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On classification of maps of a css complex into a css group

By

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Introduction

Let G be a reduced 0-connected css group (for the definition, see [5]) and (K, L) be a css pair. Denote by e_n the unit of G_n and by e the css subgroup of G consisting of all e_n, $n \geqq 0$. The set $\Pi(K, L; G)$ of all homotopy classes of maps $f: (K, L) \to (G, e)$ has a natural group structure. Then, we have a filtration

$$(1) \qquad \Pi(K, L; G) = D_0^1 \supseteqq D_1^1 \supseteqq D_2^1 \supseteqq \cdots,$$

by normal subgroups D_n^1 ($n \geqq 0$) defined in § 3. On the other hand, for each $n \geqq 1$, there are sequences of subgroups:

$$H^{n-1}(K, L; \pi_n(G)) = 'P_n^n \supseteqq 'P_{n+1}^n \supseteqq \cdots \supseteqq 'P_\infty^n$$
(the reduced $(n-1)$-st cohomology group),
$$H^n(K, L; \pi_n(G)) = P_n^n \supseteqq P_{n+1}^n \supseteqq \cdots \supseteqq P_\infty^n \supseteqq 'R_1^n \supseteqq 'R_2^n \supseteqq \cdots \supseteqq 'R_n^n = 0,$$
$$H^{n+1}(K, L; \pi_n(G)) \supseteqq R_1^n \supseteqq R_2^n \supseteqq \cdots \supseteqq R_n^n = 0$$

which are defined in § 2. Our purpose of this paper is to show that, for $1 \leqq m < n$, there are homomorphisms

$$\theta_m^{n-m}: P_{n-1}^m \to H^{n+1}(K, L; \pi_n(G))/R_{m+1}^n,$$
$$'\theta_m^{n-m}: 'P_{n-1}^m \to H^n(K, L; \pi_n(G))/'R_{m+1}^n,$$

which induce isomorphisms

$$P_{n-1}^m / P_n^m \approx R_m^n / R_{m+1}^n, \quad 'P_{n-1}^m / 'P_n^m \approx 'R_m^n / 'R_{m+1}^n$$

(§ 2, Theorem 1) and to show that

$$D^1_{n-1}/D^1_n \approx P^n_\infty/'R^n_1$$

(§ 3, Theorem 2). The homomorphisms θ^{n-m}_m and $'\theta^{n-m}_m$ are generalized cohomology operations.

In the paper [4] S. T. Hu gave a filtration (1) of $\Pi(K, L; G)$ for a finite cell complex (K, L) and a topological group G. Our filtration (1) is defined by the same manner as that of S. T. Hu, i.e., D^1_n is the set of homotopy classes of maps which are n-homotopic with 0 relative to L (see (5.2) of [4]). Then, our Theorem 2 corresponds to Theorem (5.7) of [4].

As an application, we derive some results for $\Pi(K, L; G)$ which correspond to those of F. P. Peterson [6] in the case of cohomotopy groups. We assume that (K, L) is of finite dimension and $\Pi(K, L; G)$ is abelian. If $\pi_r(G)$ and $H^r(K, L)$ are finitely generated for $r\geqq 1$, $\Pi(K, L; G)$ is finitely generated (§ 4, Proposition 1). Let $\boldsymbol{C}$ be a class of abelian groups in the sense of J. P. Serre [7]. If $H^r(K, L; \pi_r(G))$ and $H^{r-1}(K, L; \pi_r(G))$ belong to $\boldsymbol{C}$ for $r<n$, $j^*_n: \Pi(K, L; {}^nG)\to\Pi(K, L; G)$ induced by the injection $j_n: {}^nG\to G$ is a $\boldsymbol{C}$-isomorphism (for the definition of ${}^nG$, see § 2). If $H^r(K, L; \pi_r(G))$ and $H^{r+1}(K, L; \pi_r(G))$ belong to $\boldsymbol{C}$ for $r>n$ and $\Pi(K, L; G/{}^{n+1}G)$ is abelian, $p^*_n: \Pi(K, L; G)\to\Pi(K, L; G/{}^{n+1}G)$ induced by the natural map $p_n: G\to G/{}^{n+1}G$ is a $\boldsymbol{C}$-isomorphism (§ 4, Proposition 2).

§ 1. Preliminaries

Let G be a css group, N be a css normal subgroup of G, G/N be the css factor group and $q: G\to G/N$ be the natural map. The triple $(G, G/N, q)$ is a principal fibre bundle with fibre N (IV, Definition 2.1 of [1]). Then, there is a dimension preserving function $\beta: G/N\to G$ such that

(1) $q\beta a=a,\quad \beta e_0=e_0$,

(2) $\beta s_i a=s_i\beta a,\quad i\geqq 0$,

(3) $\beta\partial_i a=\partial_i\beta a,\quad i>0$,

(4) if N is $(n-1)$-connected, $\beta\partial_0 a=\partial_0\beta a$ for $a\in(G/N)_k$, $k=1,\cdots,n$ (IV, 2 of [1]). Then

$$\xi a = (\beta\partial_0 a)^{-1}(\partial_0\beta a)\quad (\in N)$$

is a twisted function of $(G, G/N, q)$. Let $\bar{W}N$ be the W-construction of N (IV, 5 of [1]). Then ξ induces a map $k_\xi: G/N \to \bar{W}N$ defined by

$$k_\xi a = [\xi\partial_0^{m-1}a,\ \xi\partial_0^{m-2}a,\ \cdots,\ \xi a] \qquad (a \in (G/N)_m)\,.$$

Consider the case where N is a reduced $(n-1)$-connected $K(\pi, n)$ $(n \geqq 1)$, i.e., $\pi_n(N) = \pi$, $\pi_n(N) = 0$ for $k \neq n$ and $N^{n-1} = \{e_k,\ k = 0, 1, \cdots, n-1\}$. Let (K, L) be a css pair. Since $\bar{W}N$ is a $K(\pi, n+1)$, there is a natural one-to-one correspondence

$$T:\ \Pi(K, L\,;\ \bar{W}N) \to H^{n+1}(K, L\,;\ \pi)\,,$$

which is defined as follows. Let $t: N_n \to \pi$ be a homomorphism defined by

$$t(x) = \text{the element of } \pi \text{ represented by } x\,.$$

Let $g: (K, L) \to (\bar{W}N, *)$ ($*$ is the base point of $\bar{W}N$) be a map. Since N is $(n-1)$-reduced, $g(\sigma)$ is written by $[e_0, e_1, \cdots, e_{n-1}, g_n(\sigma)]$ for $\sigma \in K_{n+1}$. Then the function $tg_n: \sigma \to tg_n(\sigma)$ defines a cocycle of $Z^{n+1}(K, L\,; \pi)$ which represents $T[g]$. Then k_ξ induces a transformation

$$k_\xi^\sharp:\ \Pi(K, L\,;\ G/N) \to H^{n+1}(K, L\,;\ \pi)$$

which is defined by

$$k_\xi^\sharp = T \circ k_\xi^*\,,$$

where $k_\xi^*: \Pi(K, L\,; G/N) \to \Pi(K, L\,; \bar{W}N)$ is the transformation induced by k_ξ.

Lemma 1. *The transformation $k_\xi^\sharp$ is a homomorphism.*

Proof. Define a function $\omega: (G/N \times G/N)_{n+1} \to \pi$ by

$$\omega(a \times b) = t\xi ab - t\xi a - t\xi b \qquad (a, b \in (G/N)_{n+1})\,.$$

Then ω is a cochain of $C^{n+1}(G/N \times G/N,\ G/N \cup G/N\,; \pi)$. Define a cochain $c \in C^n(G/N \times G/N,\ G/N \cup G/N\,; \pi)$ by

$$c(a' \times b') = t((\beta a'b')^{-1}(\beta a')(\beta b')) \qquad (a', b' \in (G/N)_n)\,.$$

For $a, b \in (G/N)_{n+1}$, we have

$$\sum_{i=0}^{n+1}(-1)^i c(\partial_i(a\times b)) = \sum_{i=0}^{n+1}(-1)^i t((\beta\partial_i(ab))^{-1}(\beta\partial_i a)(\beta\partial_i b))$$
$$= \sum_{i=0}^{n+1}(-1)^i t(\partial_i((\beta ab)^{-1}(\beta a)(\beta b)))$$
$$+ t((\beta\partial_0(ab))^{-1}(\beta\partial_0 a)(\beta\partial_0 b)) - t((\partial_0\beta ab)^{-1}(\partial_0\beta a)(\partial_0\beta b)) .$$

Since $\sum_{i=0}^{n+1}(-1)^i t(\partial_i((\beta ab)^{-1}(\beta a)(\beta b)) = 0$, then

$$\sum_{i=0}^{n+1}(-1)^i c(\partial_i(a\times b)) = t((\beta\partial_0(ab))^{-1}(\partial_0\beta ab)(\partial_0\beta ab)^{-1}(\beta\partial_0 a)(\beta\partial_0 b)$$
$$- t((\partial_0\beta ab)^{-1}(\beta\partial_0 a)(\beta\partial_0 b)(\beta\partial_0 b)^{-1}(\beta\partial_0 a)^{-1}(\partial_0\beta a)(\partial_0\beta b))$$
$$= t\xi ab - t((\beta\partial_0 b)^{-1}(\beta\partial_0 a)^{-1}(\partial_0\beta a)(\beta\partial_0 b)) - t((\beta\partial_0 b)^{-1}(\partial_0\beta b))$$
$$= t\xi ab - t\xi a - t\xi b .$$

This shows that $\delta c = \omega$. Let $g, g' ; (K, L) \to (G/N, e)$ be two maps. Then

$$\omega(g(\sigma)\times g'(\sigma)) = \delta c(g(\sigma)\times g'(\rho)) \qquad \text{for} \quad \sigma\in K_{n+1} .$$

Then $k_\xi^\#$ is a homomorphism.

Let K be a complex with base point $*$. The cone CK of K is obtained from $K\times I$ by identifying the subcomplex $K\times I\cup *\times I$ to $*\times 0$. Denote by r the identification map: $K\times I\to CK$. Let (K, L) be a css pair, and π be an abelian group. A natural isomorphism.

$\tau^* : H^{q+1}(CK, CL\cup K ; \pi) \to H^q(K, L ; \pi)$ (the reduced q-th cohomology group) is defined as follows. Let (E, B, p) be a fibre complex in the sense of D. M. Kan [5] such that B is a $K(\pi, q+1)$ and E is acyclic. Then the fibre of p is a $K(\pi, q)$. An element $\alpha\in H^{q+1}(CK, CL\cup K ; \pi)$ is represented by a map $f: (CK, CL\cup K) \to (B, *)$. The homotopy $h: K\times I\to B$ defined by $h = f\circ r$ is lifted to $h' : (K\times I, L\times I\cup K\times 1) \to (E, *)$. Then τ^* is defined by

$$\tau^*\alpha = \text{the element of } H^q(K, L ; \pi) \text{ represented by } h' | K\times 0.$$

Let N be a css group. Let $WN = \bar{W}N\times_\eta N$ be the twisted cartesian product whose twisted function η is defined by

$$\eta[x_0, x_1, \cdots , x_{i-1}] = x_{i-1}$$

(IV, 5 of [1]). The map $p: WN\to \bar{W}N$ defined by $p(w, x) = w$ is a fibre map in the sense of Kan, and WN is acyclic. Then, if N is a $K(\pi, q)$, the isomorphism τ^* is defined by using $(WN, \bar{W}N, p)$.

§ 2. Cohomology operations associated to a css group

Let G be a reduced 0-connected css group. Denote by ${}^nG$ the maximal css normal subgroup of G such that $({}^nG)^{n-1}=\{e_k,\ k=0,\cdots,n-1\}$. Then we have a sequence of css normal subgroups of G:

$$G={}^1G\supseteqq{}^2G\supseteqq\cdots\supseteqq{}^nG\supseteqq\cdots.$$

We put

$$B_n^m={}^mG/{}^{n+1}G\qquad(m\leqq n+1),\quad B_\infty^m={}^mG.$$

Let

$$p_{n,m}^l:B_n^l\to B_m^l\ (l-1\leqq m\leqq n\leqq\infty),\quad j_n^{m,l}:B_n^m\to B_n^l\ (l\leqq m\leqq n+1\leqq\infty)$$

be the natural map and the injection respectively. The map $p_{n,m}^l$ is a fibre map whose fibre is B_n^{m+1}. Especially, the fibre of $p_{n,n-1}^m$ is B_n^n and B_n^n is a reduced $(n-1)$-connected $K(\pi_n(G),n)$. Let (K,L) be a css pair. The map $p_{n,m}^m$ $(m\leqq n\leqq\infty)$ induces a homomorphism

$$p_{n,m}^{m\,*}:\ \Pi(K,L;B_n^m)\to\Pi(K,L;B_m^m).$$

Let $U:\Pi(K,L;B_m^m)\to H^m(K,L;\pi_m(G))$ be the natural isomorphism. Then

$$p_{n,m}^{m\,\#}=U\circ p_{n,m}^{m\,*}:\Pi(K,L;B_n^m)\to H^m(K,L;\pi_m(G))$$

is a homomorphism. Denote by $\tau_m^n:B_{n-1}^m\to\bar{W}B_n^n$ $(m\leqq n<\infty)$ the map defined by a twisted fluction ξ of the principal fibre bundle $(B_n^m,B_{n-1}^m,p_{n,n-1}^m)$ (see § 1). Then τ_m^n induces a transformation $\tau_m^{n\,*}:\Pi(K,L;B_{n-1}^m)\to\Pi(K,L;\bar{W}B_n^n)$, and the transformation

$$\tau_m^{n\,\#}=T\circ\tau_m^{n\,*}:\Pi(K,L;B_{n-1}^m)\to H^{n+1}(K,L;\pi_n(G))$$

is a homomorphism by Lemma 1. Denote by $P_n^m=P_n^m(K,L)$ the image of $p_{n,m}^{m\,\#}$ and by $R_m^n=R_m^n(K,L)$ the image of $\tau_m^{n\,\#}$. Then we have a sequence of subgroups:

$$H^m(K,L;\pi_m(G))=P_m^m\supseteqq P_{m+1}^m\supseteqq\cdots\supseteqq P_\infty^m,$$
$$H^{n+1}(K,L;\pi_n(G))\supseteqq R_1^n\supseteqq R_2^n\supseteqq\cdots\supseteqq R_n^n=0.$$

The subgroups P_n^m and R_m^n are natural, i.e., if $f:(K,L)\to(K',L')$ is a map of css pairs, then

$$f^*(P_n^m(K',L'))\subseteqq P_n^m(K,L),\quad f^*(R_m^n(K',L'))\subseteqq R_m^n(K,L).$$

Theorem 1. *For $m<n$, there is a natural homomorpnism*

$$\theta_m^{n-m} = \theta_m^{n-m}(K, L) : P_{n-1}^m \to H^{n+1}(K, L\,;\, \pi_n(G))/R_{m+1}^n$$

such that the kernel of θ_m^{n-m} is P_n^m and the image of θ_m^{n-m} is R_m^n/R_{m+1}^n.

Proof. Consider the folloging commutative diagram:

$$\begin{array}{ccccc}
 & & \Pi(K, L\,;\, B_n^m) & \overset{p_{n,m}^{m\,\#}}{\searrow} & \\
 & & \downarrow p_{n,n-1}^{m\,*} & & \\
\Pi(K, L\,;\, B_{n-1}^{m+1}) & \xrightarrow{j_{n-1}^{m+1,m*}} & \Pi(K, L\,;\, B_{n-1}^m) & \xrightarrow[p_{n-1,m}^{m\,\#}]{} & H^m(K, L\,;\, \pi_m(G)) \\
 & \underset{\tau_{m+1}^{n\,\#}}{\searrow} & \downarrow \tau_m^{n\,\#} & & \\
 & & H^{n+1}(K, L\,;\, \pi_n(G)) & &
\end{array}$$

whose row and column are exact (see [3]). Define a homomorphism θ_m^{n-m} by

$$\theta_m^{n-m}\alpha = \tau_m^{n\,\#}\circ(p_{n-1,m}^{m\,\#})^{-1}\alpha \qquad \text{mod. } R_{m+1}^n\,, \quad \alpha\in P_{n-1}^m\,.$$

This is well defined by the exactness of the row. It is clear that the image of θ_m^{n-m} is R_m^n/R_{m+1}^n and $P_n^m\subseteq$kernel θ_m^{n-m}. If $\theta_m^{n-m}\alpha=0$ for $\alpha\in P_{n-1}^m$, there are elements $\beta\in\Pi(K, L\,;\, B_{n-1}^m)$ and $\gamma\in\Pi(K, L\,;\, B_{n-1}^{m+1})$ such that $P_{n-1,m}^{m\,\#}\beta=\alpha$, $\tau_m^{n\,\#}\circ j_{n-1}^{m+1,m*}\gamma=\tau_m^{n\,\#}\beta$. By the exactness of the column, there is an element $\delta\in\Pi(K, L\,;\, B_n^m)$ such that $\beta=(j_{n-1}^{m+1,m*}\gamma)(p_{n,n-1}^{m\,*}\delta)$. Then

$$\alpha = p_{n-1,m}^{m\,\#}\beta = p_{n-1,m}^{m\,\#}\circ p_{n,n-1}^{m\,*}\delta = p_{n,m}^{m\,\#}\delta\,.$$

This shows that kernel $\theta_m^{n-m}\subseteq P_n^m$. The naturality of θ_m^{n-m} is clear.

Corollary 1. $P_{n-1}^m/P_n^m\approx R_m^n/R_{m+1}^n$.

The homomorphism θ_m^r $(m, r\geqq 1)$ defined in the proof in the above is a generalized cohomology operation associated to G. We say that R_m^{m+r} is the image of θ_m^r. If the dimension of (K, L) is s, i.e., $H^k(K, L\,;\, \pi)=0$ for each $k>s$ and for any abelian group π, then θ_m^r is trivial for $m+r\geqq s$ and $P_{s-1}^m=P_s^m=\cdots=P_\infty^m$. If G is t-connected $(t\geqq 1)$, θ_m^r is trival for $m\leqq t$ and $R_1^n=R_2^n=\cdots=R_{t+1}^n$ for $n>t$.

Let $\tau^* : H^{q+1}(CK, CL\cup K\,;\, *)\to H^q(K, L\,;\, *)$ be the natural isomorphism defined in §1. We put

$$'P_n^m = \tau^* P_n^m(CK, CL\cup K)\subseteq H^{m-1}(K, L\,;\, \pi_m(G))$$
$$'R_m^n = \tau^* R_m^n(CK, CL\cup K)\subseteq H^n(K, L\,;\, \pi_n(G))\,.$$

We define the suspension

$$'\theta_m^{n-m} : {}'P_{n-1}^m \to H^n(K, L\,;\pi_n(G))/{}'R_{m+1}^n$$

of θ_m^{n-m} by

$$'\theta_m^{n-m} = \tau^* \circ \theta_m^{n-m} \circ \tau^{*\,-1} \qquad \text{(see [2])}.$$

Corollary 2. $'P_{n-1}^m/{}'P_n^m \approx {}'R_m^n/{}'R_{m+1}^n$.

§ 3. Classification of maps of a complex into a css group

Let G be a reduced 0-connected css group and (K, L) be a css pair. Denoting by D_n^m $(m-1 \leqq n)$ the kernel of $P_{\infty,n}^{m}{}^* : \Pi(K, L\,; {}^mG) \to \Pi(K, L\,; B_n^m)$, we have a filtration

$$\Pi(K, L\,; {}^mG) = D_{m-1}^m \supseteqq D_m^m \supseteqq D_{m+1}^m \supseteqq \cdots$$

of $\Pi(K, L\,; {}^mG)$ by normal subgroups. For $m \leqq n \leqq l+1$, since D_l^n and D_l^m are the image of $j_\infty^{l+1,n*}$ and $j_\infty^{l+1,m*}$ respectively, $j_\infty^{n,m*}$ induces an epimorphism $\bar{j}_\infty^{n,m*} : D_l^n \to D_l^m$, and $\bar{j}_\infty^{n,m*}$ induces an epimorphism

$$\alpha_{l+1}^{n,m} : D_l^n/D_{l+1}^n \to D_l^m/D_{l+1}^m .$$

Since the kernel of $p_{\infty,n}^{n}{}^* : \Pi(K, L\,; {}^nG) \to \Pi(K, L\,; B_n^n)$ is D_n^n and the image of $p_{\infty,n}^{n}{}^{\#} = U \circ p_{\infty,n}^{n}{}^*$ is P_∞^n, $p_{\infty,n}^{n}{}^{\#}$ induces an isomorphism

$$\beta_n : D_{n-1}^n/D_n^n \approx P_\infty^n .$$

Then the homomorphism

$$\gamma^{n,m} = \alpha_n^{n,m} \circ \beta_n^{-1} : P_\infty^n \to D_{n-1}^m/D_n^m$$

is an epimorphism.

Lemma 2. *The kernel of* $\gamma^{n,m}$ *is* $'R_m^n$, *then*

$$D_{n-1}^m/D_n^m \approx P_\infty^n/{}'R_m^n .$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \Pi(K, L\,; {}^{n+1}G) & \to & \Pi(K, L\,; {}^mG) & \to & \Pi(K, L\,; B_n^m) \\
 & & \downarrow {\scriptstyle j_\infty^{n+1,n*}} & & \downarrow \approx & & \downarrow \\
\Pi(CK, CL \cup K\,; B_{n-1}^m) & \xrightarrow{\partial} & \Pi(K, L\,; {}^nG) & \to & \Pi(K, L\,; {}^mG) & \to & \Pi(K, L\,; B_{n-1}^m) \\
 & & \downarrow {\scriptstyle p_{\infty,n}^{n}{}^*} & & & & \\
 & & \Pi(K, L\,; B_n^n) & & & &
\end{array}$$

whose rows and column are exact. Here, ∂ is defined as follows. Let $f: (CL, CL\cup K)\to(B^m_{n-1}, e)$ represent $\alpha\in\Pi(CK, CL\cup K; B^m_{n-1})$. The homotopy $h=f\circ r: K\times I\to B^m_{n-1}$ is lifted to $h': (K\times I, L\times I\cup K\times 1)\to({}^mG, e)$. Then $\partial\alpha$ is represented by $h'|K\times 0: K\to{}^nG$. Now, by the diagram in the above, we see that the kernel of $\gamma^{n,m}$ is

$$\begin{aligned} &p^{n}_{\infty,n}{}^{\#}(\partial\Pi(CK, CL\cup K; B^m_{n-1})\cdot j^{n+1,n}_{\infty}{}^{*}\Pi(K, L; {}^{n+1}G)) \\ &\quad = p^{n}_{\infty,n}{}^{\#}(\partial\Pi(CK, CL\cup K; B^m_{n-1}))\,. \end{aligned}$$

Then, the proof is complete, if the following diagram is commutative:

$$\begin{array}{ccc} \Pi(CK, CL\cup K; B^m_{n-1}) & \xrightarrow[\tau^n_m{}^*]{} & \Pi(CK, CL\cup K; \bar{W}B^n_n) \\ \big\downarrow\partial & & \big\downarrow\tau^* \\ \Pi(K, L; {}^nG) & \xrightarrow{p^n_{\infty,n}{}^*} & \Pi(K, L; B^n_n)\,. \end{array}$$

Let α, f, h, h' be as above. Then $p^n_{\infty,n}\circ(h'|K\times 0)$ represents $p^n_{\infty,n}{}^*(\partial\alpha)$. Let $\tau^n_m: B^m_{n-1}\to\bar{W}B^n_n$ be defined by a twisted function ξ of $(B^m_n, B^m_{n-1}, p^m_{n,n-1})$ and ξ be defined by a function $\beta: B^m_{n-1}\to B^m_n$ (see §1). Then the map $l: B^m_n\to WB^n_n=\bar{W}B^n_n\times_\eta B^n_n$ defined by

$$l(b) = (\tau^n_m\circ p^m_{n,n-1}b, (\beta\circ p^m_{n,n-1}b)^{-1}\cdot b)$$

is a fibre preserving map, i.e., $\tau^n_m\circ p^m_{n,n-1}=p\circ l$. Since

$$\begin{aligned} p\circ l\circ p^m_{\infty,n}\circ h' &= \tau^n_m\circ p^m_{n,n-1}\circ p^m_{\infty,n}\circ h' \\ &= \tau^n_m\circ p^m_{\infty,n-1}\circ h' = \tau^n_m\circ h\,, \end{aligned}$$

$l\circ p^m_{\infty,n}\circ(h'|K\times 0)=l\circ p^n_{\infty,n}\circ(h'|K\times 0)$ represents $\tau^*\circ\tau^n_m{}^*\alpha$. Then

$$p^n_{\infty,n}{}^*(\partial\alpha) = \tau^*\circ\tau^n_m{}^*\alpha\,.$$

This completes the proof.

By Lemma 2 together with the definitions in §2, we have the following theorem.

Theorem 2. *Let G be a reduced 0-connected css group and (K, L) be a css pair. Then, there is a filtration*

$$\Pi(K, L; G) = D^1_0\supseteq D^1_1\supseteq D^1_2\supseteq\cdots$$

by normal subgroups such that

$$D^1_{n-1}/D^1_n \approx P^n_\infty/'R^n_1 \qquad (n \geqq 1).$$

If (K, L) *is of finite dimension,* $P^n_\infty \subseteq H^n(K, L; \pi_n(G))$ *is the intersection of the kernels of the cohomology operations* θ^l_n, $l=1, 2, \cdots$, *associated to* G. *If* G *is* $(m-1)$*-connected* $(m \geqq 1)$, *then* $D^1_0 = \cdots = D^1_{m-1}$, $'R^m_1 = 0$, *and* $'R^n_1 = \cdots = 'R^n_m$ $(n > m)$ *is the image of the suspension* $'\theta^{n-m}_m$ *of the cohomology operation* θ^{n-m}_m *associated to* G.

§ 4. Application

Let G be a reduced 0-connected css group and (K, L) be a css pair of finite dimension. We assume that $\Pi(K, L; G)$ is abelian. Let $\boldsymbol{C}$ be a class of abelian groups in the sense of J. P. Serre [7].

Proposition 1. *If* $H^r(K, L; \pi_r(G)) \in \boldsymbol{C}$ *for* $r \geqq 1$, *then* $\Pi(K, L; G) \in \boldsymbol{C}$. *Especially, if* $\pi_r(G)$ *and* $H^r(K, L)$ *are finitely generated for* $r \geqq 1$, $\Pi(K, L; G)$ *is finitely generated.*

Proof. The first part follows from Theorem 2. The second part follows from Theorem 2.2 in Appendix of [6].

Let $p_n: G \to G/^{n+1}G$ be the natural map and $j_n: {}^nG \to G$ be the injection. The maps p_n and j_n induce homomorphisms $p_n^*: \Pi(K, L; G) \to \Pi(K, L; G/^{n+1}G)$ and $j_n^*: \Pi(K, L; {}^nG) \to \Pi(K, L; G)$ respectively.

Proposition 2. (i) *If* $H^r(K, L; \pi_r(G)) \in \boldsymbol{C}$ *for* $r < n$, j_n^* *is a* $\boldsymbol{C}$*-epimorphism.*

(ii) *If* $H^{r-1}(K, L; \pi_r(G)) \in \boldsymbol{C}$ *for* $r < n$, j_n^* *is a* $\boldsymbol{C}$*-monomorphism.*

(iii) *If* $H^r(K, L; \pi_r(G)) \in \boldsymbol{C}$ *for* $r > n$, p_n^* *is a* $\boldsymbol{C}$*-monomorphism.*

(iv) *If* $H^{r+1}(K, L; \pi_r(G)) \in \boldsymbol{C}$ *for* $r > n$ *and* $\Pi(K, L; G/^{n+1}G)$ *is abelian,* p_n^* *is a* $\boldsymbol{C}$*-epimorphism.*

Proof. (i) Since the squence $\Pi(K, L; {}^nG) \xrightarrow{j_n^*} \Pi(K, L; G) \xrightarrow{p_{n-1}^*} \Pi(K, L; G/^nG)$ is exact, the image of j_n^* is D^1_{n-1}. Then the proposition follows from Theorem 2.

(ii) Since the sequence $\Pi(CK, CL \cup K; G/^nG) \xrightarrow{\partial} \Pi(K, L; {}^nG) \xrightarrow{j_n^*} \Pi(K, L; G)$ is exact, the kernel of j_n^* is $\partial\Pi(CK, CL \cup K; G/^nG)$ and $\Pi(CK, CL \cup K; G/^nG) \in \boldsymbol{C}$ by Theorem 2.

(iii) Since the kernel of p_n^* is D_n^1, the proposition follows from Theorem 2.

(iv) Denoting by D'_m the kernel of $p_{n,m}^{1*}: \Pi(K, L; G/^{n+1}G) \to \Pi(K, L; G/^{m+1}G)$ $(0 \leqq m \leqq n)$, we have a filtration

$$(1) \qquad \Pi(K, L; G/^{n+1}G) = D'_0 \supseteqq D'_1 \supseteqq \cdots \supseteqq D'_n = 0$$

such that $p_n^* D_r^1 \subseteqq D'_r$ and $P_n^r/'R_1^r \approx D'_{r-1}/D'_r$ by Theorem 2. From the definition, the diagram

$$\begin{array}{ccc} P_\infty^r/'R_1^r & \xrightarrow[\approx]{\gamma} & D_{r-1}^1/D_r^1 \\ \downarrow \eta & & \downarrow p_n^* \\ P_n^r/'R_1^r & \xrightarrow[\approx]{\gamma'} & D'_{r-1}/D'_r \end{array}$$

is commutative. Here, γ and γ' are isomorphisms induced by $\gamma^{r,1}$ defined in §3 and η is the injection. Let $S = p_n^* \Pi(K, L; G)$. Then

$$(2) \qquad D'_r \cap S = p_n^* D_r^1,$$

$$(3) \qquad D'_{r-1}/D'_r + p_n^* D_{r-1}^1 \approx P_n^r/P_\infty^r,$$

$$(4) \qquad P_n^r/P_\infty^r \in \boldsymbol{C} \qquad \text{(by Corollary 1)},$$

From the filtration (1), we have a filtration

$$\Pi(K, L; G/^{n+1}G)/S = D'_0 + S/S \supseteqq D'_1 + S/S \supseteqq \cdots \supseteqq D'_n + S/S = 0.$$

Since $D'_{r-1} + S/D'_r + S \approx D'_{r-1}/D'_r + D'_{r-1} \cap S$, the proof is complete by (2), (3) and (4).

Let $\pi_r(G)$ be finitely generated for $r \geqq 1$. We assume that $\pi_r(G) = 0$ for $1 \leqq r < m$, $m < r < n$ and $H^r(K, L) = 0$ for $r > n$. Then, in the exact sequence

$$\Pi(CK, CL \cup K; G) \xrightarrow{'p_m^*} \Pi(CK, CL \cup K; G/^{m+1}G) \xrightarrow{\partial} \Pi(K, L; {}^{m+1}G)$$
$$\xrightarrow{j_{m+1}^*} \Pi(K, L; G) \xrightarrow{p_m^*} \Pi(K, L; G/^{m+1}G) \quad \text{(see [3])},$$

p_m^* is onto and $U \circ (j_{m-1}^{m,1*})^{-1}: \Pi(K, L; G/^{m+1}G) \longrightarrow H^m(K, L; \pi_m(G))$ is an isomorphism by Proposition 2, and in the commutative diagram

$$
\begin{array}{ccccc}
 & & \Pi(CK, CL\cup K; G/{}^{m+1}G) & \xrightarrow{\partial} & \Pi(K, L; {}^{m+1}G) \\
 & & \approx \uparrow j^{m,1}_{m+1}{}^{*}\circ p^{m}_{n,m+1}{}^{*} & & \approx \uparrow j^{n,m+1}_{\infty}{}^{*} \\
H^{m}(CK, CL\cup K; \pi_m(G)) & \underset{\approx}{\xrightarrow{p_{n-1,m}{}^{*}}} & \Pi(CK, CL\cup K; {}^{m}G/{}^{n}G) & \xrightarrow{\partial} & \Pi(K, L; {}^{n}G) \\
\approx \downarrow \tau^{*} & & \downarrow \tau^{n\,\#}_{m} & & \approx \downarrow p^{n}_{\infty,n}{}^{\#} \\
H^{m-1}(K, L; \pi_m(G)) & & H^{n+1}(CK, CL\cup K; \pi_n(G)) & \underset{\approx}{\xrightarrow{\tau^{*}}} & H^{n}(K, L; \pi_n(G)),
\end{array}
$$

the homomorphisms denoted by $\approx$ are isomorphisms by Proposition 2 or by definition and the map

$$\tau^{*}\circ\tau^{n\,\#}_{m}\circ(p^{m}_{n-1,m}{}^{\#})^{-1}\circ\tau^{*-1}: H^{m-1}(K, L; \pi_m(G)) \longrightarrow H^{n}(K, L; \pi_n(G))$$

is the cohomology operation $'\theta^{n-m}_{m}$ by definition (see § 2). By putting

$$'p^{*} = \tau^{*}\circ p^{m}_{n-1,m}{}^{*}\circ(p^{m}_{n,m+1}{}^{*})^{-1}\circ(j^{m,1}_{m+1}{}^{*})^{-1}\circ{}'p^{*}_{m}$$

$$j^{*} = j^{*}_{m+1}\circ j^{n,m+1}_{\infty}{}^{*}\circ(p^{n}_{\infty,n}{}^{*})^{-1}, \quad p^{*} = U\circ(j^{m,1}_{m+1}{}^{*})^{-1}\circ p^{*}_{m},$$

we have the following exact sequence:

$$\Pi(CK, CL\cup K; G) \xrightarrow{'p^{*}} H^{m-1}(K, L; \pi_m(G)) \xrightarrow{'\theta^{n-m}_{m}} H^{n}(K, L; \pi_n(G))$$
$$\xrightarrow{j^{*}} \Pi(K, L; G) \xrightarrow{p^{*}} H^{m}(K, L; \pi_m(G)) \longrightarrow 0.$$

(cf. Theorem 3.8 of [6]).

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