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Imbedding of an abstract variety in a complete variety

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The purpose of the present paper is to prove that an arbitrary abstract variety can be imbedded in a complete variety as an open set.

As for the terminology, we shall employ the one in the sequence of papers of ours in the American Journal of Mathematics ([2] (I, II, III)). We note that we need not assume that a ground ring is a Dedekind domain. Namely, our proof is valid without any modification in the case of models over a Noetherian integral domain, models being adapted to the case. Therefore the ground ring can be replaced also by a so-called Noetherian scheme, provided that every localities are integral domains.

In §1, we state some of known theorems on birational correspondences. In §2, we discuss a special kind of birational transformation, called dilatation. In §3, we give some auxiliary results and in §4 we give the proof of our main theorem.

The writer likes to add here that there has been one contribution by J. Ohm [4] to this problem saying that if V is an abstract variety, C is a curve on V and if there is a quasi-projective open covering $\{U_i\}$ of V such that C meets all the U_i, then there is an abstract variety V' containing V as an open subset in such a way that the closure of C in V' is a complete variety.

1. Birational correspondences.

We consider from now on only models whose function fields

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are contained in a field, hence *correspondences* between models are well defined as follows.

Let M and M' be models. When M dominates M', then the map ϕ such that $P \geq \phi(P) \in M'$ for every $P \in M$ is a well defined map. This ϕ is called the projection or geometric projection or morphism, from M into M'. ($\phi(M)$ is not necessarily an open set but contains a non-empty open set of M'.) The projection ϕ is denoted by $\mathrm{proj}_{M \to M'}$ or $\mathrm{proj}_{M'}$ or proj. Now, in the general case, $M''=J(M, M')$ dominates both M and M'. The correspondence T between M and M' is defined to be $(\mathrm{proj}_{M'' \to M'}) \cdot (\mathrm{proj}_{M'' \to M})^{-1}$. This T is denoted by $T_{M \to M'}$. $T_{M \to M'}$ gives in general a many to many correspondence of spots in M and M'. Even if F is a closed set of M, $T_{M \to M'}(F)$ is not necessarily a closed set of M'; what we know in general is that if C is a constructive set of M, i.e., if C is the union of a finite number of subsets of M of the form (closed set$-$closed set), then $T_{M \to M'}(C)$ is also a constructive set of M' (cf. Chevalley [1] or Nagata-Nakai [3]).

We say that a model M of a function field L is *complete with respect to* a spot P, if, for a given function field K containing L and P, the following is true:

Every place of K dominating P has a center on M.

This property is obviously independent of the choice of K.

We say that a model M is *complete with respect to* a set M' of spots if M is complete with respect to every spots of M'.

We say that a model M is complete over a model M' if M dominates M' and if M is complete with respect to M'. (Note that if a model M dominates a model M', then M' is complete with respect to M.)

Let M be a model of a function field L. The set of places of L which have centers in M is called the *Zariski-Riemann space*[2)]

2) The name of Riemann is added because Zariski [5] called this space "Riemann manifold" in the case of a projective variety, though this is not a Riemann manifold in the usual sense in differential geometry. The writer believes that the motivation of Zariski for the terminology came from the case of a curve. Any way, the notion has nearly nothing to do with Riemann, hence the name "Zariski space" is seemingly preferable. But, unfortunately, the term "Zariski space" has been usen in a different meaning. Therefore we are proposing name "Zariski-Riemann space".

of M and is denoted by $ZR(M)$. We introduce a topology, which may be called Zariski topology, on $ZR(M)$ defining the family of subsets F of the following property to be a base of closed sets:

There exists a model M' of L which is complete over M such that F is the set of places which have centers in a certain closed set of M'.

Then we can prove easily that

Proposition 1.1. *The Zariski-Riemann space $ZR(M)$ is compact.*

The proof is just an adaption of that was given by Zariski [5] (cf. Zariski-Samuel [6]).

The following fact is easily seen.

Proposition 1.2. *If a model M is complete with respect to a model M' and if F is a closed set of M, then $T_{M\to M'}(F)$ is a closed set of M'.*

In closing this section, we add one more definition.

We say that a model M is *quasi-dominant* over another model M' if the following is true:

Whenever a spot P in M corresponds to a spot P' in M', P dominates P'. In other word, $J(M, M')$ is a subset of M.

2. Dilatation by an ideal.

When a non-zero ideal[3] $\mathfrak{a}=\{\mathfrak{a}(P)\,|\,P\in M\}$ of a model M of a function field L is given, let $(f_{P1},\cdots,f_{PN})$ be a basis for the P-component $\mathfrak{a}(P)$ and let M_P be the projective model defined by homogeneous coordinate $(f_{P1},\cdots,f_{PN})$. Then

Proposition 2.1. *$M^*=\bigcup_{P\in M}J(P, M_P)$ is a model of L which is complete over M. M^* is independent of the choice of the basis $f_{P1},\cdots,f_{PN}$ for each P.*

This M^* is called the *dialatation* of M defined by the ideal $\mathfrak{a}$.

The proof is straightforward and we omit it; cf. the case

3) An ideal of a model is a coherent sheaf of ideals in sheaf-theoretical sense.

called monoidal transformation.

We note that the above definition can be adapted to the case of fractional ideals. We note also that

Proposition 2.2. *If M is a projective model and if $\mathfrak{a}$ is defined by a homogeneous ideal $\mathfrak{a}^*$ of a homogeneous coordinate ring of M, then the dilatation M^* defined by $\mathfrak{a}$ is again projective.*

In fact, we can choose M_P independently of P (taking a basis for the module of homogeneous elements of $\mathfrak{a}^*$ of a sufficiently high degree).

We say that an ideal $\mathfrak{a}$ of a model M is *primary* (or *prime*) if the closed set $F=\{P \mid \mathfrak{a}(P) \neq P\}$ defined by $\mathfrak{a}$ is irreducible and if $\mathfrak{a}(P)$ is primary (or prime, respectively) for every P in F.

We say that a model M is *quasi-projective* if M is an open set of a projective model.

By these definitions, we have

Corollary 2.3. *If $\mathfrak{a}$ is a primary ideal of a model M and if M' is the dilatation of M defined by $\mathfrak{a}$, then for every quasi-projective open subset M^* of M, $T_{M \to M'}(M^*)$ is quasi-projective.*

We note also that if $\mathfrak{a}_P$ is a primary ideal of a spot P of a model M, then there is a uniquely determined primary ideal $\mathfrak{a}$ of M whose P-component is $\mathfrak{a}_P$. In this case, the dilatation defined by $\mathfrak{a}$ is called the *dilatation defined by* $\mathfrak{a}_P$.

In closing this section, we observe a kind of dilatation which separates two closed sets in rough speaking.

If F is a closed set of a model M, then there is an ideal $\mathfrak{a}$ of M such that F is the closed set defined by $\mathfrak{a}$. ($\mathfrak{a}$ is unique with an additional condition that $\mathfrak{a}(P)$ is semi-prime for every $P \in F$; in this case $\mathfrak{a}$ is called the *semi-prime ideal* for F.)

Now, assume that F and F' are closed subsets of a model M and that they have no common component. Let $\mathfrak{a}(F)$ and $\mathfrak{a}(F')$ be ideals of M which define F and F' respectively. Set $\mathfrak{a}=\mathfrak{a}(F)+\mathfrak{a}(F')$. Then

Proposition 2.4. *In the dilatation M^* of M defined by $\mathfrak{a}$, so-called proper transforms of F and F' have no common spots.*

Namely, if P and P' are generating spots of irreducible components of F and F' respectively, then the loci $M^*(P)$ and $M^*(P')$ do not meet each other.

The proof is easy.

3. Auxiliary results.

Lemma 3.1. *Let M and M' be models of the same function field L and let v be an arbitrary place in $ZR(J(M, M'))$. Then there is a model $M^*=M^*_v$ (depending on v) of L such that* (1) *M^* is complete over M,* (2) $M\frown M'\leq M^*$, (3) *for every quasi-projective open subset U of M, $T_{M\to M^*}(U)$ is quasi-projective and* (4) *if P^* and P' are the centers of v on M^* and M' respectively, then P^* dominates P'.*

Proof. We shall prove the assertion by induction on the rank of v. Let P be the center of v on M. If P dominates P', then $M^*=M$ is the required model. Therefore we assume that P does not dominate P'. Hence, in particular, $P\notin M\frown M'$. Let R_v be the valuation ring of v and let $\mathfrak{p}$ be the prime ideal of R_v which is next to the maximal ideal. We may assume that the center Q of $(R_v)_{\mathfrak{p}}$ in M dominates the center Q' of $(R_v)_{\mathfrak{p}}$ in M', by virtue of our induction assumption. Let A' be an affine open set of M' which contains P' and let $x'_1, \cdots, x'_n$ be a set of generators of the affine ring of A'. Then $x'_i\in Q'\leq Q$ for every i, whence there is an element f of P which is not in $\mathfrak{p}\frown P$ such that $fx'_1, \cdots, fx'_n$ are in P. Now consider the ideal $\mathfrak{a}_P=fR_v\frown P$ of P. This is a primary ideal belonging to the maximal ideal, because $\mathfrak{p}$ in next to the maximal. Let M^* be the dilatation of M defined by $\mathfrak{a}_P$, then M^* is obviously the required model.

Theorem 3.2. *Let M and M' be models of the same function field L. Then there is a model M^* of L such that* (1) *M^* is complete over M,* (2) $M\frown M'\leq M^*$, (3) *for every quasi-projective open subset U of M, $T_{M\to M^*}(U)$ is quasi-projective and* (4) *M^* is quasi-dominant over M'.*

Proof. For each $v \in ZR(J(M, M'))$, we take M_v^* given by Lemma 3.1. Then, by the compactness of $ZR(J(M, M'))$ (Proposition 1.1), we see that there are a finite number of M_v^*, say $M_{v_1}^*, \cdots, M_{v_n}^*$, such that for each $v \in ZR(J(M, M'))$ there is one i such that the center of v in $M_{v_i}^*$ dominates the center of v in M'. Then the join of all $M_{v_1}^*, \cdots, M_{v_n}^*$ is obviously the required model.

Theorem 3.3. *Let M be a model of a function field L and let M' be the projective model defined by homogeneous coordinates $(x_0, \cdots, x_n)$. For each $P \in M$, let $\mathfrak{a}(P)$ be the ideal of P generated by all elements of the form $ax_0, \cdots, ax_n$ such that $ax_0, \cdots, ax_n$ are simultaneously in P. Then the dilatation M^* of M defined by the ideal $\{\mathfrak{a}(P)\}$ of M dominates M'. A spot $P \in M$ does not dominate any spot in M' if and only if P is in the closed set F defined by the ideal $\{\mathfrak{a}(P)\}$.*

Proof. $P \in M$ dominates a spot in M' if and only if there is one i such that $x_0x_i^{-1}, \cdots, x_nx_i^{-1}$ are in P, which is equivalent to that $1 \in \mathfrak{a}(P)$. This proves the last assertion. Let P^* be an arbitrary spot in M^* and let P be the spot of M dominated by P^*. Let $\mathfrak{a}(P^*)$ be such as $\mathfrak{a}(P)$ applied to the spot P^* and to the projective model M'. Let $y_1, \cdots, y_m$ be such that $\mathfrak{a}(P) = \sum_{i=1}^m y_iP$ and $y_iy_1^{-1} \in P^*$ for every i. If $y \in \mathfrak{a}(P)$, then there are $a_1, \cdots, a_r$ such that (i) all a_ix_j are in P and (ii) $y = \sum a_ix_iz_{ij}$ $(z_{ij} \in P)$. All a_ix_j are in $\mathfrak{a}(P)$, whence all $a_ix_jy_1^{-1}$ are in $\mathfrak{a}(P^*)$, which implies that yy_1^{-1} is in $\mathfrak{a}(P^*)$. In particular, $1 = y_1y_1^{-1} \in \mathfrak{a}(P^*)$, which shows that P^* dominates a spot in M'. This completes the proof.

4. Proof of the main theorem.

Lemma 4.1. *Let M be a model of a function field L and let v be a place of L. Then there exists a model M' which contains M as an open set and such that v has a center on M'.*

Proof. We shall prove the assertion by induction on the rank of v. If v has a center on M, then we may set $M' = M$. Therefore we assume that v has no center on M. Let R_v be the valuation ring of v and let $\mathfrak{p}$ be the prime ideal of R_v which is next to the maximal ideal. By virtue of the induction, we may assume that the place w defined by $(R_v)_\mathfrak{p}$ has a center Q on M. Let M^* be a projective model carrying Q and let P^* be the center of v on M^*. By virtue of Theorem 3.2, we may assume that M^* is quasi-dominant over M. Let N^* be the set of spots in M^* which dominates properly some spots in M, i.e., $N^*=J(M, M^*)-(M\cap M^*)$. If P^* is not in the closure $\bar{N}^*$ of N^*, then we may set $M'=M\cup(M^*-\bar{N}^*)$. So, we consider the case where $P^*\in\bar{N}^*$. Since $Q\notin\bar{N}^*$ and since $\mathfrak{p}$ is next to the maximal, we see that there is an element f of P^* such that (i) $f\notin\mathfrak{p}$ and (ii) f is in the ideal which defines $\bar{N}^*$ locally at P^*. If we replace M^* by the dilatation of M^* defined by the primary ideal $fR_v\cap P^*$, then we have the situation $P^*\notin\bar{N}^*$ (note that since $fR_v\cap P^*$ is primary to the maximal ideal, new N^* coincides with the previous N^*). Thus our lemma is proved.

Lemma 4.2. Let M_1 and M_2 be models of the same function field L. Set $M=M_1\cap M_2$. If M_1-M is contained in a projective model M^*, then there is a model M_3 which contains M such that $ZR(M_3)=ZR(M_1)\cup ZR(M_2)$.

Proof. To begin with, we may assume that M_2 is quasi-dominant over M_1 by virtue of Theorem 3.2. Set $F=M-(M^*\cap M)$ and $F^*=M^*-(M^*\cap M_1)$. F^* is a closed set, $M_1=M\cup(M^*-F^*)$ and $F=T_{M^*\to M}(F^*)$. Let F_2 be the closure of F in M_2. Let H be the set of spots in M_2 which do not correspond to any spot in M_1. Set $G=T_{M^*\to M_2}(M_1-M)$.

We want to show that $H\cup F=T_{M^*\to M_2}(F^*)$.

Obviously, $F\subseteq T_{M^*\to M_2}(F^*)$. If $P\in H$, then, since $M^*-F^*\subseteq M_1$, we see that $P\in T_{M^*\to M_2}(F^*)$. Conversely, assume that $P\in M_2-(H\cup F)$ and let P_1 and P^* be corresponding spots to P in M_1 and M^* respectively (P_1 exists because $P\notin H$). By assumption, P_1 is dominated by P. Therefore we see that P_1 corresponds to P^*. Assume for a moment that P^* is in F^*. If P_1 is in M, then $P_1\in F$, whence $P=P_1\in F$, which is a contradiction. Thus $P_1\notin M$, whence $P_1\in M^*-F^*$. Since the spots P_1 and P^* are in

M^* and since they correspond to each other, we see that $P_1 = P^*$, whence $F^* \ni P^* = P_1 \notin F^*$, which is a contradiction. Therefore P^* cannot be in F^*, and we have proved the equality $F \cup H = T_{M^* \to M_2}(F^*)$.

Since F^* is closed and since M^* is complete, we see that $F \cup H$ is closed. Thus we have

$$(*) \qquad F_2 - F \subseteq H .$$

This $(*)$ being shown, we shall change the situation. Blowing up M^* and M_1 simultaneously, we can have the situation that M_1 is quasi-dominant over M_2 by Theorem 3.2. H and F are not affected obviously. By enough blowing up within the closure of $T_{M_2 \to M_1}(G)$, G can be maintained, hence the property that $M = M_1 \cap M_2$ is maintained. By the invariance of F and H,$(*)$ is obviously valid. We may assume also that M^* is quasi-dominant over M_2.

If $\mathfrak{a}$ is an ideal of M_2 whose closed set is contained in H, then we may replace M_2 by its dilatation defined by $\mathfrak{a}$. (One should apply Theorem 3.2 to M^* in order to preserve quasi-domination.) Therefore by Proposition 2.4, we may assume that F_2 does not meet the closure $\bar{G}$ of G.

Let $\{\mathfrak{a}(P)\}$ be the ideal of M_2 as is given by Theorem 3.3 with respect to the projective model M^*. Then there is an ideal $\mathfrak{b} = \{\mathfrak{b}(P)\}$ of M_2 such that (1) the closed set defined by $\mathfrak{b}$ is contained in H and (2) $\mathfrak{b}(P) = \mathfrak{a}(P)$ if $P \in H - (H \cap F_2) - (H \cap \bar{G})$. We blow up M_2 by $\mathfrak{b}$ and we get the dilatation M_2^*. Theorem 3.3 shows that $M_2^* \cap J(M_2^*, M^*)$ contains $M_2^* - T_{M_2 \to M_2^*}(F_2 \cup \bar{G})$. Let us denote by F_2^* and G^* the sets $T_{M_2 \to M_2^*}(F_2)$ and $T_{M_2 \to M_2^*}(\bar{G})$ respectively. Now we want to claim that

$$M_3 = M_1 \cup J(M_2^* - F_2^*, M^*) \cup (M_2^* - G^*)$$

is the required model.

In order to prove this, it is sufficient to show that every v in $ZR(M_1) \cup ZR(M_2)$ has one and only one center on M_3. Let P_1, P^*, P_3 and P_2 be the centers of v on M_1, $J(M_2^* - F_2^*, M^*)$, $M_2^* - G^*$ and M_2 respectively if exist.

(1) When P_1 exists: P_1 is either in M or in M_1-M. If $P_1\in M$, then we have $P_1\in M_2^*-G^*$ and therefore $P_1=P_3$. Since $P_1\notin F$, we see similarly that $P_1=P^*$ if P^* exists. Assume now that $P_1\notin M$. If P^* exists, then $P_1=P^*$ because of the facts that M^* is quasi-dominant over M_2 and that P_2 exists, hence $P_2=P^*$. P_3 does not exist in this case by the assumption that $M=M_1\cap M_2$ (whence $M_2-G-H=M$).

(2) Now we assume that P_1 does not exist. Then P_2 exists and is in $G\cup H$. Since $F_2^*\cap G^*$ is empty, at least one of P^* and P_3 exists. Thus it is sufficient to show that $P^*=P_3$ if both P^* and P_3 exist. Really, in this case, $P_2\in M_2-(F_2\cup\bar{G})$, whence P_3 dominates a spot in M^*, which shows that $P_3\in J(M_2^*, M^*)$. Therefore $P_3=P^*$, and the proof is completed.

Now we shall prove the main theorem.

Theorem 4.3. For any given model M, there is a complete model which contains M as an open subset.

Proof. Let M^* be a projective model of the function field L of M and consider $ZR(M^*)$. For each $v\in ZR(M^*)$, there is a model M_v which contains M and such that v has a center P_v on M_v. We choose such an M_v so that M_v-M is contained in an affine model. (In fact, take an affine model A such that $P_v\in A\subseteq M_v$ and replace M_v by $M\cup A$.) Then by the compactness of $ZR(M^*)$, we see that there are a finite number of models $M_1, \cdots, M_n$ such that (1) $M\subseteq M_i$ for every i, (2) M_i-M is contained in a projective model for every i which is less than n and (3) $\bigcup_i ZR(M_i)=ZR(M^*)$. We prove the theorem by induction on the number n. If $n=1$, then we have nothing to prove any more. Assume that $n>1$. We apply Lemma 4.2 to M_{n-1} and M_n and we see that there is a model M_{n-1}^* such that $M_{n-1}^*\supseteq M_{n-1}\cap M_n\supseteq M$ and $ZR(M_{n-1}^*)=ZR(M_{n-1})\cup ZR(M_n)$. Therefore we complete the proof by our induction assumption.

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REFERENCES

[1] C. Chevalley, Sur la théorie des variétés algébriques, Nagoya Math. J. 8 (1955), pp. 1-43.

[2] M. Nagata, A general theory of algebraic geometry over Dekekind domains;
I, Amer. J. Math. **78** (1956), pp. 78-116,
II, ibid. **80** (1958), pp. 382-420,
III, ibid. **81** (1959), pp. 401-435.

[3] M. Nagata and Y. Nakai, Algebraic geometry, Kyoritsu, Tokyo (1957). (In Japanese)

[4] J. Ohm, Semi-projective completions of abstract curves, Ill. J. Math. 5 (1961), pp. 345-350.

[5] O. Zariski, The compactness of the Riemann manifold of an abstract field of algebraic functions, Bull. Amer. Math. Soc. **50** (1944), pp. 683-691.

[6] O. Zariski and P. Samuel, Commutative algebra, II, van Nostrand, New York (1961).