J. Math. Kyoto Univ.
4-3 (1965) 607-615.

# On the regularization of the second order random distribution*

By

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(Communicated by Professor K. Itô, March 1, 1964)

## §1. Introduction

Let us start with some preliminary notations and definitions to explain our problem.

Let $\mathcal{D}$ be the space of all $C^\infty$ complex valued functions, each being defined on the real line and vanishing outside of a compact set. We shall consdel $\mathcal{D}$ as a linear topological space with the Schwarz topology [4]. Any linear continuous functional on $\mathcal{D}$ is called a (Schwarz) distribution and the set of all distributions is denoted by $\mathcal{D}'$.

Let $\Omega(B, P)$[1] be a probability space. The totality of complex valued random variables with the finite second moment constitutes a Hilbert space $H\equiv L^2(\Omega, B, P)$ with the following inner roduct:

$$(X, Y)=E(X\overline{Y})=\int_\Omega X(\omega)\overline{Y}(\omega)dP(\omega)$$

An $H$-valued continuous linear functional on $\mathcal{D}$ is called a *second order random distribution* [1], [2], [3]. We shall denote with $\mathcal{D}'_H$ the totality of second order random distributions.

A second order continuous stochastic process $X(t)$ is regarded as a second order random distribution as

$$X(\phi)=\int_{-\infty}^{\infty}\phi(t)X(t)dt\equiv\lim_{n\to\infty}\sum_{k=-\infty}^{\infty}\phi\left(\frac{k}{n}\right)X\left(\frac{k}{n}\right)\frac{1}{n}\quad \phi\in\mathcal{D},$$

*This work was supported by the Natinal Science Foundation.

1) We assume that $P$ is a complete measure.

where the sun is actually a finite sum because of the compactness of the support of $\phi$ and the limit is understood in the sense of the norm limit in $H$.

The derivatives of $X \in \mathcal{D}'_H$ are defined by

$$X^{(k)}(\phi) = (-1)^k X(\phi^{(k)})$$

similarly to the case of $D'$.

A complex valued function $X(\phi, \omega)$ of $\phi \in \mathcal{D}$ and $\omega \in \Omega$ is called a *sample-wise random distribution* [2], [3], if it is measurable in $\omega$ for each $\phi$ and belongs to $\mathcal{D}'$ as a function of $\phi \in D$ for almost every $\omega \in \Omega$.

A stochastic process $X(t, \omega)$ measurable in $(t, \omega) \in R^1 \times \Omega$ is regarded as a sample-wise random distribution as

$$X(\phi, \omega) = \int_{-\infty}^{\infty} \phi(t) X(t, \omega) dt$$

if $X(\phi, \omega)$ is locally summable in $t$ for almost every $\omega$.

The derivative of a sample-wise random distribution $X(\phi, \omega)$ is naturally defined

$$X^{(k)}(\phi, \omega) = (-1)^k X(\phi^{(k)}, \omega)$$

It is well-known that any second order continuous stochastic process $X(t)$ has a measurable version $\widetilde{X}(t, \omega)$ such that $\widetilde{X}(t, \omega)$ is measurable in $(t, \omega)$ and that

$$P(\widetilde{X}(t, \omega) = X(t, \omega)) = 1$$

for every $t$. The main aim of this paper is to prove an analogons foct for randam distributions, namely we are going to pasve that, given a second order random distribution $X(\phi)$, we have a unique sample-wise random distribution $\widetilde{X}(\phi, \omega)$ such that

$$P(\widetilde{X}(\phi, \omega) = X(\phi, \omega)) = 1$$

for every $\phi \in \mathcal{D}$. $\widetilde{X}(\phi, \omega)$ will be called the *regularization* of $X(\phi)$.

In the case of second order *stationary* random distributions, the existence of such a regularization follows at once from the fact [3] that it is the derivative of a certain order $n$ of a second order continuous stochastic process with the $n^{th}$ order stationary increments. We shall establish a similar fact for the non-stationary case in Section 3 to get our regularization of the second order random distribution in Section 4. Section 2 is devoted to an inequality which will be useful in Section 3. It should be noted that our technique used here has something similar to that due to K. Urbanik [5].

The author wishes to express her sincere thanks to Professer K. Ito for his valuable suggestions.

## §2. The subspace $\mathcal{D}_N$ of $\mathcal{D}$.

Let $\mathcal{D}_N$ be the set of all functions in $\mathcal{D}$ that vanishes outside of $[-N, N]$. $\mathcal{D}_N$ is topologized by the following neighborhood basis at O:

$$V(p, \varepsilon) = \{\phi \in D_N;\ \|\phi\|_p < \varepsilon\},\ p=0, 1, 2, \cdots,\ \varepsilon>0,$$

where

$$\|\phi\|_p = \max\{|\phi^{(k)}(t)|;\ k \leqq p,\ t \in R^1\}.$$

Let $\|\ \|$ denote the mean square norm i.e.

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Then we have

**Lemma 1.** $\|\phi\|_p \leqq (2N)^{p+\frac{1}{2}} \|\phi^{(p+1)}\|$ $\phi \in \mathcal{D}_N$.

**Paoof.** For $\phi \in \mathcal{D}_N$, we have

$$\phi^{(i)}(t) = \int_{-N}^{t} \phi^{(i+1)}(s)\, ds, \tag{2.1}$$

so that we have, for $i=0, 1, \cdots, p$

$$\|\phi^{(i)}\|_0 \leqq 2N \|\phi^{(i+1)}\|_0 \leqq (2N)^2 \|\phi^{(i+2)}\|_0 \leqq \cdots \leqq (2N)^{(p-i)} \|\phi^{(p)}\|_0$$

and so

$$\|\phi^{(i)}\|_0 \leqq (2N)^p \|\phi^{(p)}\|_0.$$

Applying the Schwarz inequality to (2.1), we have

$$\|\phi^{(p)}\|_0 \leqq \sqrt{2N}\,\|\phi^{(p+1)}\|$$

and so

$$\|\phi^{(i)}\|_0 \leqq (2N)^{p+\frac{1}{2}}\|\phi^{(p+1)}\| \quad \text{for } i=0, 1, 2, \cdots, p.$$

Thus we have

$$\|\phi\|_p = \max(\|\phi\|_0,\ \|\phi'\|_0,\ \cdots,\ \|\phi^{(p)}\|_0) \leqq (2N)^{p+\frac{1}{2}}\|\phi^{(p+1)}\|.$$

## §3. Representation of second order random distributions

In this section we shall express second order random distributions in terms of the derivatives of second order continuous stochastic processes.

**Theorem 1.** A second order random distribution $X$ can be written in the form

$$X = \sum_n Y_n^{(p_n)}$$

where $Y_n$, $n=1, 2, \cdots$ are second order continuous stochastic processes such that any compact set intersects only a finite number of $\mathrm{Supp}(Y_n) \equiv \overline{\{t;\ Y_n(t) \neq 0\}}$, $n=1, 2, \cdots$.

**Proof.**

**Case 1.** *Supp X is compact.* Suppose that $A \equiv \mathrm{Supp}\ X$ lies in the open interval $(-N, N)$. Take two open sets $U$ and $V$ such that $A \subset U \subset \overline{U} \subset V \subset \overline{V} \subset (-N, N)$ and choose $\alpha$ from $\mathcal{D}$ such that

$$\alpha = 1 \quad \text{on} \quad \overline{U}, \quad = 0 \ \text{on} \quad \overline{V}^c.$$

Then we have

$$X(\phi) = X(\alpha\phi) \quad \phi \in \mathcal{D}. \tag{3.1}$$

The restriction of $X \in \mathcal{D}'_H$ on $\mathcal{D}_N$ is also an $H$-valued continuous linear functional on $\mathcal{D}_N$. Hence, for every positive number $\varepsilon$, there exists an integer $p$ and a positive number $\delta$ such that, if $\phi \in \mathcal{D}_N$ and $\|\phi\|_n \leqq \delta$, then $\|X(\phi)\| < \varepsilon$. Therefore

$$\|X(\phi)\| < \frac{\varepsilon}{\delta}\|\phi\|_p \quad \phi \in \mathcal{D}_N .$$

Combining this inequality with the Lemma 1 in Section 2, we get

(3. 2) $$\|X(\phi)\| < K\|\phi^{(p+1)}\| \quad \phi \in \mathcal{D}_N,$$

where $$K = \frac{\varepsilon}{\delta}(2N)^{p+\frac{1}{2}} .$$

Consider a linear subspace of $\mathcal{D}_N$:

$$M = \{\psi;\ \psi = \phi^{(p+1)},\ \phi \in \mathcal{D}_N\}$$

and define $\xi$ by

(3. 3) $$\xi(\psi, \omega) = X(\phi, \omega) \quad \text{if} \quad \psi = \phi^{(p+1)}.$$

$\xi(\psi)$ is well-defined since, for $\psi \in M$, there exists one and only one $\phi \in \mathcal{D}_N$ such that $\psi = \phi^{(p+1)}$. It is clear that $\xi$ is linear and satisfies

(3. 4) $$\|\xi(\psi)\| \leq K\|\psi\| \quad \psi \in M.$$

Hence $\xi$can be extended to an $H$-valued contionuous linear functional on $\bar{M}$(=the $L^2(R^1)$-closure of $M$). Further we can extend $\xi$ onto $L^2(R^1)$ by

$$\xi(f) = \xi(\mathrm{Proj}_{\bar{M}} f) \quad f \in L^2(L^1).$$

Then $\xi$ is linear and continuous on $L^2(R^1)$, moreover

(3. 5) $$\|\xi(f)\| \leqq K\|\mathrm{Proj}_{\bar{M}} f\| \leqq K\|f\|.$$

Define a process $\eta$ by.

(3. 6) $$\eta(t) = \xi(\chi_{[\min(-N,t),t]})$$

where $\chi_E$ is the indicator function of $E$. Then we heve $\|\eta(t) - \eta(s)\| \leqq K|t-s|$ by (3.5). This implies that $\eta(t)$ is a second order continuous stochastic process. We shall now prove that for $\phi \in \mathcal{D}_N$,

(3. 7) $$\eta(\phi', \omega) = -\xi(\phi, \omega) \text{ a.e.}$$

Define $\tau_n$ by $\tau_n(t) = t_i^n$ for $t_i^n \leqq t < t_{i+1}^n$, $i = 0, 1, \cdots, n-1$,

where $$t_i^n = -N + 2N\frac{i}{n} .$$

$$\left\|\int\eta(t)\phi'(t)dt-\sum_{i=0}^{n-1}\eta(t_i^n)\phi'(t_i^n)(t_{i+1}^n-t_i^n)\right\|$$

$$=\left\|\int_{-N}^{N}\eta(t)\phi'(t)dt-\int_{-N}^{N}\eta(\tau_n(t))\phi'(\tau_n(t))dt\right\|$$

$$\leqq\left\|\int_{-N}^{N}|\eta(\tau^n(t))|\,|\phi'(\tau_n(t))-\phi'(t)|\,dt\right\|$$

$$+\left\|\int_{-N}^{N}|\eta(\tau_n(t))-\eta(t)|\,|\phi(t)|\,dt\right\|$$

$$\leqq\sqrt{2N}\left(\int_{-N}^{N}|\phi'(\tau_n(t))-\phi'(t)|^2E|\eta(\tau_n(t))|^2dt\right)^{1/2}$$

$$+\sqrt{2N}\left(\int_{-N}^{N}|\phi'(t)|^2E|\eta(\tau_n(t))-\eta(t)|^2dt\right)^{1/2}$$

$$\leqq 2N\max_{t\in[-N,N]}\|\eta(t)\|\max_{\substack{|t-s|\leqq\frac{2N}{n}\\ t,s\in[-N,N]}}|\phi'(t)-\phi'(s)|+(2N)^2\frac{k}{n}\|\phi'\|_0\,.$$

Hence,

$$\int\eta(t)\phi'(t)dt=\lim_{n\to\infty}\sum_{i=0}^{n-1}\eta(t_i^n)\phi'(t_i^n)(t_{i+1}^n-t_i^n). \tag{3.8}$$

On the other hand

$$\left\|\sum_{i=0}^{n-1}\eta(t_i^n)\phi'(t_i^n)\phi'(t_{i+1}^n-t_i^n)+\xi(\phi)\right\|$$

$$=\left\|\xi\left(\sum_{i=0}^{n-1}\chi_{[-N,t_i^n]}(s)\phi'(t_i^n)(t_{i+1}^n-t_i^n)\right)+\xi(\phi)\right\|$$

$$=\left\|\xi\left(\sum_{i=0}^{n-1}\chi_{[-N,t_i^n]}(s)\phi'(t_i^n)(t_{i+1}^n-t_i^n)+\phi(s)\right)\right\|$$

$$\leqq K\left\|\sum_{i=0}^{n-1}\chi_{[-N,t_i^n]}(s)\phi'(t_i^n)(t_{i+1}^n-t_i^n)+\phi(s)\right\|$$

$$\leqq K\left\|\int_s^N\phi'(\tau_n(t))dt-\int_s^N\phi'(t)dt\right\|+K\left\|\int_{s-\frac{2N}{n}}^{s}\phi'(\tau_n(t))dt\right\|$$

$$\leqq 2NK\sup_{\substack{|t-s|\leqq\frac{2N}{n}\\ t,s\in[-N,N]}}|\phi'(t)-\phi'(s)|+\frac{2NK}{n}\|\phi'\|_0\,.$$

Hence

$$\xi(\phi)=-\lim_{n\to\infty}\sum_{i=0}^{n-1}\eta(t_i^n)\phi'(t_i^n)(t_{i+1}^n-t_i^n). \tag{3.9}$$

By (3.8) and (3.9) we get (3.7).

Recalling the definition (3. 3) of $\xi$, we have for $\phi \in \mathcal{D}_N$,

$$X(\phi, \omega) = \xi(\phi^{(p+1)}, \omega) = -\eta(\phi^{(p+2)}, \omega) \text{ a.e.}$$

Hence by (3.1), for any $\phi \in D$

$$X(\phi) = X(\alpha\phi) = -\eta((\alpha\phi)^{p+2)}) = -\sum_{k=0}^{p+2}\binom{p+2}{k}\eta(\alpha^{(p+2-k)}\phi^{(k)})$$
$$= -\sum_{k=0}^{p+2}\binom{p+2}{k}(\alpha^{(p+2-k)}\eta)(\phi^{(k)}) = \sum_{k=0}^{p+2}\binom{p+2}{k}(-1)^{k+1}(\alpha^{(p+2-k)}\eta)^{(k)}(\phi).$$

Therefore $X = \sum_{k=0}^{p+2} Y_k^{(k)}$ where $Y_k = (-1)^{k+1}\binom{p+2}{k}(\alpha^{(p+2-k)}\eta)$. Since the support of $(\alpha^{(p+2-k)}\eta)$ lies in $\overline{V}$, the proof is completed in our seccial case.

*General Case.* We can reduce the general case to the specsal case discussed above by virtue of the famous

**Partion of unity:** Let $\{O_i,\ i=1, 2, \cdots\}$ be a countable covering of $R^1$ by open sets $\{O_i\}$. Then there exist functions $\{\alpha_i\}$ having the following properties:

(i) $\alpha_i \geqq 0$, $\sum \alpha_i = 1$ on $R^1$

(ii) $\alpha_i \in C^\infty(R^1)$ and its support lies in $O_i$

(iii) every compact set intersects only a finite number of supports of $\alpha_i$.

Take a countable covering $\{O_i\}$ of $R^1$ by bounded open intervals. Let $\{\alpha_i\}$ form a partition of unity corresponding to this covering. Define $X_i(\phi) = X(\alpha_i\phi)$ $\phi \in \mathcal{D}$. Then $X_i$ is a second order random distribution whose support lies in $O_i$. Further the series $\sum_i X_i$ is locally finite and converges to $X$, since

$$\sum_i X_i(\phi) = \sum_i X(\alpha_i\phi) = X(\sum_i \alpha_i\phi) = X(\phi).$$

To complete the proof of theorem 1, we need only apply the result obtained in Case 1 to vach $X_i$.

## §4. Regularization of second order random distributions

We are now in a position to prove

**Theorem 2.** Any second order randon distripobutn has a unique regulzation.

**Proof.** Let $X(\phi)$ be a given second order rondom distribution. Let

(4.1) $$X(\phi)=\sum_n Y_n^{(p_n)}(\phi)$$

be the reprsentation of $X(\phi)$ in the sense of Theorem 1. Since $Y_n(t)$ is a second order continuous stochastic process, it has a measurable version $\widetilde{Y}_n(t, \omega)$. Noticing

$$E\left[\int_a^b |\widetilde{Y}_n(t, \omega)|\,dt\right]=\int_a^b E[|\widetilde{Y}_n(t, \omega)|]\,dt=\int_a^b E[|Y_n(t)|]\,dt,$$

we see that $\widetilde{Y}_n(t, \omega)$ is locally integrable in $t$ for almost every $\omega$. Therefore it defines a sample-wise random distribution $\widetilde{Y}_n(\phi, \omega)$. It is easy to see that

$$P(Y_n^{(p_n)}(\phi, \omega)=\widetilde{Y}_n^{(p_n)}(\phi, \omega))=1,\ n=1, 2, \cdots$$

for every $\phi\in\mathscr{D}$. Setting $\widetilde{X}(\phi, \omega)=\sum_n \widetilde{Y}_n^{(p_n)}(\phi, \omega)$ we get a sample-wise random distribution $\widetilde{X}(\phi, \omega)$ such that

(4.2) $$P(X, \omega)=\widetilde{X}(\phi, \omega))=1$$

for every $\phi\in\mathscr{D}$. $\widetilde{X}(\phi, \omega)$ is clearly a regularization of $X(\phi)$.

Let $\overline{X}(\phi, \omega)$ be any other regularization of $X(\phi)$. We shall prove

(4.3) $$P(\overline{X}(\phi, \omega)=\widetilde{X}(\phi, \omega)\text{ for every }\phi\in\mathscr{D})=1,$$

in order to show the uniqueness of the regularization of $X(\phi)$. Let $\{\phi_j\}$ be a countable dense set in $\mathscr{D}$. Since we have

$$P(\overline{X}(\phi_j, \omega)=X(\phi_j, \omega)=\widetilde{X}(\phi_j, \omega))=1\text{ for }j=1, 2, \cdots$$

by the definition of regularization, we have

(4.4) $$P(\overline{X}(\phi_j, \omega)=\widetilde{X}(\phi_j, \omega),\ j=1, 2, \cdots)=1.$$

But both $\overline{X}(\phi, \omega)$ and $\widetilde{X}(\phi, \omega)$ belong to $\mathscr{D}'$ for almost every $\omega$, (4.3) follows at once from (4.4).

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