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Connected and not arcwise connected invariant sets for some 2-dimensional dynamical systems

By

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Abstract

The invariant Cantor set of Smale's horseshoe map is imbedded into a connented and not arcwise connected, invariant, compact set with zero Lebesgue measure.

1. Introduction

Strange attractors of some 2-dimensional dynamical systems seem to be a connented and not arcwise connected, invariant, compact set with zero Lebesgue measure, which will be called an N-set in this paper ([1], [2], [3]). The purpose of this paper is to prove the existence of an N-set for some 2-dimensional dynamical systems involving Smale's horseshoe map. We shall consider the 2-dimensional diffeomorphism T:

$$T(x,y) = (x', y'), \quad x' = \varphi(x,y), y' = \psi(x,y),$$

where $x, x', y, y' \in R$ and $\varphi(x,y)$ and $\psi(x,y)$ are once continuously differentiable with respect to x and y. In the following, $DT(P)$ denotes the Jacobi's matrix of T for $P = (x,y)$, that is,

$$DT(P) = \begin{pmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \end{pmatrix}$$

and $|K|$ denotes the area of the subset K of R^2.
Our main theorem is the following.

Theorem 1. *Assume that conditions* (i) $\sim$ (iii) *hold*:
(i) *there exists a compact, simply connected set K with piecewisely smooth boundary such that $T(K) \subset K$,*
(ii) $|T^n(K)|$ *approaches zero as n tends to infinity,*

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(iii) *K contains at least two distinct fixed points, and for one of them, say P_1, the eigenvalues of $DT(P_1)$, say λ_1 and λ_2, satisfy that*

$$\lambda_1 < -1 < \lambda_2 < 0.$$

Then the set $\Omega := \bigcap_{n=0}^{\infty} T^n(K)$ *is an N-set.*

Remark 1. An N-set is not a manifold, because a connected manifold is arcwise connected. In the arcwise connected space S, given two points P_1 and P_2 can be connected by a simple, continuous arc in S, and hence the connected and not arcwise connected space S contains a pair of P_1 and P_2 which cannot be connected by any simple, continuous arc in S. For example, $S := \{(x, y); y = \sin\frac{1}{x}, 0 < x < 1\} \cup \{(0, y); -1 \le y \le 1\}$ is connected and not arcwise connected in itself.

Remark 2. P_1 is said to be inversely unstable if $\lambda_1 < -1 < \lambda_2 < 0$. On the other hand, if $0 < \lambda_1 < 1 < \lambda_2$, then P_1 is directly unstable. It will be shown afterward that Theorem 1 does not hold if P_1 is directly unstable.

The proof of Theorem 1 is stated in Section 2, and applications such as Poincare map associated with some Duffing type equations, a quasi-linear map motivated by Henon map and Smale's horseshoe map in Sections 3, 4 and 5, respectively.

2. Proof of Theorem 1

It is obvious that Ω is an invariant, compact, connected set with zero Lesbesgue measure, and hence it is remained to prove that Ω is not arcwisely connected. To the contrary, suppose that Ω is arcwisely connected. Now, let P_1 and P_2 be the two distinct fixed points in K, and hence P_1 and P_2 belong to Ω. By Remark 1, we can take a simple, continuous arc c in Ω, which joins P_1 and P_2. Since P_1 is inversely unstable, there exists an invariant, unstable 1-dimensional manifold around P_1, that is, a T-invariant, smooth curve l in Ω such that l contains P_1 in its inside and the tangent vector along l at P_1 is the eigenvector of $DT(P_1)$ associated with λ_1 [4, Theorem 5.1]. Since λ_1 is negative, any point Q on one hand side of l with respect to P_1 is mapped into the another hand side of l by T, if Q is sufficiently close to P_1.

First of all we shall claim that there exists no simple closed curve in Ω. In fact, if such a curve h exists, then the interior of h contains a small disk H with $|H| > 0$. Since $h \subset \Omega \subset T^n K$ and $T^n K$ is simply connected, it follows that $H \subset T^n K$, which implies that $|H| < |T^n K|$. Therefore, (ii) shows that $|H| = 0$, which is a contradiction.

Now we shall take a small open neighbourhood U of P_1 such that U minus a component of l containing P_1 is divided into the two disjoint open sets U_1 and U_2, and moreover set c_0 to be the connected component of $c \cap U$ containg

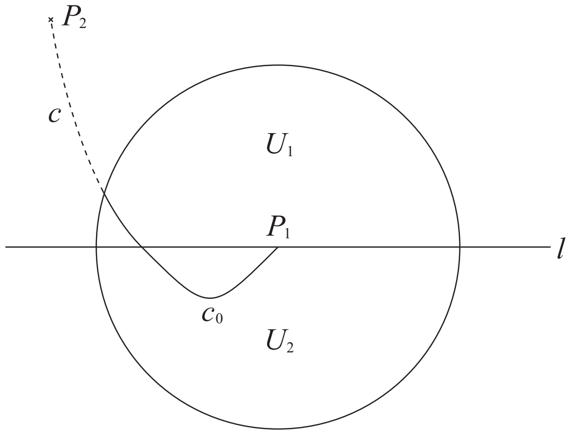

Figure 1.

P_1 (see Figure 1).

We shall show that either $c_0 \subset U_1 \cup l$ or $c_0 \subset U_2 \cup l$. In fact, if c_0 contains not only a point P_3 in U_1, but also a point P_4 in U_2, c_0 must intersect with l at some point Q between P_3 and P_4. In the case where Q is distinct from P_1, both some part of l and some part of c_0 joins P_1 and Q together, and hence we can take a simple closed curve joining P_1 and Q as the part of $c_0 \cup l$. In the case where $Q = P_1$, the part of c_0 itself is a simple closed curve. Anyhow, the both cases contradict our first claim.

Next we shall show that c_0 is not identical with any part of l in any neighbourhood of P_1. To the contrary, suppose that c_0 is identical with some part of l in a neighbourhood V of P_1. Since c_0 contains P_1 as one of the terminal points, $c_0 \cap V$ is contained in one hand side of l with respect to P_1. Then $T(c_0 \cap V)$ is contained in another hand side of l with respect to P_1, which implies that c_0 and Tc_0 are distinct from each other, and hence c and Tc are distinct from each other. Since both c and Tc join P_1 and P_2 together, we can take a simple closed curve in Ω as a part of $c \cup Tc$, which is a contradiction.

The remaining case is either that c_0 is contained in $U_1 \cup l$ and contains one point P_5 of U_1, or that c_0 is contained in $U_2 \cup l$ and contains one point of U_2. We shall treat the former case without loss of generality. Since P_5 may be assumed to belong to any small neighbourhood of P_1 by the above argument, it follows from the continuity of T that TP_5 belongs to U minus a component of l containing P_1, which implies that TP_5 belongs to U_2, because λ_2 is negative. Therefore c_0 and Tc_0 are distinct from each other, and hence there arises a contradiction. The proof is completed. □

We shall prove the assertion of Remark 2 such that (iii) cannot be replaced by $0 < \lambda_1 < 1 < \lambda_2$. For example, let consider the map $T(x, y) = (x', y')$ on K

defined by $x' = \sin \frac{\pi}{2} x, y' = ye^{-y-x^2-1}$, where $K = \{-1 \leq x \leq 1, \ -1 \leq y \leq 1\}$. It is seen that $TK \subset K$ and T has the four fixed points $(0,0), (-1,0), (1,0)$ and $(0,-1)$, and moreover that for $P = (0,0)$, $DT(P)$ has the two eigenvalues $\lambda_1 = e^{-1}$ and $\lambda_2 = \frac{\pi}{2}$. Setting $T^n(x,y) = (x_n, y_n)$ for positive integer n, we may verify that $|x_n| \geq |x|$ and $|y_n| \leq |y_{n-1}|e^{-x_n^2}$ for $n \geq 1$, which implies that $|y_n| \leq e^{-nx^2}$. Therefore we can show that $\Omega = \{(0,y); -1 \leq y \leq 0\} \cup \{(x,0); -1 \leq x \leq 1\}$, which is arcwisely connected.

3. Duffing type equations

We shall consider the application of Theorem 1 to Duffing type equations:

$$\dot{x} = y, \ \dot{y} = -\varepsilon\lambda y - (1 + \varepsilon \cos 2t)x - ax^2 - x^3 \quad \left(\cdot = \frac{d}{dt}\right) \tag{1}$$

where ε, λ and a are positive constants. The Poincare mapping T for (1) is defined by $(x_2, y_2) = T(x_1, y_1)$:

$$x_2 = x(\pi, x_1, y_1), \quad y_2 = y(\pi, x_1, y_1),$$

where the pair of $x(t, x_1, y_1)$ and $y(t, x_1, y_1)$ is a solution of (1) which passes (x_1, y_1) at time $t = 0$.

Theorem 2. *Assume that $a > 2$ and $0 < \lambda < \frac{1}{4}$. If ε is sufficiently small, then T has an N-set Ω. Moreover Ω is globally stable, that is, for any point P of R^2, $T^n(P)$ approaches Ω as n tends to infinity.*

Proof. First of all we shall show that conditions (ii) and (iii) are statisfied. The appearance of positive damping term implies (ii). We may prove that the null solution of (1) is inversely unstable, by the same argument as in [6, Lemma 2]. The existence of nontrivial π-periodic solutions follows from the perturbation theory for ε. In fact, when $\varepsilon = 0$, (1) is reduced to

$$\dot{x} = y, \quad \dot{y} = -x - ax^2 - x^3,$$

which has the constant solution $x_1 = \frac{-a+\sqrt{a^2-4}}{2}$. Since the characteristic exponents are non zero real numbers, it follows that (1) has a π-periodic solution $x(t)$ for small ε, which is close to x_1. Thus we get (iii). Now we shall prove (i). The solutions of (1) is uniform-ultimately bounded [5], that is, there exists a disk D_0 such that for any disk D there is a positive number n such that $T^n(D) \subset D_0$, where n may depend on D. Therefore there is a positive number m such that $T^m(D_0) \subset D_0$. By the fixed point theorem of L.E.J.Brower, there exists a point P_0 such that $T^m(P_0) = P_0$. We shall take a large disk $D_1 \supset D_0$ such that $D_1 \supset \bigcup_{k=0}^{m-1} \{T^k P_0\}$, which implies that $T(D_1) \cap D_1 \neq \emptyset$, and hence that $T^i(D_1) \cap T^{i+1}(D_1) \neq \emptyset$ for $i \geq 1$. Furthermore we may assume that

$T^m(D_1) \subset D_1$ for the previous m, and hence setting $E = \bigcup_{i=0}^{m-1} T^i(D_1)$, we can see that $T(E) \subset E$. Letting J_i be the boundary of $T^i(D_1)$ for $0 \le i \le m-1$, we shall apply [7, Theorem9.1] in order that the infinite component $R^2 - E$ has for boundary a Jordan curve J contained in $\bigcup_{i=0}^{m-1} J_i$. Letting K be the interior of J, we can see that $T(K) \subset K$, because $K \supset E \supset T(E) \supset T(J)$. Thus, Theorem 1 guarantees that $\Omega := \bigcap_{n=0}^{\infty} T^n(K)$ is an N-set. Now, let P be any point P of R^2. Since $T^n(P)$ remains in D_0 for large n and since $D_0 \subset D_1 \subset E \subset K$, it follows that $T^n(P)$ remains in K for large n, which implies that $T^n(P)$ approaches Ω as n tends to infinity. The proof is completed. $\square$

Remark 3. The system (1) is something of the artificial, and we shall consider the Duffing equation, which describes the dynamics of electric current of some electric circuits,

$$\dot{x} = y, \quad \dot{y} = -ky - x^3 + B_0 + B\cos t,$$

where k, B_0 and B are positive constants. It is difficult to prove the existence of inversely unstable periodic solutions for this system; the experimental results of [1] suggests that the existence of inversely unstable periodic solutions implies the existence of the attracting N-set.

4. Quasi-linear map

We shall consider the map $T(x, y) = (x', y')$

$$x' = y + f(x), \quad y' = -bx \tag{2}$$

where $f(x)$ is continuously differentiable with respect to x and bounded on R, and b is a constant such that $0 < b < 1$. Clearly, $P(x, y)$ is a fixed point of (2) if and only if

$$(1 + b)x = f(x) \tag{3}$$

where $y = -bx$. Applying Theorem 1 to (2), we shall obtain the following result.

Theorem 3. *Assume that* (3) *has at least two distinct solutions, one of which, say x_1, satisfies that*

$$f'(x_1) < -(1 + b). \tag{4}$$

Then (2) *has an N-set.*

Proof. Let $M = \sup_{x\in R} |f(x)|$, and m be a positive number such that $m > M/(1-b)$ and the interval $[-m, m]$ contains the two solutions of (3). Moreover set $K = \{|x| \le m, |y| \le bm\}$. Then we can see that $TK \subset K$, because $|x'| \le |y| + |f(x)| \le bm + M \le m$ for $(x, y) \in K$. Since

$$(DT)(P) = \begin{pmatrix} f'(x) & 1 \\ -b & 0 \end{pmatrix},$$

where $P = (x, y)$, the determinant of $DT(P)$ is equal to b, which implies (ii) of Theorem 1, because $0 < b < 1$. Clearly T has two distinct fixed points in K, one of which is $P_1 = (x_1, bx_1)$, and the eigenvalues of $DT(P_1)$, say λ_1 and λ_2, is the following:

$$\lambda_1 = \frac{f'(x_1) - \sqrt{A}}{2}, \quad \lambda_2 = \frac{f'(x_1) + \sqrt{A}}{2},$$

where $A = (f'(x_1))^2 - 4b > 0$. The inequality such that $\lambda_1 < -1 < \lambda_2 < 0$ is equal to

$$\frac{4b}{f'(x_1) + \sqrt{A}} < -2 < f'(x_1) + \sqrt{A},$$

which is furthermore identical with the inequality

$$f'(x_1) + \sqrt{A} > -2b,$$

which holds by (4). Thus the proof is completed. □

We shall show an example for Theorem 3:

$$x' = y + \cos ax, \quad y' = -bx, \tag{5}$$

where a and b are positive constants, $0 < b < 1$, and

$$a \ge (1+b)\sqrt{1 + \frac{\pi^2}{4}}. \tag{6}$$

This map has an N-set in the domain $K = \{|x| \le \frac{1}{1-b}, |y| \le \frac{b}{1-b}\}$. Indeed, let c be the positive number such that $c \sin \sqrt{c^2 - 1} = 1$, where $1 \le c \le \sqrt{1 + \frac{\pi^2}{9}}$, and set $d = \frac{a}{c} - 1$, which is positive by (6). By an elementary calculus we may verify that the equation $(1+b)x = \cos ax$, which is (3) in our case (5), has at least two solutions x_1 and x_2 such that $-\frac{\pi}{a} < x_2 < 0 < x_1 < \frac{\pi}{2a}$, if $0 < b < d$. Furthermore we shall show that (4) holds in our case, that is, $-a \sin ax_1 < -(1+b)$, which is identical with $1 - (1+b)^2 x_1^2 > \left(\frac{1+b}{a}\right)^2$. Since $0 < x_1 < \frac{\pi}{2a}$, this inequality is satisfied by (6). Thus our assertion holds.

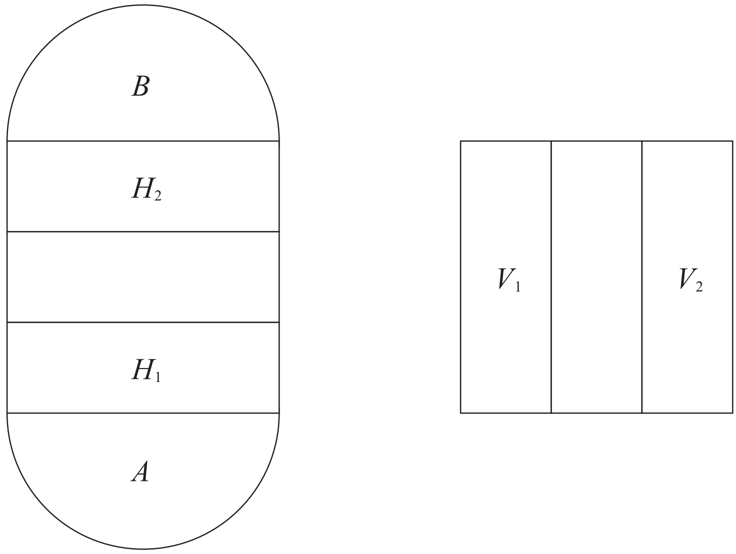

Figure 2.

5. Smale's horseshoe map

Smale's horseshoe map itself is well known, and it has an invariant Cantor set Λ, which contains many periodic points with arbitrary period [3]. We shall show that Λ is embedded into an N-set. We shall state an example of Smale's horseshoe map instead of the generalized definition.

Let consider our plane with coodinate (x, y), and set the following: $S = \{(x, y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$, $H_1 = \{0 \le x \le 1,\ 0 \le y \le \frac{1}{3}\}$, $H_2 = \{0 \le x \le 1,\ \frac{2}{3} \le y \le 1\}$, $A = \{0 \le x \le 1,\ -\sqrt{x - x^2} \le y \le 0\}$, $B = \{0 \le x \le 1,\ 1 \le y \le 1 + \sqrt{x - x^2}\}$, $K = A \cup B \cup S$, $V_1 = \{0 \le x \le \frac{1}{3},\ 0 \le y \le 1\}$, $V_2 = \{\frac{2}{3} \le x \le 1,\ 0 \le y \le 1\}$ (see Figure 2). The diffeomorphism T on K into K is defined so that $T(H_1) = V_1$, $T(H_2) = V_2$, $T(S) \subset S \cup B$, $T(B) \subset A$, $T(A) \subset A$ and T is linear on $H_1 \cup H_2$; the boundary of K is arcwisely smooth. Moreover it is assumed that the absolute value of the determinant of $DT(P)$ is less than one for each $P \in K$, which implies (ii). By the argument of symbolic dynamical system we know that T has two fixed points in S, say P_1 and P_2, such that $P_1 \in H_1 \cap V_1$ and $P_2 \in H_2 \cap V_2$. Since the eigenvalues of T on H_2 is -3 and $-\frac{1}{3}$, P_2 is inversely unstable. Therefore Theorem 1 guarantees that $\Omega := \bigcap_{n=1}^{\infty} T^n(K)$ is an N-set, which contains Λ.

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