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Invariants of a group under a semi-reductive action

By

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In a paper [2], we proved that if a group G acts on a ring[1] R which is finitely generated over a pseudo-geometric ring K, and if the action of G is semi-reductive, then for any G-stable ideal $\mathfrak{a}$ of R, the set $I_G(R/\mathfrak{a})$ of G-invariants in $R/\mathfrak{a}$ is a K-algebra of finite type under the following assumption: R is graded, the action of G preserves the gradation and the module of elements of degree zero in R is a finite K-module.

The purpose of the present note is to prove the result without assuming anything on gradation, but assuming a condition on K that if P is a normal local ring which is a ring of quotients of a K-algebra of finite type, then P is analytically irreducible. We really prove it under a weaker condition of the action of G.

1. Notation and the main result.

Let K be, throughout this paper, a pseudo-geometric ring such that every normal locality over a homomorphic image of K is analytically irreducible. Note that any field or a Dedekind domain of

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1) In this note, a ring will mean a commutative ring with identity. When K is a ring, a K-algebra of finite type means that it is a finitely generated ring over the canonical image of K in the ring. A locality over K' means that it is a local ring which is a ring of quotients of a finitely generated integral domain over K'. A normal ring is an integral domain which is integrally closed in its field of quotients.

characteristic zero satisfies this condition. R denotes always a K-algebra of finite type and G a group acting on R as a group of K-automorphisms. For an element f of R, $\mathfrak{a}(f)$ denotes the ideal $\sum_{\sigma\in G}(f^\sigma-f)R$. The notation $I_G(\ )$ stands for the set of G-invariants.

We say that the action of G has *SR-property* if the following condition is satisfied:

For every element f of R, there is a G-invariant F in R of the form $F=f^n+c_1f^{n-1}+\cdots+c_n$ $(n\geq 1,\ c_i\in\mathfrak{a}(f)^i)$.

Note that semi-reductive actions have this property.[2] Now we can state the main theorem as follows:

Main theorem. *If the action of G has SR-property, then* (1) *$I_G(R)$ is a K-algebra of finite type,* (2) *for each prime ideal $\mathfrak{p}$ of $I_G(R)$, there is a prime ideal of R which lies over $\mathfrak{p}$, namely, the natural map from* Spec(R) *into* Spec$(I_G(R))$ *is surjective,*[3] *and* (3) *if $\mathfrak{a}$ is a G-stable ideal in R, then for each element f' of $I_G(R/\mathfrak{a})$, there is a natural number n such that f'^n is in $I_G(R)/(\mathfrak{a}\cap I_G(R))$, hence, in particular, $I_G(R/\mathfrak{a})$ is integral over $I_G(R)/(\mathfrak{a}\cap I_G(R))$.*

2. Proof of the main theorem.

Lemma 1. *If S is an K-algebra that there is an over ring R (which is of finite type by our convention) which is integral over S, then S itself is of finite type.*

This is well known and easily proved, and therefore we omit the proof.

We shall make use of the main result (Theorem 1) in our paper [3].

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2) That SR-property is weaker than semi-reductivity can be seen by the action of the additive group G_a of K on the polynomial ring $K[x]$ defined by $x\to x+b$ (for each $b\in G_a$).

3) Note that this is also equivalent to that for any ideal $\mathfrak{b}$ of $I_G(R)$, $\mathfrak{b}$ and $\mathfrak{b}R\cap I_G(R)$ have the same radical.

Proof of (3). Take an element f of R such that f modulo $\mathfrak{a}=f'$. Then $\mathfrak{a}(f)\subseteq\mathfrak{a}$. Then taking a G-invariant $F=f^n+c_1f^{n-1}+\cdots+c_n$ $(c_i\in\mathfrak{a}(f)\subseteq\mathfrak{a})$, we see that $(F$ modulo $\mathfrak{a})=f'^n\in I_G(R)/(\mathfrak{a}\cap I_G(R))$. This proves (3).

Proof of (2). We have only to show that if $h_1,\cdots,h_s$ are in $I_G(R)$, then every element f of $(\sum h_iR)\cap I_G(R)$ is nilpotent modulo $\sum h_iI_G(R)$. We shall prove this by induction on s. When $s=1$; $f=h_1r$ $(r\in R)$. Since f is G-invariant, $r^\sigma-r\in 0:h_1R$ $(\sigma\in G)$. Thus $\mathfrak{a}(r)h_1=0$. Take G-invariant $r^n+c_1r^{n-1}+\cdots+c_n$ $(c_i\in\mathfrak{a}(r))$. Then $f^n=h_1^nr^n=h_1^n(r^r+c_1r^{n-1}+\cdots+c_n)\in h_1I_G(R)$. When $s>1$; Let ϕ be the natural homomorphism from R onto R/h_1R. Then $\phi(f)\in\sum_{i=2}^s\phi(h_iR)$. Therefore, by induction assumption, $\phi(f^t)\in\sum\phi(h_i)I_G(\phi(R))$. Considering f^t instead of f, we may assume that $f=\sum h_ir_i$ with $\phi(r_i)\in I_G(\phi(R))$ for $i\geq 2$, hence in particular, for $i=s$. Then there is a natural number u such that $\phi(r_s^u)$ is in $\phi(I_G(R))$. Then, considering f^u instead of f, we may assume that $r_s\in I_G(R)$. Then $f-h_sr_s$ is in $(\sum_{i=1}^{s-1}h_iR)\cap I_G(R)$ and, by our induction assumption, we have $f-h_sr_s$ is in the radical of $\sum_{i\leq s-1}h_iI_G(R)$. This proves (2).

Proof of (1). Assume for a moment that there is a pair of R and G such that $I_G(R)$ is not a K-algebra of finite type. Choose such a pair so that the Krull dimension (=altitude) of R is smallest among those R. Thus we may assume that:

(A) For any other pair of R and G, say R' and G', if the Krull dimension of R' is less that that of R, then $I_{G'}(R')$ is of finite type.

Next, take the set of G-stable ideals $\mathfrak{a}$ of R such that $I_G(R/\mathfrak{a})$ is not of finite type. Since R is noetherian, there is a maximal member, say $\mathfrak{a}^*$. Then, considering $R/\mathfrak{a}^*$ instead of R, we may assume furthermore that:

(B) If $\mathfrak{a}$ is a G-stable ideal of R and if $\mathfrak{a}\neq 0$, then $I_G(R/\mathfrak{a})$ is of finite type.

Under the circumstance, we have the following

Lemma 2. *If* $\mathfrak{a}\neq 0$ *is a G-stable ideal of R, then* $I_G(R)/(\mathfrak{a}\cap I_G(R))$ *is a K-algebra of finite type.*

Proof. $I_G(R/\mathfrak{a})$ is integral over $I_G(R)/(\mathfrak{a}\cap I_G(R))$ and is a K-algebra of finite type. Therefore we prove the assertion by Lemma 1.

Now we go back to the proof of (1).

Case I). Assume that there is a non-zero element h of $I_G(R)$ which is a zero-divisor in R. Set $\mathfrak{a}=0: hR$. Then $\mathfrak{a}$ is a G-stable ideal and $\mathfrak{a}\neq 0$. By Lemma 2, both $A_1=I_G(R)/(hR\cap I_G(R))$ and $A_2=I_G(R)/(\mathfrak{a}\cap I_G(R))$ are K-algebra of finite type. Therefore there is a subring A of $I_G(R)$ which is of finite type and such that $A_1=A/(hR\cap A)$ and $A_2=A/(\mathfrak{a}\cap A)$. Since $I_G(R/\mathfrak{a})$ is a finite module over A_2, there is a module basis $\bar{b}_1, \cdots, \bar{b}_t$ for the module. Let $b_1, \cdots, b_t$ be representatives of them in R. We shall show that $I_G(R)=A[hb_1, \cdots, hb_t]$. Since $b_i^\sigma-b_i\in\mathfrak{a}$ for every $\sigma\in G$ and since $h\mathfrak{a}=0$, we see that each hb_i is G-invariant. Therefore we see that $A[hb_1, \cdots, hb_t]\subseteq I_G(R)$. Take, conversely, an arbitrary element f of $I_G(R)$. By our choice of A, there is an element a of A such that $f-a\in hR\cap I_G(R)$. Write $f-a=hb$ $(b\in R)$. Since $f-a$ is G-invariant, we see that $b^\sigma-b\in\mathfrak{a}$ for every $\sigma\in G$. Thus $\bar{b}=b$ modulo $\mathfrak{a}$ belongs to $I_G(R/\mathfrak{a})$. Therefore $\bar{b}=\sum c_i\bar{b}_i$ with $c_i\in A$. Set $b'=\sum c_ib_i$. Then $b-b'\in\mathfrak{a}$ and $hb=hb'$. Thus $f-a=hb'\in A[hb_1, \cdots, hb_t]$. Thus $I_G(R)=A[hb_1, \cdots, hb_t]$ and this is of finite type, which settles the case.

Case II). Now we assume that there is no non-zero element of $I_G(R)$ which is a zero-divisor in R. Hence, in particular, $I_G(R)$ is an integral domain. Let $\mathfrak{r}$ be the radical (=maximal nilpotent ideal) of R. Then $\mathfrak{r}$ is G-stable and $\mathfrak{r}\cap I_G(R)=0$. Therefore if $\mathfrak{r}\neq 0$, then we see that $I_G(R)$ is of finite type by Lemma 2. Thus we assume that $\mathfrak{r}=0$. We show furthermore

Lemma 3. $I_G(R)$ *is noetherian.*

Proof. Let $\mathfrak{b}$ be an arbitrary non-zero ideal of $I_G(R)$ and let

b be a non-zero element of $\mathfrak{b}$. Then $bR \cap I_G(R) = bI_G(R)$ because b is not a zero-divisor. Therefore Lemma 2 shows that $I_G(R)/bI_G(R)$ is of finite type, hence is noetherian. Thus $\mathfrak{b}$ has a finite basis, and $I_G(R)$ is noetherian.

Now we go back again to the proof of (1). Let Q and Q_G be the total quotient ring of R and the field of quotients of $I_G(R)$ respectively. Since non-zero elements of $I_G(R)$ are not zero-divisors in R, we see that Q_G is contained in Q. Let R^* be the integral closure of R in Q and let R_1 be the ring generated by $R^* \cap Q_G$ over R. Since R is pseudo-geometric, R^* is a finite R-module, hence R_1 is also a finite R-module. Furthermore, since R_1 is generated, as a module over R, by a finite number of elements of Q_G, we see that

Lemma 4. *There is a non-zero element h of $I_G(R)$ such that $hR_1 \subseteq R$.*

Now we extend the action of G on R_1. Then we claim that

Lemma 5. *The action of G on R_1 has SR-property.*

Proof. Take an arbitrary element f of R_1. We consider $\mathfrak{a}_1(f) = \sum_{\sigma \in G}(f^\sigma - f)R_1$ and also $\mathfrak{a}(hf)$ (h being as in Lemma 4). Since h is G-invariant, we see that $h\mathfrak{a}_1(f) = \mathfrak{a}(hf)R_1$. Since the action of G has SR-property on R, there is a G-invariant $(hf)^n + c_1(hf)^{n-1} + \cdots + c_n$ with $c_i \in \mathfrak{a}(hf)^i \subseteq h^i\mathfrak{a}_1(f)^i$. Thus we have a G-invariant $f^n + c_1'f^{n-1} + \cdots + c_n'$ with $c_i' = c_i h^{-i} \in \mathfrak{a}_1(f)^i$, and the lemma is proved.

Now, since $I_G(R_1)$ contains $R^* \cap Q_G$, we have $I_G(R_1) = R^* \cap Q_G$. Therefore, by virtue of (2), Lemma 3 (applied to R_1) and by the main theorem of [3], we see that $I_G(R_1)$ is a K-algebra of finite type. Let a be an arbitrary element of $I_G(R_1)$. Then, with h in Lemma 4, we see that $ha \in R \cap Q_G = I_G(R)$. Hence, in particular, $ha^n \in I_G(R)$ for every natural number n. Since $I_G(R)$ is noetherian by Lemma 3, we see that a is integral over $I_G(R)$. Thus $I_G(R_1)$ is integral over $I_G(R)$. Therefore $I_G(R)$ is of finite type by Lemma 1. This completes the proof of our main theorem.

3. Supplementary remarks.

The last step of our proof of the main theorem proves the following:

Proposition 1. *Assume that a group G acts on R and also a ring R' containing R for which there is an element d of $I_G(R)$ such that $dR' \subseteq R$ and d is not a zero-divisor in R'. If $I_G(R)$ is noetherian, then $I_G(R')$ is integral over $I_G(R)$.*

We shall prove here also the following:

Proposition 2. *Assume that the action of G on R has SR-property. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are G-stable ideals in R such that $\mathfrak{a}_1+\mathfrak{a}_2=R$, then there is an $f \in I_G(R)$ such that $f \in \mathfrak{a}_1$ and $f-1 \in \mathfrak{a}_2$. Therefore the natural map* $\mathrm{Spec}(R) \to \mathrm{Spec}(I_G(R))$ *gives a one-one correspondence between the set of all minmal G-stable closed sets in* $\mathrm{Spec}(R)$ *and the set of all maximal ideals in $I_G(R)$.*

Proof. $\mathfrak{a}_1+\mathfrak{a}_2=R$ implies that there is a g in $\mathfrak{a}_1$ such that $1-g \in \mathfrak{a}_2$. $\mathfrak{a}(g)=\mathfrak{a}(1-g)$. Therefore $\mathfrak{a}(g) \subseteq \mathfrak{a}_1 \cap \mathfrak{a}_2$. Take G-invariant $f=g^n+c_1g^{n-1}+\cdots+c_n$ $(c_i \in \mathfrak{a}(g))$. Then we see that $f \in \mathfrak{a}_1$ and $f-1 \in \mathfrak{a}_2$. This proves the assertion.

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