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A probabilistic interpretation of equilibrium charge distributions

By

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1. Background (2-Dimensional Brownian Motion).

Consider a standard 2-dimensional Brownian motion with sample paths $t\to x(t)$, probabilities $P_a(B)$, and expectations $E_a(f)$. Given closed $K\subset R^2$, the hitting time $\mathfrak{m}=\inf(t>0: x(t)\in K)$ is a stopping time, and in case $P_\cdot(\mathfrak{m}<\infty)\equiv 1$, the hitting probability $P_a[x(\mathfrak{m})\in B]$ coincides with the classical harmonic measure of the arc $B\subset\partial K$ as viewed from $a\in R^2-K$. A point $a\in\partial K$ is *singular* if $P_a(\mathfrak{m}>0)=1$; according to Blumenthal's 01 law, the alternative is $P_a(\mathfrak{m}=0)=1$. $P_a[x(\mathfrak{m})\in db]$ is loaded up on the non-singular points of ∂K.

Kakutani's alternative states that *either* $P_\cdot(\mathfrak{m}<\infty)\equiv 1$ *or* $P_\cdot(\mathfrak{m}=\infty)\equiv 1$ according as the (logarithmic) capacity of K:

$$C(K)=\inf_{\substack{e\geq 0\\ e(K)=1}}\exp\left[-\int_{K\times K} lg|b-a|\,e(da)e(db)\right]$$

is positive or not.

A domain $D\subset R^2$ is Greenian if R^2-D is of positive logarithmic capacity; in that case, $P_\cdot(\mathfrak{e}<\infty)\equiv 1$, $\mathfrak{e}$ being the exit time $\inf(t>0: x(t)\notin D)$, and on $D\times D$,

$$E_a[\text{measure }(t<\mathfrak{e}: x(t)\in db)]=G(a,b)\,db,$$

G being the Green function for $\Delta/2$ and db the element of area. A subcompact $K\subset D$ of positive capacity has a Newtonian equilibrium

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$$=C(K)E_{e_1}\left(\begin{array}{l}\displaystyle\int f\circ x(t)\,dt\\ t<\mathfrak{m}_1\\ \mathfrak{m}(w_t^+)<\mathfrak{e}(w_t^+)<\varepsilon+\mathfrak{m}(w_{t+\varepsilon}^+)\\ \varepsilon<\mathfrak{e}(w_t^+)\end{array}\right)$$

$$=C(K)E_{e_1}\left(\begin{array}{lcl}\displaystyle\int f\circ x(t)\,dt & + & \displaystyle\int f\circ x(t)\,dt\\ (\mathfrak{f}-\varepsilon)\vee 0\leq t<\mathfrak{f} & & (\mathfrak{m}_1-\varepsilon)\vee \mathfrak{f}^*\leq t<\mathfrak{m}_1\\ \varepsilon<\mathfrak{e}(w_t^+) & & \varepsilon<\mathfrak{e}(w_t^+)<\varepsilon+\mathfrak{m}(w_{t+\varepsilon}^+)\end{array}\right)^{*},$$

$\mathfrak{f}^*$ denoting (for the moment) the last leaving time $\max(t<\mathfrak{m}_1: x(t)\notin D)$ from $M-D$ before the first return to ∂K. Multiplying by ε^{-1}, letting $\varepsilon\downarrow 0$ under the expectation sign, and using 4), one finds

$$C(K)^{-1}\int f\,de^{p}=E_{e_1}[f\circ x(\mathfrak{f}),\ \mathfrak{f}>0]$$
$$+E_{e_1}[f\circ x(\mathfrak{m}_1),\ \mathfrak{m}(w_{\mathfrak{m}_1}^+)>\mathfrak{e}(w_{\mathfrak{m}_1}^+)].$$

But both $P_{e_1}(\mathfrak{f}>0)=1$ and $P_{e_1}(\mathfrak{m}(w_{\mathfrak{m}_1}^+)=0)=P_{e_1}(\mathfrak{m}=0)=1$ since e_1 is loaded up on the non-singular points of ∂K. 1) is now obvious.

4. Proof of 2).

2) is now to be proved for a self-dual motion: the backward motion is introduced for this purpose, and at the same time a new proof of 1) is obtained.

Choose $0<f\in C^\infty(M)$ with $\int f\,de=1$ and compose the sample path x with the inverse function $\mathfrak{t}^{-1}$ of $\mathfrak{t}(t)=\int_0^t f\circ x(s)\,ds$, obtaining a new diffusion $x^0=x(\mathfrak{t}^{-1})$ with $\boldsymbol{G}^0=f^{-1}\boldsymbol{G}$ and $e^0=fe$.** $e^0(M)=1$ and since a time substitution does not change hitting probabilities such as $P.(\mathfrak{m}<\mathfrak{e})$ or Green functions such as G, it is legitimate to suppose $e(M)=1$ from the beginning, as will be done below.

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*) $a\vee b$ means the bigger of a and b.

**) [4,6,10] contain information about such time substitutions.

Because e was stable, it is possible to define a non-negative shift-invariant distribution Q of total mass $+1$ on the class of all sample paths $t\in R^1\to x(t)$ according to the rule:

$$Q[x(t_0)\in da,\ x(t_1)\in db_1,\ \cdots,\ x(t_n)\in db_n]$$
$$=e(da)P_a[x(t_1-t_0)\in db_1,\ \cdots,\ x(t_n-t_{n-1})\in db_n]$$
$$-\infty<t_0<t_1<\cdots<t_n,\ a,\ b_1,\ \cdots,\ b_n\in M,\ n\geq 1.$$

G. HUNT [5] now defines the *backward* motion $[x^*(t)\equiv x(-t):\ t\in R^1,\ Q]$, dual to the *forward* motion $[x,\ Q]$. $[x^*,\ Q]$ is the diffusion associated with dual $\boldsymbol{G}^*$ of $\boldsymbol{G}$ relative to the stable volume element e; it hits each subregion *i.o.* since $Q[x(t)\notin D,\ t\geq 0]$ is unchanged by time reversal; also, it has the same stable distribution $e^*=e$ as the forward motion since

$$e(da)P_a^*[x(t)\in db]=e(db)P_b[x(t)\in da].^{**}$$

Both motions have the same Greenian domains, and the associated Green functions are related according to the rule: $G^*(a,b)=G(b,a)$. $[x,\ Q]$ is self-dual if it is identical in law to $[x^*,\ Q]$; this happens if and only if $\boldsymbol{G}=\boldsymbol{G}^*$.

2) and the new proof of 1) are immediate from the fact that if e_1^* is the stable distribution of hits on ∂K via $M-D$ for the backward motion, then

5) $$e^p=C(K)e_1^*$$

and

6) $$e_1^*(db)=P_{e_1}[x(\mathfrak{f})\in db].$$

Beginning with the proof of 5), the backward stable mass $e^*(db)=e(db)$ attached to a small volume $db\subset K$ can be computed up to a positive multiplicative constant $C(K)$ (identified later as the capacity of K) in terms of e_1^*, the backward exit time $\mathfrak{e}^*$ from D, and the backward Green function G^*:

**) $P_a^*(B)$ and $E_a^*(f)$ denote backward probabilities and expectations.

$$e(db) = C(K) E^*_{e_1^*}[\text{measure } (t < \mathfrak{e}^* : x^* \in db)]$$
$$= C(K) \int_{\partial K} e_1^*(da) G^*(a, b) e(db).$$

It follows that

$$p^*(a) \equiv C(K) \int_{\partial K} G(a, b) e_1^*(db) = 1$$

at almost all points $a \in K$ relative to e. Because G is smooth on $D \times D$ apart from a pole on the diagonal, $p^* \leq 1$ on the whole of K and, as such, is smaller than the equilibrium potential $p = P.(\mathfrak{m} < \mathfrak{e})$ on D. But also, if p_n^* is the analogue of p^* for the closed $1/n$ neighbourhood K_n of K, then $p_n^* \equiv 1$ on K (e is positive on opens), and so $p_n^* \geq p$ on the whole of D. As is easy to prove, the stàble distribution of backward hits on ∂K_n via $M-D$ tends to e_1^* as $n \uparrow \infty$, and now the identification of p^* with p on the whole of D follows by standard methods. 5) is now proved and 6) follows from 1), but a direct proof 6) is also possible.

UENO [9: 122] proved that $P_a[x(\mathfrak{m}_n) \in db]$ tends geometrically fast to $e_1(db)$ as $n \uparrow \infty$, uniformly for $a \in \partial K$; thus, the chain of hits $[x(\mathfrak{m}_n) : n \geq 1, Q]$ is mixing, and

$$P_{e_1}[x(\mathfrak{f}) \in db] = \lim_{n \uparrow \infty} \int_{\partial K} Q[x(\mathfrak{m}_n) \in da] P_a[x(\mathfrak{f}) \in db]$$
$$= \lim_{n \uparrow \infty} Q[x(\mathfrak{m}_n + \mathfrak{f}(w^+_{\mathfrak{m}_n}))\in db]$$
$$= \lim_{n \uparrow \infty} Q[x^*(\mathfrak{m}^*_{-n}) \in db),$$

$\mathfrak{m}^*_{-1} > \mathfrak{m}^*_{-2} > etc.$ being the successive hitting times to ∂B via $M-D$ for the backward motion during the past $t \leq 0$. But the 2-sided chain of forward hits $[x(\mathfrak{m}_n) : n \in Z^1, Q]$ is mixing under the (forward) shift, so by G. D. BIRKHOFF's ergodic theorem, it is also mixing under the backward shift, and by a second application of BIRKHOFF's theorem,

$$P_{e_1}[x(\mathfrak{f}) \in db] = \lim_{n \uparrow \infty} Q[x^*(\mathfrak{m}^*_{-n}) \in db]$$
$$= \lim_{n \uparrow \infty} Q[x^*(\mathfrak{m}_{+n}) \in db] = e_1^*(db),$$

completing the proof.

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