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A condition for the extension of a complex line bundle for a family of Kähler surfaces

By

Shigeo NAKANO

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It is well known that the surfaces of degree 3 in a projective 3-space contain straight lines, while some of the surfaces of degree 4 do, and some do not, contain straight lines. In view of this fact, we are led to the following question: Let a differentiable family $\mathcal{V}\rightarrow M$ of compact complex analytic manifolds and an analytic submanifold W of V_0 be given. (V_t denotes the member of $\mathcal{V}$ corresponding to $t\in M$.) Then under what condition does there exist a submanifold $\mathcal{W}$ of $\mathcal{V}$, which forms a family of complex manifolds $\{W_t | t\in M\}$, W_t being a submanifold of V_t and $W_0=W$?

In the case where W is of co-dimension 1 in V_0, the problem is divided into two parts: extension of the line bundle $[W]$ defined over V_0 to a family of bundles over $\mathcal{V}$, and the extension of the cross section of $[W]$ defining the divisor W to a family of cross sections.

As for the first part, Kodaira and Spencer gave a condition in [3], §13. We shall give here another condition, which may be called the differentiated form of theirs.

§1. $\barwedge$ operation of Fröhlicher and Nijenhuis

In [1] and [2], Fröhlicher and Nijenhuis defined a kind of multiplication between a scalar differential form and a vector differential form, and studied its properties.

Let X be a differentiable manifold and ω, L be scalar and vector differential forms of degrees q and l respectively, then $\omega\barwedge L$ is a scalar form of degree $(q+l-1)$ defined by

(1.1) $\omega\barwedge L(u_1, \cdots, q_{q+l-1})$
$$=\frac{1}{(q-1)!l!}\sum_{\sigma}\operatorname{sgn}(\sigma)\ \omega(L(u_{\sigma(1)}, \cdots, u_{\sigma(l)}), u_{\sigma(l+1)}, \cdots, u_{\sigma(q+l-1)}),$$

where $u_1, \cdots, u_{q+l-1}$ are variable tangent vectors to X at the point under consideration, and $\sum$ ranges over all permutations σ on suffixes $1, \cdots, q+l-1$.

In the case where X has a complex analytic structure, the tangent vector bundle is the Whitney sum of holomorphic and anti-holomorphic tangent bundles, and a differential form decomposes into a sum of terms of various types. We consider a scalar form ω of type (r, s) with $r\geqq 1$, and a holomorphic vector form L of type $(0, l)$[1]. Then $\omega\barwedge L$ is the scalar form of type $(r-1, s+l)$ given by

(1.2) $\omega\barwedge L(u_1, \cdots, u_{r-1}, \bar v_1, \cdots, \bar v_{s+l})$
$$=\frac{1}{(r-1)!(s+l)!}\sum_{\substack{\sigma\in\mathfrak{S}_{r-1}\\ \tau\in\mathfrak{S}_{s+l}}}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$
$$\times\omega(L(\bar v_{\tau(1)}, \cdots, \bar v_{\tau(l)}), u_{\sigma(1)}, \cdots, u_{\sigma(r-1)}, \bar v_{\tau(l+1)}, \cdots).$$

(The meaning of $\sum$ is clear.) For a holomorphic vector form M which has the expression $M=(M^\alpha)$, where

$$M^\alpha = \frac{1}{r!s!}\sum M^\alpha_{\beta_1\cdots\beta_r,\bar\gamma_1\cdots\bar\gamma_s}dz^{\beta_1}\wedge\cdots\wedge dz^{\beta_r}\wedge d\bar z^{\gamma_1}\wedge\cdots\wedge d\bar z^{\gamma_s}$$

with respect to local parameters $(z^1, \cdots, z^n)$ and to the basis $\left(\frac{\partial}{\partial z^\alpha}\right)$ of holomorphic tangent space and with $r\geqq 1$, we define

(1.3) $$S(M) = \frac{1}{(r-1)!s!}\sum_{\alpha} M^\alpha_{\alpha\beta_1\cdots\beta_{r-1},\bar\gamma_1\cdots\bar\gamma_s}dz^{\beta_1}\wedge\cdots\wedge dz^{\beta_{r-1}}\wedge d\bar z^{\gamma_1}\wedge\cdots\wedge d\bar z^{\gamma_s}.$$

$S(M)$ is a scalar form on X. In the case of (1.2), we easily verify

(1.4) $$\omega\barwedge L = (-1)^{q+l+1}S(\omega\wedge L) \qquad (q = r+s).$$

(The meaning of $\omega\wedge L$ is clear.) It is also easy to see

1) Here, a holomorphic vector form means a (C^∞-) differential form with values in the holomorphic tangent bundle.

$$\bar\partial S(M) = S(\bar\partial M) . \tag{1.5}$$

Hence we obtain

$$\bar\partial(\omega \barwedge L) = -\bar\partial\omega \barwedge L + (-1)^{q-1}\omega \barwedge \bar\partial L . \tag{1.6}$$

This shows that (holomorphic) vector valued Dolbeault cohomology group of type $(0, l)$ operaties on scalar Dolbeault group of X. In other words there is a natural bilinear mapping

$$H^s(X, \Omega^r) \times H^l(X, \Theta) \to H^{s+l}(X, \Omega^{r-1}) \qquad (r \geqq 1) ,$$

induced by $\barwedge$. (Here Ω^r denotes the sheaf of germs of holomorphic r-forms on X and Θ the sheaf of germs of holomorphic tangent vector fields on X.)

§ 2. Family of line bundles

Let $\mathcal{B} \to \mathcal{V} \overset{\varpi}{\to} M$ be a differentiable family of complex line bundles over a family $\mathcal{V}$ of compact Kähler manifolds parametrized by M. (For the definition, see Kodaira-Spencer [3]. Generally we follow these authors in terminology and notation.)
Since we concern ourselves with structures sufficiently near a particular one, we can assume that M is covered by a single coordinate neighborhood and $\mathcal{V}$ is covered by coordinate neighborhoods $\{\mathcal{U}_j\}$ such that $\varpi(\mathcal{U}_j) = M$. In $\mathcal{U}_j$ we have local coordinates $(z_j^1, \cdots, z_j^n, t^1, \cdots, t^m)$, where (t) is a system of coordinates on M and $(z_j^1, \cdots, z_j^n)$ form, for each fixed (t), a system of complex analytic local coordinates on V_t, the complex structure corresponding to $(t) \in M$.

We can also assume that $\mathcal{V}$ is diffeomorphic to $V_0 \times M$, where $V_0 = \varpi^{-1}(0)$. In terms of local coordinates, the diffeomorphism $\mathcal{V} \cong V_0 \times M$ can be expressed as

$$\begin{cases} z_j^\alpha = f_j^\alpha(\zeta_j, t) \\ t^\lambda = t^\lambda \end{cases} \qquad (\alpha = 1, \cdots, n ; \lambda = 1, \cdots, m) , \tag{2.1}$$

where (ζ_j) denotes a system of local coordinates on V_0, and f_j^α is C^∞ in ζ and t. We can assume $f_j^\alpha(\zeta_j, 0) = \zeta_j^\alpha$. The equations (2.1) can be solved as

$$(2.2) \qquad \begin{cases} \zeta_j^\alpha = g_j^\alpha(z_j, t) \\ t^\lambda = t^\lambda . \end{cases}$$

Replacing M by a smaller neighborhood of (0) if necessary, we can assume $\det(\partial f_j^\alpha/\partial\zeta_j^\beta) \neq 0$ and define $\varphi_{j\bar\beta}^\gamma$ by

$$\frac{\partial f_j^\alpha}{\partial\bar\zeta_j^\beta} = \sum_\gamma \frac{\partial f_j^\alpha}{\partial\zeta_j^\gamma} \varphi_{j\bar\beta}^\gamma(\zeta, t) .$$

Then $\varphi = \{\varphi_j\}$, with $\varphi_j = {}^t(\varphi_j^1, \cdots, \varphi_j^n)$ and $\varphi_j^\gamma = \sum \varphi_{j\bar\beta}^\gamma d\bar\zeta_j^\beta$, is a Θ_0-valued differential form of type (0, 1), which satisfies the relation

$$\bar\partial_0\varphi = [\varphi, \varphi]$$

and which characterizes the family $\mathcal{V}$ of complex structures. ($\bar\partial_t$ denotes the exterior differentiation with respect to anti-holomorphic coordinate in the structure V_t).

$$(2.3) \qquad \eta_\lambda = \left(\frac{\partial\varphi}{\partial t^\lambda}\right)_{t=0}$$

determines an element of $H^1(V_0, \Theta_0)$, which is nothing else than $\rho_0\left(\frac{\partial}{\partial t^\lambda}\right)$ (Kodaira-Spencer [4]).

Now we shall consider the Chern classes of the bundles B_t. As cohomology classes $\in H^2(X, \boldsymbol{Z})$ (X = the underlying differentiable manifold of V_t), these Chern classes are all the same. Hence its image in $H^2(X, \boldsymbol{C})$ is represented by a differential form

$$(2.4) \qquad \Phi(\zeta) = \sqrt{-1} \sum_{\alpha,\beta} \Phi_{\alpha\bar\beta}(\zeta) d\zeta^\alpha \wedge d\bar\zeta^\beta ,$$

which is real, closed and of type (1, 1) with respect to the structure V_0, and has periods which are integers.

Since we concern ourselves with a family of Kähler manifolds, we may assume that we have a family of Kähler metrics on $\{V_t\}$, depending differentiably on t. Denote by $\pi_t^{(r,s)}$ and H_t the operators of projection of differential forms to the part of type (r, s) and to the harmonic part in the Kähler structure of V_t. Then the condition that Φ represents the Chern class of B_t implies that

$$(2.5) \qquad H_t \pi_t^{(2,0)} \Phi = 0 \qquad \text{for} \quad t \in M .$$

Proposition 1. *Notations being as above, we have*

(2.6) $$H_s\{(H_s\pi_s^{(1,1)}\Phi)\bar{\wedge}\eta\} = 0$$

for any $s\in M$ *and for those* Θ_s*-valued* $\bar\partial_s$*-closed differential forms* η *which determine cohomology classes in* $\rho_s(T_M)$.

Proof. We consider the case $s=0$. We can take Φ to be harmonic and we have

$$\Phi = \sqrt{-1}\sum \Phi_{\alpha\bar\beta}(\zeta)d\zeta^\alpha\wedge d\bar\zeta^\beta = \sqrt{-1}\sum \Phi_{\alpha\bar\beta}(g(z,t))dg^\alpha\wedge d\bar g^\beta.$$

Hence

$$\pi_t^{(0,2)}\Phi = \sqrt{-1}\sum \Phi_{\alpha\bar\beta}(g(z,t))\frac{\partial g^\alpha}{\partial\bar z^\rho}\frac{\partial\bar g^\beta}{\partial\bar z^\sigma}d\bar z^\rho\wedge d\bar z^\sigma,$$

and

(2.7) $$H_t\left\{\sqrt{-1}\sum \Phi_{\alpha\bar\beta}(g(z,t))\frac{\partial g^\alpha}{\partial\bar z^\rho}\frac{\partial\bar g^\beta}{\partial\bar z^\sigma}d\bar z^\rho\wedge d\bar z^\sigma\right\} = 0.$$

Since $z^\alpha=f^\alpha(g(z,t),t)$, we have

$$0 = \sum\frac{\partial f^\alpha}{\partial\zeta^\gamma}\frac{\partial g^\gamma}{\partial\bar z^\beta}+\sum\frac{\partial f^\alpha}{\partial\bar\zeta^\gamma}\frac{\partial\bar g^\gamma}{\partial\bar z^\beta} = \sum\frac{\partial f^\alpha}{\partial\zeta^\gamma}\left(\frac{\partial g^\gamma}{\partial\bar z^\beta}+\sum\varphi^\gamma_{\bar\delta}\frac{\partial\bar g^\delta}{\partial\bar z^\beta}\right) = 0,$$

and

$$\frac{\partial g^\gamma}{\partial\bar z^\beta}+\sum\varphi^\gamma_{\bar\delta}\frac{\partial\bar g^\delta}{\partial\bar z^\beta} = 0.$$

Putting this into the expression (2.7) and taking the value of its derivative at $(t)=(0)$ with respect to t^λ, we obtain

$$H_0(\Phi\bar\wedge\eta_\lambda) = \left(\frac{\partial}{\partial t^\lambda}\pi_t^{(0,2)}\Phi\right)_{t=0} = 0,$$

where

$$\eta_\lambda = \left(\frac{\partial\varphi}{\partial t^\lambda}\right)_{t=0}.$$

This argument holds good for general value of s. We have only to observe that $H_s\pi_s^{(1,1)}\Phi$ must be the harmonic form representing the Chern class of B_s.

§3. Sufficiency

Suppose we have a differentiable family $\mathcal{V}\xrightarrow{\varpi}M$ of *compact*

Kähler surfaces, and suppose a complex line bundle B_0 over $V_0 = \varpi^{-1}(0)$ is given. Let Φ be the harmonic form of type (1, 1), which represents (the image of) the Chern class of B_0. Our purpose is to prove

Proposition 2. *Under the situations of this paragraph, and making use of previous notation, let the condition* (2.6) *hold for* $s \in M$ *near enough to* 0, *and for* η *belonging to* $\rho_s(T_M)$, *then there exists an open neighborhood* U *of* 0 *on* M *such that* B_0 *can be extended to a family of line bundles* $\mathcal{B}$ *over* $\mathcal{V}|U$.

For the proof, we first note that Prop. 13.2 of Kodaira-Spencer [3] can be applied to our case, since $\dim H^2(V_t, \Omega_t)$ is constant because V_t are Kähler. Therefore, we have only to prove $H_s\pi_s^{(0,2)}\Phi = 0$ for s near enough to 0.

We take a differentiable family $\Psi^{(1)}(\ ,t), \cdots, \Psi^{(p)}(\ ,t)$ of bases of $H^0(V_t, \Omega_t^2)$. Such a family exists since $\dim H^0(V_t, \Omega_t^2)$ is independent of t. We set

$$(3.1) \qquad u_r(t) = \int_{V_t} \Psi^{(r)}(z, t) \wedge \Phi(g(z, t))$$

and try to prove $\quad u_r(t) = 0 \qquad (r = 1, \cdots, p)$.

For the purpose we consider $(\partial u_r/\partial t^\lambda)_{t=s}$. Since we fix λ throughout, we omit λ. We write $\xi_j^\alpha = f_j^\alpha(\zeta_i, s)$. Then (ξ_j^α) is a system of analytic local parameters on V_s. We have $\zeta_j^\beta = g_j^\beta(\xi_j, s)$. We put

$$(3.2) \qquad h_j^\alpha(\xi_j, t; s) = f_j^\alpha(g_j(\xi_j, s), t),$$

then (ξ) and $(z) = (h(z, t; s))$ are in the same relationship as (ζ) and (z) in §2. (Only V_s takes the place of V_0.)

Define $\psi^\gamma = \sum \psi^\gamma{}_{\bar\beta}(\xi, t; s) d\bar\xi^\beta$ by

$$(3.3) \qquad \frac{\partial h^\alpha}{\partial \bar\xi^\beta} = \sum \frac{\partial h^\alpha}{\partial \xi^\gamma} \psi^\gamma{}_\beta(\xi, t; s),$$

then $\psi = (\psi^\gamma)$ has the same meaning as φ in §2, with respect to s. Thus $\psi(s) = 0$ and $\left(\frac{\partial \psi}{\partial t}\right)_{t=s}$ is the $\bar\partial_s$-closed Θ_s-valued form which represents $\rho_s\left(\frac{\partial}{\partial t}\right)$. Omitting the suffix r for simplicity, we have

$$u(t)=\int_{V_s}\Psi_{12}(h(\xi,t;s),t)\sum_{\alpha,\beta}\left(\frac{\partial h^1}{\partial\xi^\alpha}d\xi^\alpha+\frac{\partial h^1}{\partial\bar\xi^\alpha}d\bar\xi^\alpha\right)\wedge\left(\frac{\partial h^2}{\partial\xi^\beta}d\xi^\beta+\frac{\partial h^2}{\partial\bar\xi^\beta}d\bar\xi^\beta\right)\wedge$$

$$\wedge\sum_{\lambda,\mu,\rho,\sigma}\Phi_{\lambda\bar\mu}(g(\xi,s))\left(\frac{\partial g^\lambda}{\partial\xi^\rho}d\xi^\rho+\frac{\partial g^\lambda}{\partial\bar\xi^\rho}d\bar\xi^\rho\right)\wedge\left(\frac{\partial\bar g^\mu}{\partial\xi^\sigma}d\xi^\sigma+\frac{\partial\bar g^\mu}{\partial\bar\xi^\sigma}d\bar\xi^\sigma\right)$$

$$=\int_{V_s}\sum\Psi_{12}\Phi_{\lambda\bar\mu}\begin{vmatrix}\dfrac{\partial h^1}{\partial\xi^1} & \dfrac{\partial h^1}{\partial\xi^2} & \dfrac{\partial h^1}{\partial\bar\xi^1} & \dfrac{\partial h^1}{\partial\bar\xi^2}\\ \dfrac{\partial h^2}{\partial\xi^1} & \dfrac{\partial h^2}{\partial\xi^2} & \dfrac{\partial h^2}{\partial\bar\xi^1} & \dfrac{\partial h^2}{\partial\bar\xi^2}\\ \dfrac{\partial g^\lambda}{\partial\xi^1} & \dfrac{\partial g^\lambda}{\partial\xi^2} & \dfrac{\partial g^\lambda}{\partial\bar\xi^1} & \dfrac{\partial g^\lambda}{\partial\bar\xi^2}\\ \dfrac{\partial\bar g^\mu}{\partial\xi^1} & \dfrac{\partial\bar g^\mu}{\partial\xi^2} & \dfrac{\partial\bar g^\mu}{\partial\bar\xi^1} & \dfrac{\partial\bar g^\mu}{\partial\bar\xi^2}\end{vmatrix} d\xi^1\wedge d\xi^2\wedge d\bar\xi^1\wedge d\bar\xi^2$$

$$=\int_{V_s}\sum\Psi_{12}\cdot\Phi_{\lambda\bar\mu}\det\left(\frac{\partial h^\alpha}{\partial\xi^\beta}\right)\left\{\left(\frac{\partial g^\lambda}{\partial\bar\xi^1}-\sum_\rho\frac{\partial g^\lambda}{\partial\xi^\rho}\psi^\rho_{\bar 1}\right)\left(\frac{\partial\bar g^\mu}{\partial\bar\xi^2}-\sum_\sigma\frac{\partial\bar g^\mu}{\partial\xi^\sigma}\psi^\sigma_{\bar 2}\right)\right.$$

$$\left.-\left(\frac{\partial g^\lambda}{\partial\bar\xi^2}-\sum\frac{\partial g^\lambda}{\partial\xi^\rho}\psi^\rho_{\bar 2}\right)\left(\frac{\partial\bar g^\mu}{\partial\bar\xi^1}-\sum_\sigma\frac{\partial\bar g^\mu}{\partial\xi^\sigma}\psi^\sigma_{\bar 1}\right)\right\}d\xi^1\wedge d\xi^2\wedge d\bar\xi^1\wedge d\bar\xi^2\,.$$

Hence we obtain

$$\text{(3.4)}\qquad \left(\frac{\partial u(t)}{\partial t}\right)_{t=s}=\int_{V_s}\left\{\frac{\partial\Psi_{12}(z,t)}{\partial t}+\sum\frac{\partial\Psi_{12}}{\partial z^\rho}\frac{\partial h^\rho}{\partial t}\right.$$

$$\left.+\Psi_{12}\frac{\partial}{\partial t}\left(\log\det\left(\frac{\partial h^\alpha}{\partial\xi^\beta}\right)\right)\right\}_{t=s}d\xi^1\wedge d\xi^2\wedge\pi_s^{(0,2)}\Phi$$

$$-\int_{V_s}\Phi\wedge(\pi_s^{(1,1)}\Phi\barwedge\eta).$$

Now we have $\Phi=H_s\Phi+dX$ and $X=\Xi_1+\Xi$, where Ξ_1 is of type (1, 0) and Ξ of type (0, 1). Hence

$$\pi_s^{(2,0)}\Phi=H_s\pi_s^{(2,0)}\Phi+\partial_s\Xi_1\,,$$

$$\pi_s^{(1,1)}\Phi=H_s\pi_s^{(1,1)}\Phi+\partial_s\Xi_1+\bar\partial_s\Xi\,,$$

$$\pi_s^{(0,2)}\Phi=H_s\pi_s^{(0,2)}\Phi+\bar\partial_s\Xi\,.$$

As we easily verify, the relation

$$\Psi\wedge(\Phi\barwedge\eta)=-(\Psi\barwedge\eta)\wedge\Phi$$

holds. Hence we have

$$\int_{V_s} \Psi \wedge (\pi_s^{(1,1)}\Phi \,\bar{\wedge}\, \eta) = \int_{V_s} \Psi \wedge (H_s\pi_s^{(1,1)}\Phi \,\bar{\wedge}\, \eta) - \int_{V_s} (\Psi \,\bar{\wedge}\, \eta) \wedge \partial_s \Xi$$
$$= \int_{V_s} \partial_s(\Psi \,\bar{\wedge}\, \eta) \wedge \Xi ,$$

because of our assumption $\int_{V_s} \Psi \wedge (H_s\pi_s^{(1,1)}\Phi \,\bar{\wedge}\, \eta) = 0$.

On the other hand, we have $\partial h^\alpha/\partial\xi^\beta = \delta^\alpha_\beta$ for $t=s$. Hence

$$\left[\frac{\partial}{\partial t}\log\det\left(\frac{\partial h^\alpha}{\partial \xi^\beta}\right)\right]_{t=s} = \left[\frac{\partial^2 h^1}{\partial t\partial\xi^1} + \frac{\partial^2 h^2}{\partial t\partial\xi^2}\right]_{t=s},$$

$$\left[\frac{\partial\Psi_{12}}{\partial t} + \sum_\rho \frac{\partial\Psi_{12}}{\partial z^\rho}\frac{\partial h^\rho}{\partial t} + \Psi_{12}\frac{\partial}{\partial t}\left(\log\det\left(\frac{\partial h^\alpha}{\partial\xi^\beta}\right)\right)\right]_{t=s} d\xi^1 \wedge d\xi^2$$
$$= \left[\frac{\partial\Psi_{12}}{\partial t} + \sum_\rho \frac{\partial}{\partial\xi^\rho}\left(\Psi_{12}\frac{\partial h^\rho}{\partial t}\right)\right]_{t=s} d\xi^1 \wedge d\xi^2.\ ^{2)}$$

Now

$$\int_{V_s}\left[\frac{\partial\Psi_{12}}{\partial t} + \sum_\rho \frac{\partial}{\partial\xi^\rho}\left(\Psi_{12}\frac{\partial h^\rho}{\partial t}\right)\right]_{t=s} d\xi^1 \wedge d\xi^2 \wedge \bar\partial_s \Xi$$
$$= -\int_{V_s} \bar\partial_s\left\{\left[\frac{\partial\Psi_{12}}{\partial t} + \sum_\rho \frac{\partial}{\partial\xi^\rho}\left(\Psi_{12}\frac{\partial h^\rho}{dt}\right)\right]_{t=s} d\xi^1 \wedge d\xi^2\right\} \wedge \Xi$$
$$= \int_{V_s} \sum \frac{\partial}{\partial\xi^\rho}\left(\Psi_{12}\frac{\partial^2 h^\rho}{\partial\bar\xi^\beta\partial t}\right)_{t=s} d\xi^1 \wedge d\xi^2 \wedge d\bar\xi^\beta \wedge \Xi .$$

Since $\left(\frac{\partial^2 h^\rho}{\partial\bar\xi^\beta\partial t}\right)_{t=s} = \eta^\rho{}_{\bar\beta}$, this integral is equal to

$$\int_{V_s} \partial_s(\Psi \,\bar{\wedge}\, \eta) \wedge \Xi$$

Putting these into (3. 4), and showing the suffx r explicitly, we obtain

$$\frac{\partial u_r(s)}{\partial s} = \int_{V_s}\left[\frac{\partial\Psi^{(r)}_{12}}{\partial t} + \sum_\rho \frac{\partial}{\partial\xi^\rho}\left(\Psi^{(r)}\frac{\partial h^\rho}{\partial t}\right)\right]_{t=s} d\xi^1 \wedge d\xi^2 \wedge H_s\pi_s^{(1,1)}\Phi .$$

The harmonic part of $\left[\frac{\partial\Psi^{(r)}_{12}}{\partial t} + \sum_\rho \frac{\partial}{\partial\xi^\rho}\left(\Psi^{(r)}\frac{\partial h^\rho}{\partial t}\right)\right]_{t=s} d\xi^1 \wedge d\xi^2$ is equal to $\sum_q a_{rq}(s)\Psi^{(q)}(\ ,s)$, where $a_{rq}(s)$ are differentiable functions of s. Hence $\{u_r(s)\}$ satisfy the system of differential equations

2) This expression is a well defined differential form on V_s.

$$\frac{\partial u_r(s)}{\partial s} = \sum_q a_{rq}(s)u_q(s) .$$

Since $u_r(0)=0$, we see that $u_r(s)=0$ in a neighborhood of 0.

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