let $a_- < a_\pm$ be parameters separated by only one miniwall $a_0$, and denote $M_{\pm} = \bar{M}(a_{\pm})$ and $M_0 = \bar{M}(a_0)$. Since a rank-two $a_\pm$-semistable sheaf of type $(c_1, c_2)$ is $a_0$-semistable, there are natural morphisms $f_- : M_\rightarrow \rightarrow M_0$ and $f_+ : M_\rightarrow \rightarrow M_0$ when $M = H_+ - H_-$. Since $M_+$ is effective and $C$ equals $n_0 M$ with sufficiently large $n_0$, by [1, Prop. 3.14].

Remark that the canonical divisors of $M_{\pm}$ are $\mathbb{Q}$-Cartier. Indeed, $M_-$ equal $R/\text{SL}(N)$, where $R$ is a subscheme of the Quot-scheme parameterizing quotient sheaves $\mathcal{O}_X(-M) \rightarrow E^-$ on $X$. Let $E^\pm_R$ be the universal quotient sheaf on $X_R$. From descent lemma [3, Theorem 4.2.15], det $R\text{Hom}_{X_R/R}(E^\pm_R, E^\mp_R)$ descends to a line bundle on $M_-$, which we denote by det $R\text{Hom}_{X_{M_-}/M_-}(E^-, E^-)$.

It is known that $K_{M_-|M_-, \text{Sing}(M_-)}$ equals $R\text{Hom}_{X_{M_-, M_-}}(E^-, E^-)$ from deformation theory. Since $M_-$ is normal, we have

$$K_{M_-} = \text{det} R\text{Hom}_{X_{M_-}/M_-}(E^-, E^-),$$

$(2.1)$
so it is $\mathbb{Q}$-Cartier.

Let $\eta$ be an element of
\[ A^+(W) = \{ \eta \in \text{NS}(X) \mid \eta \text{ defines } W, \ 4c_2 - c_1^2 + \eta^2 \geq 0 \text{ and } \eta \cdot H_+ > 0 \} \].

After [1, Definition 4.2] we define
\[ T_{\eta} = M(1, (c_1 + \eta)/2, n) \times M(1, (c_1 - \eta)/2, m), \]
where $n$ and $m$ are numbers defined by
\[ n + m = c_2 - (c_1^2 - \eta^2)/4 \text{ and } n - m = \eta \cdot (c_1 - K_X)/2 + (2a_0 - 1)\eta \cdot C, \]
and $M(1, (c_1 + \eta)/2)$ is the moduli scheme of rank-one torsion-free sheaves on $X$ with Chern classes ($(c_1 + \eta)/2, n$). We also denote $T = \coprod T_{\eta}$, where $\eta$ runs over $A^+(W)$. If $F_{T_{\eta}}$ (resp. $G_{T_{\eta}}$) is the pull-back of a universal sheaf of $M(1, (c_1 + \eta)/2, n)$ (resp. $M(1, (c_1 - \eta)/2, m)$) to $X_{T_{\eta}}$, then we have the following.

**Proposition 2.1** ([9], Section 5). We have isomorphisms
\[
\begin{align*}
(2.2) \quad P_- & \cong \coprod_{\eta \in A^+(W)} P_{T_{\eta}} \left( \text{Ext}^1_{X_{T_{\eta}}/T_{\eta}} (F_{T_{\eta}}, G_{T_{\eta}}(K_X)) \right) \quad \text{and} \\
(2.3) \quad P_+ & \cong \coprod_{\eta \in A^+(W)} P_{T_{\eta}} \left( \text{Ext}^1_{X_{T_{\eta}}/T_{\eta}} (G_{T_{\eta}}, F_{T_{\eta}}(K_X)) \right).
\end{align*}
\]

There are line bundles $L_1$ (resp. $L'_1$) on $P_-$ (resp. $P_+$) with exact sequences
\[
\begin{align*}
0 & \rightarrow F_T \otimes L_1 \rightarrow E_{M_-}|_{P_-} \rightarrow G_T \otimes L_2 \rightarrow 0 \quad \text{and} \\
0 & \rightarrow G_T \otimes L'_1 \rightarrow E_{M_+}|_{P_+} \rightarrow F_T \otimes L'_2 \rightarrow 0
\end{align*}
\]
such that $L_1 \otimes L_2^{-1} = \mathcal{O}_{P_-}(1)$, which means the tautological line bundle of the right side of (2.2), and $L_1' \otimes L_2'^{-1} = \mathcal{O}_{P_+}(1)$. Here $E_{M_-}$ is a universal family of $M_-$, which exists etale-locally.

**Claim 1.** It holds that
\[ K_{M_-}|_{P_- \times T_{\eta}} = - (\eta \cdot K_X) \mathcal{O}_{P_-}(1) + \text{ (some line bundle on } T), \quad \text{and} \]
\[ K_{M_+}|_{P_+ \times T_{\eta}} = (\eta \cdot K_X) \mathcal{O}_{P_+}(1) + \text{ (some line bundle on } T). \]

**Proof.** Suppose that $A^+(W) = \{ \eta \}$ for simplicity. From the definition of $\eta$, one can check $c_1(F_t) - c_1(G_t) = \eta$. By Proposition 2.1 and (2.1),
\[
\begin{align*}
K_{M_-}|_{P_-} & = \det RH \text{om}_{X_{P_-}/P_-} (E_{M_-}|_{P_-}, E_{M_-}|_{P_-}) \\
& = \det RH \text{om}_{X_T/T}(F_T, F_T) + \det (RH \text{om}_{X_T/T}(F_T, G_T) \otimes L_2 \otimes L_2^{-1}) \\
& \quad + \det (RH \text{om}_{X_T/T}(G_T, F_T) \otimes L_1 \otimes L_1^{-1}) + \det RH \text{om}_{X_T/T}(G_T, G_T) \\
& = -\chi(F_t, G_t) \cdot \mathcal{O}_{P_-}(1) + \chi(G_t, F_t) \cdot \mathcal{O}_{P_+}(1) + \text{ (line bundle on } T) \\
& = -(\eta \cdot K_X) \mathcal{O}_{P_-}(1) + \text{ (line bundle on } T). \]
\]
One can calculate $K_{M_+}|_{P_+}$ similarly.

Remark that, since $\eta \cdot H_+ > 0$, one can verify that $\eta \cdot K_X < 0$ if and only if $K_X$ does not lie in $W = W$, and $H_-$ and $K_X$ lie in the same connected components of $\text{NS}(X)_{\mathbb{R}} \setminus W$. The next lemma ends the proof of 1.1.

**Lemma 2.1.** The map $f_+: M_+ \to f_+(M_+)$ is proper.

**Proof.** It suffices to show that $f_+^{-1}(f_+(M_+)) = M_+$. Suppose not. Then some $[E] \in M_+ \setminus M_+$ satisfies $f_+([E]) = f_+([E'])$. Since $a_+ > 0$ is generic, $E'$ is denoted with an exact sequence

$$0 \to F' \to E' \to G' \to 0$$

with rank-one torsion-free sheaves $F'$ and $G'$ such that $P_a(F'(n)) = P_a(G'(n))$ for any $a$. $[E]$ and $[E']$ are S-equivalent with respect to $a_0$-stability, so (2.4) implies that $E$ cannot be $a_+-$stable, which is a contradiction.

**Corollary 2.1.** Any sequence of flips occurring from the variation of polarizations and moduli schemes of sheaves on a surface $X$ always stops after finitely many modifications.

**Proof.** In fixing a polarization $H$, we claim that only finitely many walls pass across the segment $L$ connecting $H$ and $K_X$. Indeed, $L \cap \partial \text{Amp}(X)$ is empty or equals $\{H_0\}$ with a $Q$-divisor $H_0$ from the cone theorem. Thus the claim follows from the fact [6, Lem. 1.5] that only finitely many walls intersect with a fixed polyhedral cone in $\text{Amp}(X)$ spanned by $Q$-divisors. One can readily check the corollary from this claim.

We end with proving some facts in Introduction.

**Lemma 2.2.** Assume $K_X$ is nef, and fix $c_1$ and a polarization $H_0$. If $c_2$ is sufficiently large with respect to $X$ and $H_0$, then for $H \in \mathcal{L} = \{(1-t)H_0 + tK_X | t \in [0,1]\}$, $M(H)$ are mutually isomorphic in codimension one.

**Proof.** Let $H_-$ and $H_+$ be any ample line bundles lying in adjacent chambers and separated by a wall $W$ passing through a point $L$ on $\mathcal{L}$. Since $K_X$ is nef, one can find an effective divisor $H \in H^0(\mathcal{O}_X(NK_X))$ with some $N > 0$ such that $H$ is the disjoint union of some finite smooth curves. Then similarly to [7, Lem. 2.2] we can show that, for a divisor $F$ with $2F - c_1 \sim \eta$,

$$h^0(\mathcal{O}_X(K_X - (2F - c_1))) \leq d_1(X) + N|K_X \cdot \eta|,$$

where $d_1$ is a constant depending only on $X$. By the proof of [7, Lem. 2.1], it holds that

$$|K_X \cdot \eta| \leq \left(2K_X^2 + \left(\frac{K_X \cdot L}{(L^2)^{1/2}}\right)\right) \left(2c_2 - c_1^2\right)^{1/2}.$$

When $K_X^2 > 0$, $|K_X \cdot L|/(L^2)^{1/2}$ is bounded for any $L \in \mathcal{L}$. When $K_X^2 = 0$, one can check that

$$|K_X \cdot L|^2 \leq |K_X - H_0|^2 / H_0^2.$$
Thus some $d_2$ depending only on $X$ and $H_0$ satisfies
\begin{equation}
|K_X \cdot \eta| \leq d_2(X, H_0) \cdot (4c_2 - c_1^2)^{1/2}.
\end{equation}
In the same way as [7, Thm. 2.3], we can show that
\[
\dim(P_\times T_\eta) \leq (3/4)(4c_2 - c_1^2) + d_3(X, H_0) + d_4(X, H_0) \cdot (4c_2 - c_1^2)^{1/2}
\]
with some constant $d_3$ and $d_4$ depending only on $X$ and $H_0$ by using (2.5), (2.6) and (2.7), and this implies the lemma.

Claim 2. Suppose $K_X$ is nef, and let $H_X$ be an ample line bundle $H_X$ such that no wall of type $(c_1, c_2)$ divides $K_X$ and $H_X$. Then the canonical divisor of $M(H_X)$ is nef.

Proof. From [3, Prop. 8.3.1] $2K_{M(H_X)} = p_*(\Delta(E_{M(H_X)}) \cdot K_X)$, and $p_*(\Delta(E_{M(H_X)}) \cdot H_X)$ is nef. When $H_X$ is sufficiently close to $K_X$, the assertion holds.

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