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On almost sure asymptotic sample properties of diffusion processes defined by stochastic differential equation

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§1. Introduction

The asymptotic properties of solutions to stochastic differential equations appear to be of current interest. Much of this interest can be traced to the increasing use of stochastic differential equations in the formulation of problems of the physical and engineering sciences. A number of interesting results have appeared. For example, Kac [1] and McKean [2] study the winding of the solution paths of simple osillators driven by "white noise" around the origin in the phase space of the solutions.

Related to the present manuscript is the work of Khas'minskii [3] in which a Lyapunov stability in a probabilistic sense is defined and studied for the solution process of a stochastic differential equation with "white noise" coefficients. In particular if the Markov solution process is denoted by $\{X(t, \omega), P_{s,x}\}$, then he defines stability of the equilibrium solution $X \equiv 0$, as

$$(1.1) \qquad \lim_{x\to 0} P_{s,x}\{\operatorname{Sup}_{t>s}|X(t)|>\varepsilon\}=0$$

A similar definition is given for asymptotic stability. He obtains sufficient conditions for (1.1) analogous to those of the second method of Lyapunov in ordinary differential equation theory. In particular, if there exists a twice continuously differentiable positive definite function $V(s, x)$, such that

$$\frac{\partial V}{\partial s}+L_{s,x}V\leqslant 0, \tag{1.2}$$

where $L_{s,x}$ is the elliptic Backward Diffusion operator associated with the given stochastic differential equation, then (1.1) holds. The purpose of the present paper is to study the sample asymptotic behavior and the almost sure stability of the equilibrium solution of stochastic differential equations. We obtain a sufficient condition in terms of exponentially decaying second moments that guarantees the sample solutions, themselves, decay exponentially with probability one. This, of course, implies almost sure stability of the equilibrium solution for linear homogeneous stochastic differential equations. Moreover, it is shown that the second moments of the solution process satisfy a differential relation which yields a sufficient condition for their exponential decay. Results similar to those obtained in this work for a different class of stochastic systems may by seen in [4].

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§2. Preliminaries

Let $\{X(t, \omega);\ t\geqslant t_0>0\}$ denote the solution process defined by the stochastic differential equation

$$dX(t, \omega)=m(t, X(t, \omega))dt+\sigma(t, X(t, \omega))dZ(t, \omega). \tag{2.1}$$

We further assume that

$$\begin{aligned} X&=[X_1, \cdots, X_n]\\ m&=[m_1, \cdots, m_n]\\ \sigma&=(\sigma_{ij}),\ i, j=1, \cdots, n\\ Z&=[Z_1, \cdots, Z_n]. \end{aligned} \tag{2.2}$$

The components of the vector Z-process are independent Wiener processes for which

$$(2.3)\qquad \begin{aligned} &P(Z_i(0)=0)=1\\ &E\{Z_i(t)\}=0\\ &E(Z_i(s)Z_i(t)\}=\min(s,t),\ i=1,\cdots,n. \end{aligned}$$

The stochastic integral equation denfining the meaning of (2.1), [5], is

$$(2.4)\qquad X(t,\omega)=X_0+\int_{t_0}^{t} m(\tau,X(\tau,\omega))d\tau+\int_{t_0}^{t}\sigma(\tau,X(\tau,\omega))d_\tau Z(\tau,\omega).$$

Conditions guaranteeing the existence, uniqueness and sample continuity of the process defined by (2.4) on any finite interval are known. The conditions also guarantee the existence and absolute integrability of the second moments on any finite interval. Furthermore, it is known that the processes so determined have no finite killing time, that is, they do not terminate at some finite time. This last fact guarantees that the processes exist over the entire positive time axis. Hence, it is not vacuous to consider questions concerning the asymptotic properties of these processes.

Sufficient condinions for the properties above are

a) Uniform Lipschitz condition.

$$(2.5)\qquad \begin{aligned} &\|m(t,\xi_2)-m(t,\xi_1)\|_2\leqslant K\|\xi_2-\xi_1\|_2\\ &\|\sigma(t,\xi_2)-\sigma(t,\xi_1)\|_2\leqslant K\|\xi_2-\xi_1\|_2 \end{aligned}$$

b) Uniform Growth condition

$$\begin{aligned} &\|m(t,\xi)\|_2\leqslant K(1+\|\xi\|_2^2)^{1/2}\\ &\|\sigma(t,\xi)\|_2\leqslant K(1+\|\xi\|_2^2)^{1/2}, \end{aligned}$$

where $t\geqslant 0$, ξ_1, $\xi_2\in R_n$, $K>0$ is a constant independent of t and the norms are defined as

$$(2.6)\qquad \begin{aligned} &\|y\|_2=[\sum_{i=1}^{n} y^2]^{1/2}\\ &\|A\|_2=[\sum_{i,j=1}^{n} a_{ij}^2]^{1/2}. \end{aligned}$$

It is well known that the solution process defined by (2.1) or (2.4) is associated with the Backward Diffusion operator $L^{-}_{\tau,y}$ defined by

$$(2.7)\qquad L^{-}_{\tau,y}=\frac{\partial}{\partial\tau}+\frac{1}{2}\sum_{i,j=1}^{n}b_{ij}(\tau,y)\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^{n}m_i(\tau,y)\frac{\partial}{\partial y_i},$$

where $(b_{ij})=\sigma\cdot\sigma^T$. Superscript T denotes transpose.

In fact if $F(\xi)$ is any twice continuously differentiable function, it has been shown that [6]

$$(2.8)\qquad dF(X(t,\omega)) = \left[L^{-}_{t,\xi}F(\xi)dt+\sum_{i,j=1}^{n}\sigma_{ij}(t,\xi)\frac{\partial F(\xi)}{\partial\xi_i}dZ_j(t,\omega)\right]_{\xi=X(t,\omega)}.$$

The symbolic formula (2.8) attains its meaning only through the associated stochastic integral equality. If F is bounded then its expectation exists and satisfies.

$$(2.9)\qquad \frac{d}{dt}E_{\tau,y}\{F(X(t,\omega))\}\,|_{t=\tau}=L^{-}_{\tau,y}F(y)$$

We shall be concerned, in this paper, with the equilibrium solution defined as

$$(2.10)\qquad X_1\equiv X_2\equiv\cdots\equiv X_n\equiv 0$$

A sufficient condition guaranteeing the existence of the equilibrium solution for the stochastic equation (2.4) (or, symbolically, (2.1)) is that

$$(2.11)\qquad \begin{aligned} m(t,0)&\equiv[0,\cdots,0]\\ \sigma(t,0)&=(0), \end{aligned}$$

identically in t.

It immediately follows from (2.5) and (2.11) that

$$(2.12)\qquad \begin{aligned} \|m(t,\xi)\|_2&\leqslant K\|\xi\|_2\\ \|\sigma(t,\xi)\|_2&\leqslant K\|\xi\|_2, \end{aligned}$$

for $\xi\in R_n$, identically in t.

We are interested in the asymptotic approach of the sample solutions to the equilibrium solution, as well as the stability of the equilibrium solution. We define almost sure stochastic versions of Lyapunov stability and psuedo-asymptotic stability. Lyapunov stabilty is a uniform convergence with respect to the initial conditions, hence we display the explicit dependence of the solution process on the initial conditions in the following two definitions. We shall display this dependence, whenever it appears appropriate.

Definition 1. *Almost Sure Lyapunov Stability.* The equilibrium solution of a stochastic system is said to possess the property of almost sure Lyapunov stability if

$$P\{\lim_{\delta\downarrow 0}\ \operatorname{Sup}_{\|x_0\|<\delta}\ \operatorname{Sup}_{t_0\leqslant t}\|X(t: x_0, t_0, \omega)\|=0\}=1 \tag{2.13}$$

Definition 2. *Almost Sure Psuedo-Asymptotic Stability Relative to* R_n. The equilibrium solution of a stochastic system is said to possess almost sure psuedo-asymptotic stability relative to R_n, if $x_0 \in R_n$ implies

$$P\{\lim_{T\uparrow\infty}\ \operatorname{Sup}_{T<t}\|X(t: x_0, t_0, \omega)\|=0\}=1. \tag{2.14}$$

The norm "$\| \ \|$" without subscript refers to the absolute norm,

$$\|X\|=\sum_{i=1}^{k}|X_i|.$$

We use this norm in the following without comment.

§3. Main Results

We shall now prove the two theorems that are the main results of this paper. The proofs follow directly along the lines set by Ito [5] and Doob [7].

Theorem 1. Let the stochastic system defined by (2.4) satisfy (2.5). Let the second moments, which exist, of the solution process decay exponentially in the sense that there exists constants a, $b>0$ such that for $x_0 \in R_n$, $t \geqslant t_0$,

$$E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}\leqslant a^2\|x_0\|_2^2 \exp[-2b(t-t_0)]. \tag{3.1}$$

If follows that the sample paths decay exponentially with probability one. In particular, there exists constants α, $\beta>0$ and a positive integer $M(\omega)$ depending upon the sample path for which $t>M(\omega)$ implies

$$\|X(t: x_0, t_0, \omega)\|<\alpha\|x_0\|_2 \exp[-\beta(t-t_0-1)] \tag{3.2}$$

on an ω-set of probability measure one.

Proof. The proof is based upon estimates derived using the properties of the stochastic integral.

Clearly for any positive integers $N<N'$,

$$\sup_{N\leqslant t\leqslant N'}\|X(t, \omega)\|\leqslant \sup_{N\leqslant t\leqslant N'}\|X(t, \omega)-X(N, \omega)\|+\|X(N, \omega)\|.$$

Hence, for any $\varepsilon_N>0$, we have

$$\begin{aligned} &P_{t_0, x_0}\{\sup_{N\leqslant t\leqslant N'}\|X(t, \omega)\|\geqslant\varepsilon_N\}\leqslant \\ &P_{t_0, x_0}\left\{\sup_{N\leqslant t\leqslant N'}\|X(t, \omega)-X(N, \omega)\|\geqslant\frac{\varepsilon_N}{2}\right\}+P_{t_0, x_0}\left\{\|X(N, \omega)\|\geqslant\frac{\varepsilon_N}{2}\right\} \end{aligned} \tag{3.3}$$

It is useful to recall, from the Schwarz inequality, that

$$\|y\|\leqslant\sqrt{n}\,\|y\|_2, \quad \text{for vectors.}$$

$$\|A\|\leqslant n\|A\|_2, \quad \text{for matrices.}$$

For the second term on the right side of the inequality (3.3), we have from Chebychef's inequality and (3.1)

$$\begin{aligned} P_{t_0, x_0}\left\{\|X(N, \omega)\|\geqslant\frac{\varepsilon_N}{2}\right\}&\leqslant P_{t_0, x_0}\left\{\|X(N, \omega)\|_2\geqslant\frac{\varepsilon_N}{2\sqrt{n}}\right\} \\ &\leqslant\frac{4n}{\varepsilon_N^2}E_{t_0, x_0}\{\|X(N, \omega)\|_2^2\} \\ &\leqslant\frac{4n}{\varepsilon_N^2}a^2\|x_0\|_2^2 \exp[-2b(N-t_0)]. \end{aligned} \tag{3.4}$$

It follows from (2.4),

$$\begin{aligned} &P_{t_0, x_0}\left\{\sup_{N\leqslant t\leqslant N'}\|X(t, \omega)-X(N, \omega)\|\geqslant\frac{\varepsilon_N}{2}\right\} \\ &\leqslant P_{t_0, x_0}\left\{\sup_{N\leqslant t\leqslant N'}\left\|\int_N^t m(\tau, X(\tau, \omega))d\tau\right\|\geqslant\frac{\varepsilon_N}{4}\right\} \\ &+P_{t_0, x_0}\left\{\sup_{N\leqslant t\leqslant N'}\left\|\int_N^t \sigma(\tau, X(\tau, \omega))dZ(\tau, \omega)\right\|\geqslant\frac{\varepsilon_N}{4}\right\}. \end{aligned} \tag{3.5}$$

The uniform conditions (2.12) and the Markov inequality yield

$$
\begin{aligned}
(3.6)\qquad & P_{t_0, x_0}\left\{\operatorname*{Sup}_{N \leqslant t \leqslant N'} \left\| \int_N^t m(\tau, X(\tau, \omega))\, d\tau \right\| \geqslant \frac{\varepsilon_N}{4}\right\} \\
& \leqslant P_{t_0, x_0}\left\{\operatorname*{Sup}_{N \leqslant t \leqslant N'} \int_N^t \| m(\tau, X(\tau, \omega)) \| d\tau \geqslant \frac{\varepsilon_N}{4}\right\} \\
& = P_{t_0, x_0}\left\{\int_N^{N'} \| m(\tau, X(\tau, \omega)) \| d\tau \geqslant \frac{\varepsilon_N}{4}\right\} \\
& \leqslant P_{t_0, x_0}\left\{\sqrt{n}\, K \int_N^{N'} \| X(\tau, \omega) \|_2 d\tau \geqslant \frac{\varepsilon_N}{4}\right\} \\
& \leqslant \frac{4\sqrt{n}\, K}{\varepsilon_N} \int_N^{N'} E_{t_0, x_0}\{\| X(\tau, \omega) \|_2\}\, d\tau,
\end{aligned}
$$

where the last inequality, through the interchange of expectation and integration holds because of the measurability of the solution process and the finitness of the integrals involved.

In the same fashion, applying the usual semi-martingale property and the property of the second moments of generalized stochastic integrals, we obtain

$$
\begin{aligned}
(3.7)\qquad & P_{t_0, x_0}\left\{\operatorname*{Sup}_{N \leqslant t \leqslant N'} \left\| \int_N^t \sigma(\tau, X(\tau, \omega))\, dZ(\tau, \omega) \right\| \geqslant \frac{\varepsilon_N}{4}\right\} \\
& \leqslant \sum_{i,j=1}^{n} P_{t_0, x_0}\left\{\operatorname*{Sup}_{N \leqslant t \leqslant N'} \left| \int_N^t \sigma_{ij}(\tau, X(\tau, \omega))\, dZ_j(\tau, \omega) \right| \geqslant \frac{\varepsilon_N}{4n^2}\right\} \\
& \leqslant \sum_{i,j=1}^{n} \frac{16n^4}{\varepsilon_N^2} E_{t_0, x_0}\left\{\left| \int_N^{N'} \sigma_{ij}(\tau, X(\tau, \omega))\, dZ_j(\tau, \omega) \right|^2\right\} \\
& = \sum_{i,j=1}^{n} \frac{16n^4}{\varepsilon_N^2} \int_N^{N'} E_{t_0, x_0}\{\sigma_{ij}^2(\tau, X(\tau, \omega))\}\, d\tau \\
& = \frac{16n^4}{\varepsilon_N^2} \int_N^{N'} E_{t_0, x_0}\{\| \sigma(\tau, X(\tau, \omega)) \|_2^2\}\, d\tau \\
& \leqslant \frac{16n^4 K^2}{\varepsilon_N^2} \int_N^{N'} E_{t_0, x_0}\{\| X(\tau, \omega) \|_2^2\}\, d\tau .
\end{aligned}
$$

We may now combine the results (3.3)–(3.7), and apply the hypotheses (3.1) to yield

$$
P_{t_0, x_0}\left\{\operatorname*{Sup}_{N \leqslant t \leqslant N'} \| X(t, \omega) \| \geqslant \varepsilon_N\right\}
$$

$$\text{(3.8)} \qquad \leqslant \frac{4\sqrt{n}K}{\varepsilon_N}\int_N^{N'} a\|x_0\|_2 \exp[-b(\tau-t_0)]\,d\tau$$

$$+\frac{1}{\varepsilon_N^2}(16n^4K^2+4n)\int_N^{N'} a^2\|x_0\|_2^2 \exp[-2b(\tau-t_0)]\,d\tau.$$

We now choose ε_N as,

$$\varepsilon_N = a\|x_0\|_2 \exp\left[-\frac{b}{2}(N-t_0)\right],\quad N\geqslant 1,$$

and allow $N'\to\infty$ to obtain,

$$\text{(3.9)} \qquad P_{t_0,x_0}\left\{\operatorname*{Sup}_{N\leqslant t}\|X(t,\omega)\|\geqslant a\|x_0\|_2 \exp\left[-\frac{b}{2}(N-t_0)\right]\right\}$$

$$\leqslant \frac{4\sqrt{n}K}{b}\exp\left[-\frac{b}{2}(N-t_0)\right]$$

$$+\frac{(8n^4K^2+2n)}{b}\exp[-b(N-t_0)].$$

However, the expression on the right hand side of the inequality (3.9) is the N-th term of a convergent series. Hence, by the Borel-Cantelli lemma, there exists a positive integer $M(\omega)$, depending upon the sample, for which $N\geqslant M(\omega)$ implies

$$\text{(3.10)} \qquad \operatorname*{Sup}_{N\leqslant t}\|X(t,\omega)\| < a\|x_0\|_2 \exp\left[-\frac{b}{2}(N-t_0)\right]$$

on an ω-set of probability measure one.
Now, if $[T]$ denotes the greatest integer in T for $T>M(\omega)$, then $T-1<[T]\leqslant T$ imples from (3.10),

$$\text{(3.11)} \qquad \|X(T,\omega)\| \leqslant \operatorname*{Sup}_{T\leqslant t}\|X(t,\omega)\| \leqslant \operatorname*{Sup}_{[T]\leqslant t}\|X(t,\omega)\|$$

$$\leqslant a\|x_0\|_2 \exp\left[-\frac{b}{2}([T]-t_0)\right] \leqslant a\|x_0\|_2 \exp\left[-\frac{b}{2}(T-t_0-1)\right].$$

The theorem is proved.
An immediate consequence of Theorem 1 is that the sample solutions satisfy (2.14) of Definition 2. It also follows, if the components of the vector m and the matrix σ are linear (therefore homogeneous by 2.11), then the equilibrium solution is almost surely

Lyapunov stable according to Definition 1. Indeed, the sum of two solutions is a solution since generalized stochastic integrals satisfy the linearity properties of ordinary integrals. Furthermore the solutions are unique relative to the initial conditions, with probability one. Therefore, one can form a fundamental matrix of n linearly independent solutions of (2.4) and prove stability from Theorem 1 by the usual methods of ordinary differential equation theory, [e.g. see. [8]].

We can also establish (2.13) directly using estimates obtained in Theorem 1. If $\chi(t, \omega)$ denotes a fundmental matrix of solutions $X^i(t, \omega)$, $i=1, \cdots, n$, where the j^{th} component of X^i satisfies $X_j^i(t_0, \omega)$ $=\delta_j^i$ (Kronecker delta), then for a solution with initial condition x_0, we have

$$X(t: x_0, t_0, \omega)=x_0\chi(t, \omega),$$

with probability one.

Therefore, from the estimates in (3.8) we have,

$$\begin{aligned} P\{\operatorname*{Sup}_{\|x_0\|<\delta}\operatorname*{Sup}_{t_0\leqslant t}\|X(t: x_0, t_0, \omega)\|>\varepsilon\} &\leqslant P\{\delta\operatorname*{Sup}_{t_0\leqslant t}\|\chi(t, \omega)\|>\varepsilon\} \\ &\leqslant\sum_{i=1}^{n} P\left\{\operatorname*{Sup}_{t_0\leqslant t}\|X^i(t, \omega)\|\geqslant\frac{\varepsilon}{n\delta}\right\} \\ &\leqslant\sum_{i=1}^{n}\delta\left[\frac{4n^{3/2}Ka}{b\varepsilon}+\delta\left(\frac{2n^3+8n^6}{b}\right)\frac{a^2}{\varepsilon^2}\right]. \end{aligned}$$

The limit as $\delta\downarrow 0$ may be taken under the probability measure since the events are monotonically decreasing in δ. This establishes stability as given in Definition 2 for linear stochastic differential equations.

It is desirable to know when the condition (3.1) of Theorem 1 holds. The following theorem yields a sufficient condition as a simple corollary.

Theorem 2. Let the stochastic system (2.4) satisfy the uniform conditions (2.5). Then the differential of the expectation $E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}$ satisfies

$$\begin{aligned} (3.12)\qquad & dE_{t_0, x_0}\{\|X(t, \omega)\|_2^2\} \\ &= E_{t_0, x_0}\{\|\sigma(t, X(t, \omega))\|_2^2+2(X(t, \omega), m(t, X(t, \omega))\}\, dt \end{aligned}$$

where (,) denotes inner product.

Proof. The proof is a slight modification of proofs in Doob [7: chap VI, Sec 3]. We shall first establish a lemma.

Lemma 1. For $0<t-\tau<k$,

$$\int_{\tau}^{t} E_{t_0,x_0}\{\|X(s,\omega)-X(\tau,\omega)\|_2^2\}ds \leqslant K'(t-\tau)^2[E_{t_0,x_0}\{\|X(\tau,\omega)\|_2^2\}+1], \tag{3.13}$$

where K' is a constant depending only upon k.

Proof. Applying the properties of generalized stochastic integral as well as (2.5) to the system (2.4), yields.

$$\begin{aligned} E_{t_0,x_0}\{\|X(t,\omega)-X(\tau,\omega)\|_2^2\} &\leqslant 2E_{t_0,x_0}\left\{\left\|\int_{\tau}^{t} m(s,X(s,\omega))ds\right\|_2^2\right\} \\ &+2E_{t_0,x_0}\left\{\left\|\int_{\tau}^{t}\sigma(s,X(s,\omega))dZ(s,\omega)\right\|_2^2\right\} \\ &\leqslant 2(t-\tau)E_{t_0,x_0}\left\{\int_{\tau}^{t}\|m(s,X(s,\omega))\|_2^2 ds\right\} \\ &+2nE_{t_0,x_0}\left\{\int_{\tau}^{t}\|\sigma(s,X(s,\omega))\|_2^2 ds\right\} \\ &\leqslant 2((t-\tau)+n)K^2\int_{\tau}^{t}[E_{t_0,x_0}\{\|X(s,\omega)\|_2^2\}+1]ds \\ &\leqslant 4((t-\tau)+n)K^2\left\{(t-\tau)[E_{t_0,x_0}\{\|X(\tau,\omega)\|_2^2\}+1]\right. \\ &\left.+\int_{\tau}^{t}E_{t_0,x_0}\{\|X(s,\omega)-X(\tau,\omega)\|_2^2\}ds\right\}, \end{aligned} \tag{3.14}$$

since $A^2\leqslant 2(A-B)^2+2B^2$.

We are interested in t close to τ, thus for some choice of k, $0<k<1$, we assume $t-\tau<k$. Settnig $c=4K^2(k+n)$, we have

$$E_{t_0,x_0}\{\|X(t,\omega)-X(\tau,\omega)\|_2^2\}\leqslant c\int_{\tau}^{t}E_{t_0,x_0}\{\|X(s,\omega)-X(\tau,\omega)\|_2^2\}ds +c(t-\tau)[E_{t_0,x_0}\{\|X(\tau,\omega)\|_2^2\}+1]. \tag{3.15}$$

Integration of (3.15) with respect to t yields,

$$\int_{\tau}^{t}E_{t_0,x_0}\{\|X(s,\omega)-X(\tau,\omega)\|_2^2\}ds \leqslant ce^{c(t-\tau)}\frac{(t-\tau)^2}{2}[E_{t_0,x_0}\{\|X(\tau,\omega)\|_2^2\}+1],$$

which is (3. 13) for $K'=\frac{c}{2}e^{ck}$.

The lemma is proved.

It is interesting to note that the "Backward" inequality

$$\int_{\tau}^{t} E_{t_0, x_0}\{\|X(t, \omega)-X(s, \omega)\|_2^2\}\, ds \leqslant K'(t-\tau)^2 [E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}+1]$$

may be established in exactly the same fashion as (3.13). We may now proceed with the proof of theorem.

For $h>0$, we first consider

$$\begin{aligned} E_{t_0, x_0}\{\|X(t+h, \omega)\|_2^2-\|X(t, \omega)\|_2^2\} &= E_{t_0, x_0}\{\|X(t+h, \omega)-X(t, \omega)\|_2^2\} \\ &+2E_{t_0, x_0}\{(X(t, \omega),\ X(t+h, \omega)-X(t, \omega))\}. \end{aligned} \tag{3.16}$$

It will be convenient to write

$$\Delta m(s, t, \omega)=m(s, X(s, \omega))-m(s, X(t, \omega))$$

$$\Delta\sigma(s, t, \omega)=\sigma(s, X(s, \omega))-\sigma(s, X(t, \omega)).$$

The second term on the right of the equality (3.16) may be written as

$$\begin{aligned} E_{t_0, x_0}\{(X(t, \omega),\ X(t+h, \omega)-X(t, \omega))\} &= E_{t_0, x_0}\left\{(X(t, \omega),\ \int_t^{t+h} m(s, X(t, \omega))ds)\right\} \\ &+E_{t_0, x_0}\left\{(X(t, \omega),\ \int_t^{t+h} \Delta m(s, t, \omega)ds)\right\} \\ &+E_{t_0, x_0}\left\{(X(t, \omega),\ \int_t^{t+h} \sigma(s, X(s, \omega))dZ(s, \omega))\right\}. \end{aligned} \tag{3.17}$$

By the Schwarz inequality, the last expectation on the right hand side of (3.17) exists absolutely, since σ satisfies (2.5) and the second moments are finite. Thus, we may write

$$\begin{aligned} &E_{t_0, x_0}\left\{(X(t, \omega), \int_t^{t+h} \sigma(s, X(s, \omega))dZ(s, \omega))\right\} \\ &=E_{t_0, x_0}\left\{(X(t, \omega),\ E_{t_0, x_0}\left\{\int_t^{t+h} \sigma(s, X(s, \omega))dZ(s, \omega) \mid X(t, \omega)\right\}\right\}=0. \end{aligned}$$

It is interesting to note that this last equality does not follow from the usual L_2 theory of generalized stochastic integrals, which requires the existence of fourth order moments.

Applying Lemma 1 to the second term on the right of (3.17) gives us

$$\begin{aligned}
&\left|E_{t_0, x_0}\left\{(X(t, \omega), \int_t^{t+h} \Delta m(s, t, \omega) ds)\right\}\right| \\
&\quad \leqslant E_{t_0, x_0}\left\{\|X(t, \omega)\|_2 \left\|\int_t^{t+h} \Delta m(s, t, \omega) ds\right\|_2\right\} \\
&\quad \leqslant h^{1/2} E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}^{1/2} E_{t_0, x_0}\left\{\int_t^{t+h} \|\Delta m(s, t, \omega)\|_2^2 ds\right\}^{1/2} \\
&\quad \leqslant O(h^{3/2}) E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}^{1/2} (E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\} + 1)^{1/2},
\end{aligned}$$

where $O(h^{3/2})$ is independent of any random quantity and is uniform in t. We now proceed to study the first term on the right of the equality (3.16), and we write

$$\begin{aligned}
&E_{t_0, x_0}\{\|X(t+h, \omega) - X(t, \omega)\|_2^2\} \\
&\quad = E_{t_0, x_0}\left\{\left\|\int_t^{t+h} m(s, X(t, \omega)) ds + \int_t^{t+h} \sigma(s, X(t, \omega)) dZ(s, \omega)\right\|_2^2\right\} \\
&\quad + 2E_{t_0, x_0}\left\{\left(\int_t^{t+h} m(s, X(t, \omega)) ds + \int_t^{t+h} \sigma(s, X(t, \omega)) dZ(s, \omega),\right.\right. \\
&\qquad \left.\left.\int_t^{t+h} \Delta m(s, t, \omega) ds + \int_t^{t+h} \Delta\sigma(s, t, \omega) dZ(s, \omega)\right)\right\} \\
&\quad + E_{t_0, x_0}\left\{\left\|\int_t^{t+h} \Delta m(s, t, \omega) ds + \int_t^{t+h} \Delta\sigma(s, t, \omega) dZ(s, \omega)\right\|_2^2\right\}.
\end{aligned} \tag{3.18}$$

The first term on the right of (3.18) may be written as

$$\begin{aligned}
&E_{t_0, x_0}\left\{\left\|\int_t^{t+h} m(s, X(t, \omega)) ds\right\|_2^2\right\} + E_{t_0, x_0}\left\{\int_t^{t+h} \|\sigma(s, X(t, \omega))\|_2^2 ds\right\} \\
&\quad \leqslant h^2 K^2 (E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\} + 1) + E_{t_0, x_0}\left\{\int_t^{t+h} \|\sigma(s, X(t, \omega))\|_2^2 ds\right\}
\end{aligned}$$

since the cross product expectations are zero as shown above, and the components of the vector Z-process are assumed independent.

Using the Schwarz inequality, we can bound the last term on the right of (3.18) by

$$2E_{t_0,x_0}\left\{\left\|\int_t^{t+h}\Delta m(s,t,\omega)ds\right\|_2^2\right\}+2E_{t_0,x_0}\left\{\left\|\int_t^{t+h}\Delta\sigma(s,t,\omega)dZ(s,\omega)\right\|_2^2\right\}$$
$$\leqslant 2hE_{t_0,x_0}\left\{\int_t^{t+h}\|\Delta m(s,t,\omega)\|_2^2ds\right\}+2nE_{t_0,x_0}\left\{\int_t^{t+h}\|\Delta\sigma(s,t,\omega)\|_2^2ds\right\}$$
$$\leqslant 2(h+n)K^2K'h^2[E\{\|X(t,\omega)\|_2^2\}+1]$$
$$=O(h^2)[E\{\|X(t,\omega)\|_2^2\}+1],$$

upon application of Lemma 1, and the properties of generalized stochastic integrals.

For the middle term on the right side of (3.18) we have upon applications of the Schwarz inequality and the estimates above.

$$2E_{t_0,x_0}\left\{\left|\left(\int_t^{t+h}m(s,X(t,\omega))ds+\int_t^{t+h}\sigma(s,X(t,\omega))dZ(s,\omega),\right.\right.\right.$$
$$\left.\left.\left.\int_t^{t+h}\Delta m(s,t,\omega)ds+\int_t^{t+h}\Delta\sigma(s,t,\omega)dZ(s,\omega)\right)\right|\right\}$$
$$\leqslant 2E_{t_0,x_0}\left\{\left\|\int_t^{t+h}m(s,X(t,\omega))ds+\int_t^{t+h}\sigma(s,X(t,\omega))dZ(s,\omega)\right\|_2\cdot\right.$$
$$\left.\cdot\left\|\int_t^{t+h}\Delta m(s,t,\omega)ds+\int_t^{t+h}\Delta\sigma(s,t,\omega)dZ(s,\omega)\right\|_2\right\}$$
$$\leqslant 2E_{t_0,x_0}\left\{\left\|\int_t^{t+h}m(s,X(t,\omega))ds+\int_t^{t+h}\sigma(s,X(t,\omega))dZ(s,\omega)\right\|_2^2\right\}^{1/2}$$
$$\times E_{t_0,x_0}\left\{\left\|\int_t^{t+h}\Delta m(s,t,\omega)ds+\int_t^{t+h}\Delta\sigma(s,t,\omega)dz(s,\omega)\right\|_2^2\right\}^{1/2}$$
$$\leqslant 2(h^2K^2+h)^{1/2}[2(h+n)K^2K'h^2]^{1/2}[E_{t_0,x_0}\{\|X(t,\omega)\|_2^2\}+1]$$
$$=O(h^{3/2})[E_{t_0,x_0}\{\|X(t,\omega)\|_2^2\}+1].$$

We may now combine our results above, interchanging integration and expectation to yield

$$\left|E_{t_0,x_0}\{\|X(t+h,\omega)\|_2^2-\|X(t,\omega)\|_2^2\}-\int_t^{t+h}E_{t_0,x_0}\{\|\sigma(s,X(t,\omega))\|_2^2\}ds\right.$$
$$\left.-2\int_t^{t+h}E_{t_0,x_0}\{(X(t,\omega),\ m(s,X(t,\omega)))\}ds\right| \tag{3.19}$$
$$\leqslant O(h^{3/2})[E_{t_0,x_0}\{\|X(t,\omega)\|_2^2\}+1],$$

which yields the desired result.

Corollary. If there exists a constant $\beta>0$, such that

(3.20) $$\|\sigma(t, \xi)\|_2^2+2(\xi, m(t, \xi))\leqslant-\beta\|\xi\|_2^2,$$

then $E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}$ is exponentially decreasing as $t\to\infty$.

Proof. From (3.12), we have

$$dE_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}\leqslant-\beta E\{\|X(t, \omega)\|_2^2\}\, dt.$$

Thus, integration of

$$d\log E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}=\frac{dE_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}}{E_{t_0, x_0}\{\|X(t, \omega)\|_2^2\}}\leq-\beta\, dt$$

yields the desired result.

Example. Let us consider the simple linear system,

$$dx_1(t, \omega)=[ax_1(t, \omega)+bx_2(t, \omega)]\, dt+\sigma_1 x_1(t, \omega)dZ_1(t, \omega)$$
$$dx_2(t, \omega)=[cx_1(t, \omega)+dx_2(t, \omega)]\, dt+\sigma_2 x_2(t, \omega)dZ_2(t, \omega),$$

where $b+c=0$.

The condition (3.20) requires that there exists $\beta>0$ for which

$$(\sigma_1^2+2a)x_1^2+(\sigma_2^2+2d)y_1^2\leqslant-\beta(x_1^2+y_1^2).$$

In that case, not only do the sample solutions decay exponentially with probability one but the equilibrum solution is stable in the Lyapunov sense with probability one.

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