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On spectral functions related to birth and death processes

By

Masatoshi FUKUSHIMA

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§1. Introduction

Let λ_i, $i=0, 1, 2, \cdots$; μ_i, $i=1, 2, \cdots$, be sequences of strictly positive real numbers. We consider a Jacobi matrix

(1) $\quad A=(a_{ij},\ i, j\geqq 0)$, where, $a_{00}=-\lambda_0$, $a_{01}=\lambda_0$,

$a_{ii}=-(\mu_i+\lambda_i)$, $a_{i,i+1}=\lambda_i$, $a_{i,i-1}=\mu_i$, $i=1, 2, \cdots$, and $a_{ij}=0$, if $|i-j|>1$.

Let us fix a positive integer N and consider a matrix $A^0=(a_{ij},\ i, j\geqq N)$ which is obtained from A by deleting its first N columns and N rows.

Our problem is to compare the spectral function corresponding to $-A$ with that corresponding to $-A^0$, and investigate the properties of the former by those of the latter.

In the case that $N=1$, S. Karlin and J. McGregor have introduced a probabilistic method to solve such a problem [3], [5]. For the general N, but in the special case that $\lambda_i=\lambda$, $i\geqq 0$; $\mu_i=i\mu$, $i\leqq N$, $\mu_i=N\mu$, $i\geqq N$, with positive λ, μ, they have also solved this problem and determined the spectral function corresponding to the N-server queueing process with the Poisson input and the exponential service time [4].

Our problem is connected with a physical problem about the vibrating lattice points [1], [6], [7]. Let $\{x_n, n=0, 1, 2, \cdots\}$ be a sequence of strictly increasing numbers. $(x_0=0)$. Assume that a particle with mass m_n is located at x_n, $n=0, 1, 2, \cdots$, and that these particles are connected with a weightless string whose elastic coefficient is uniformly equal to one. We consider small oscillations with x_0 as free boundary and denote by $u_n(t)$ the displacement of the n-th particle at time t. It is clear that the vector $\boldsymbol{u}(t)$, whose components are $u_n(t)$, $n=0, 1, 2, \cdots$, satisfies the equation

$$\ddot{\boldsymbol{u}}(t)=A\boldsymbol{u}(t), \qquad \text{with}$$

$$\lambda_n=\frac{1}{m_n(x_{n+1}-x_n)}, \ n\geqq 0; \quad \mu_n=\frac{1}{m_n(x_n-x_{n-1})}, \ n\geqq 1,$$

If the values of m_n, x_n $(n\leqq N-1)$ are changed, what changes of the asymptotic behaviours of motions may occur? Our results will give a partial answer to this physical question.

Now, let us define the spectral functions of $-A$ and $-A^0$, and state our results about our problem. We shall use the notations in [2], [3].

For simplicity, we shall assume, throughout this paper, the

(2) *assumption.* λ_i, $i=0, 1, 2, \cdots$; μ_i, $i=1, 2, \cdots$, *are bounded.*

If we introduce the sequence

$$\text{(3)} \qquad \Pi_0=1, \quad \Pi_n=\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}, \quad n=1, 2, \cdots,$$

and the Hilbert space $L^2(\Pi)=\{f: \sum_{n\geqq 0} f(n)^2\Pi_n<+\infty\}$ with the inner product $(f, g)=\sum_{n\geqq 0} f(n)g(n)\Pi_n$, then $-Af(\cdot)=-\sum_{j\geqq 0} a_{\cdot j}f(j)$ becomes a bounded, self-adjoint, positive definite operator on $L^2(\Pi)$. We define the spectral function $\psi(x)$ of $-A$ as $\psi(x)=(E_x e, e)$, where E_x is the resolution of identity corresponding to $-A$ and $e(i)=\delta_{i0}$, $i=0, 1, 2, \cdots$. Analogously, we can define the spectral function $\alpha(x)$ of $-A^0$, considering $-A^0$ as a bounded, self-adjoint, positive definite operator acting on $L^2(\Pi^0)=\{f: \sum_{n\geqq 0} f(N+n)^2\Pi^0_{N+n}<+\infty\}$, where

$$\text{(4)} \qquad \Pi^0_N=1, \quad \Pi^0_{N+n}=\frac{\lambda_N\lambda_{N+1}\cdots\lambda_{N+n-1}}{\mu_{N+1}\cdots\mu_{N+n}}, \quad n=1, 2, \cdots.$$

Next, we introduce a system of polynomials $\boldsymbol{Q}_n(s)$, $n \geqq 0$ associated with $-A$, as follows:

$$
\begin{aligned}
&\boldsymbol{Q}_0(s)=1, \\
(5) \qquad &s\boldsymbol{Q}_0(s)=\lambda_0 \boldsymbol{Q}_0(s)-\lambda_0 \boldsymbol{Q}_1(s), \\
&s\boldsymbol{Q}_n(s)=-\mu_n \boldsymbol{Q}_{n-1}(s)+(\lambda_n+\mu_n)\boldsymbol{Q}_n(s)-\lambda_n \boldsymbol{Q}_{n+1}(s), \quad n \geqq 1.
\end{aligned}
$$

Our aim is the proof of the following two theorems. We shall denoted by $\mathfrak{S}$ the spectrum of $-A^0$, that is, the set of points of increase of $\alpha(x)$, which is a compact subset of the non negative real line.

Theorem 1. *In* $[0, +\infty)-\mathfrak{S}$, $\psi(s)$ *increases only with jumps, and* $s \in [0, +\infty)-\mathfrak{S}$ *is a jump point of* $\psi(s)$ *if and only if,*

$$
(6) \qquad \boldsymbol{Q}_N(s)-\mu_N \boldsymbol{Q}_{N-1}(s) R_{NN}^0(s)=0,
$$

where $\displaystyle R_{NN}^0(s)=\int_0^{+\infty} \frac{d\alpha(x)}{x-s}.$

Theorem 2. *If in an interval* $(a, b) \subset \mathfrak{S}$, $d\alpha(x)$ *has a density* $\alpha'(x)$, *which is strictly positive in* (a, b), *and uniformly Hölder continuous in any closed subinterval of* (a, b), *then* $d\psi(x)$ *has also a density* $\psi'(x)$ *in* (a, b) *and*

$$
(7) \qquad \psi'(x)=\frac{1}{\Pi_N} \cdot \\
\times \frac{\alpha'(x)}{\left(\boldsymbol{Q}_N(x)-\mu_N \boldsymbol{Q}_{N-1}(x) \int_{[0,+\infty)}^{*} \frac{1}{y-x} d\alpha(y)\right)^2+\mu_N^2 \pi^2 \boldsymbol{Q}_{N-1}(x)^2 \alpha'(x)^2},
$$

$x \in (a, b)$, *where* $\displaystyle \int_{[0,+\infty)}^{*} \frac{1}{y-x} d\alpha(y)=\lim_{\varepsilon \downarrow 0} \int_{[0,+\infty) \cap \{|y-x|>\varepsilon\}} \frac{d\alpha(y)}{y-x}.$

In §2, we consider the birth and death process $\boldsymbol{X}$ with the generator A, and give some relations between resolvents of A and A^0. In §3, we prove two theorems stated above, using the formula (21) in §2.

In the case that $N=1$, our procedure is nothing but that developed by S. Karlin and J. McGregor [3], who also obtained the formula (21), for general N, by an inductive method [4]. But our procedure makes clear the probabilistic meaning of the denominator of (21) (which is also, up to a constant, the left hand side of (6) in theorem 1). It is, in fact, up to a constant and for negative s, the determinant of a matrix expressing the generator of a Markov process on $\{0, 1, 2, \cdots, N-1\}$, which is obtained from $\boldsymbol{X}$ through killing with rate $-s$ and time change with the sojourn time of $\boldsymbol{X}$ in the set $\{0, 1, 2, \cdots, N-1\}$.

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§2. Relations between resolvents of A and A^0

Put $S=\{0, 1, 2, \cdots\}$ and $S^0=\{N, N+1, \cdots\}$. Considered as an operator on $L^2(\Pi)$, A is bounded, so that following quantities are well defined.

(8) $$P_{ij}(t)=(e^{(i)}, \exp(tA)\cdot e^{(j)})\Pi_j, \quad i, j\in S, \ t>0,$$

where $\quad e^{(i)}(k)=\delta_{ki}/\Pi_i, \quad k=0, 1, 2, \cdots.$

$P_{ij}(t)$ is an unique function satisfying the following conditions.

$$P_{ij}(t)\geqq 0, \ i, j\in S, \ t>0; \ \sum_{j\in S}P_{ij}(t)\leqq 1, \ i\in S, \ t>0;$$
$$P_{ij}(t)\to\delta_{ij}, \ (t\downarrow 0), \ i, j\in S; \quad P(t)P(s)=P(t+s), \ t, s>0;$$
$$\frac{d}{dt}P(t)=AP(t)=P(t)A, \ t>0;$$

where $P(t)$ is the matrix with elements $P_{ij}(t)$, $i, j\in S$. We call $P_{ij}(t)$ the transition function associated with A. Exactly in the same way, we can associate the transition function $P^0_{ij}(t)$, $i, j\in S^0$, $t>0$, with A^0. We put, for $s<0$,

(9) $$\begin{aligned} R_{ij}(s) &=\int_0^{+\infty}e^{st}P_{ij}(t)\,dt, \qquad i, j\in S, \\ R^0_{ij}(s) &=\int_0^{+\infty}e^{st}P^0_{ij}(t)\,dt, \qquad i, j\in S^0, \end{aligned}$$

and call them resolvents of A and A^0 respectively.

Now consider a right continous Markov process $\boldsymbol{X}=\{X_t, \mathfrak{B}_t, \boldsymbol{P}_i, i\in S\}$ such that $\boldsymbol{P}_i(X_t=j)=P_{ij}(t)$, $i, j\in S$, $t>0$. We denote by σ_{N-1} the first passage time to the set $\{N-1\}$ of the process $\boldsymbol{X}$. Then it is easy to see that

$$\boldsymbol{P}_i(X_t=j, t<\sigma_{N-1})=P^0_{ij}(t), \quad i, j\in S^0, \ t>0.$$

Let us prove the fundamental

Lemma 1. *Consider the* $N\times N$ *matrices*

$$(10) \qquad R(s)=(R_{ij}(s),\ 0\leqq i, j\leqq N-1), \quad s<0,$$

$$(11) \qquad U(s)=\begin{array}{c} \\ N-1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ \\ \end{array}\!\!\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\quad\quad\quad}^{N-1}} & \\ -s & & & \vdots \\ & -s & 0 & 0 \\ & 0 & -s & \vdots \\ \hline \multicolumn{3}{c|}{\cdots\cdots 0 \cdots\cdots} & -s+\lambda_{N-1}\mu_N\int_0^{+\infty}(1-e^{st})P^0_{NN}(t)\,dt \end{array}\right), \quad s<0.$$

Then it holds that

$$(12) \qquad R(s)-R(u)+R(s)(U(s)-U(u))R(u)=\mathbf{0}, \quad s<0,\ u<0,$$

where $\mathbf{0}$ is $N\times N$ *zero matrix.*

Proof. A simple analysis of the sample paths of $\boldsymbol{X}$ shows that

$$P_{ij}(t)=\lambda_{N-1}\int_0^t P_{i,N-1}(t')P^0_{Nj}(t-t')\,dt', \quad i\leqq N-1,\ j\geqq N,\ t>0,$$

$$P_{jk}(t)=\mu_N\int_0^t P^0_{jN}(t')P_{N-1,k}(t-t')\,dt', \quad j\geqq N,\ k\leqq N-1,\ t>0.$$

Taking Laplace transforms, we have,

$$(13) \qquad R_{ij}(s)=\lambda_{N-1}R_{i,N-1}(s)R^0_{Nj}(s), \quad i\leqq N-1,\ j\geqq N,\ s<0,$$

$$(14) \qquad R_{jk}(s)=\mu_N R^0_{jN}(s)R_{N-1,k}(s), \quad j\geqq N,\ k\leqq N-1,\ s<0.$$

From (13), (14) and the resolvent equations for $R_{ij}(s)$ and $R^0_{ij}(s)$, it follows that, for $0\leqq i$, $k\leqq N-1$, $s<0$, $u<0$,

$$\begin{aligned} R_{ik}(s)-R_{ik}(u) &= (s-u)\sum_{j\geqq 0}R_{ij}(s)R_{jk}(u) \\ &= (s-u)\sum_{0\leqq j\leqq N-1}R_{ij}(s)R_{jk}(u)+\lambda_{N-1}\mu_N R_{i,N-1}(s)R_{N-1,k}(u) \\ &\quad \times(R^0_{NN}(s)-R^0_{NN}(u)). \end{aligned}$$

Thus, (12) is proved.

Let us put $A_k=(a_{ij}: 0\leqq i, j\leqq k)$, $k\geqq 0$.

Lemma 2. *Consider the matrix*[1)]

$$\mathfrak{G}=\left(\begin{array}{c:c} A_{N-2} & \begin{array}{c} 0 \\ \vdots \\ 0 \\ \lambda_{N-2} \end{array} \\ \hdashline 0\cdots 0\ \mu_{N-1} & -(\lambda_{N-1}+\mu_{N-1}-\lambda_{N-1}\mu_N R^0_{NN}(0)) \end{array}\right). \tag{15}$$

Then we have

$$-(\mathfrak{G}-U(s))R(s)=I_N, \quad s<0,^{2)} \tag{16}$$

where I_N is the N-square identity matrix.

Proof. Since $\lim_{s\to-\infty} sR(s)=-I_N$, $R(s)$ is regular for all s less than some $s_0<0$. We put, for $s<s_0$, $\mathfrak{G}=-R(s)^{-1}+U(s)$, which is independent of $s<s_0$ by virtue of the equation (12). (12) again assures that for all $s<0$ (16) is true. We can calculate elements of $\mathfrak{G}$ by the formulas

$$s(I_N-U(s)R(s))\xrightarrow[s\to-\infty]{}\mathfrak{G},\quad s(1+sR_{ii}(s))\xrightarrow[s\to-\infty]{}a_{ii}$$

and
$$s^2R_{ij}(s)\xrightarrow[s\to-\infty]{}a_{ij},\quad i\neq j.$$

Corollary.

$$\det(R(s))=\frac{1}{\lambda_0\lambda_1\cdots\lambda_{N-1}}\frac{1}{\boldsymbol{Q}_N(s)-\mu_N\boldsymbol{Q}_{N-1}(s)R^0_{NN}(s)},\quad s<0, \tag{17}$$

where $\boldsymbol{Q}_{N-1}$ and $\boldsymbol{Q}_N$ are polynomials defined by (5).

Proof. By (11) and (15), we have

$$\det(-\mathfrak{G}+U(s))=\det(-A_{N-1}-sI_N)-\lambda_{N-1}\mu_N R^0_{NN}(s)\det(-A_{N-2}-sI_{N-1}).$$

Since $\det(-A_k-sI_{k+1})=\lambda_0\lambda_1\cdots\lambda_k\boldsymbol{Q}_{k+1}(s)$,[3)] (17) follows from Lemma 2.

1) When $N=1$, we consider $\mathfrak{G}$ as $\mathfrak{G}=-(\lambda_0-\lambda_0\mu_1R^0_{00}(0))$.

2) (12) and (16) are our version of the formulas obtained by Neveu [8], where more general Markov chains are treated.

3) cf. [1].

Lemma 3. *Let $\widetilde{R}_{ij}(s)$, $i, j \geqq 1$, $s<0$, be the resolvents of $\widetilde{A}=(a_{ij};\ i, j\geqq 1)$ and $\widetilde{R}(s)$ be the $(N-1)$-square matrix $(\widetilde{R}_{ij}(s)$, $1\leqq i, j\leqq N-1)$. Then,*

$$\det((R(s))=R_{00}(s)\det(\widetilde{R}(s)),\quad s<0, \tag{18}$$

Proof. We have

$$\widetilde{R}_{ij}(s)=\boldsymbol{E}_i\left(\int_0^{\sigma_0} e^{st}\chi_{\{j\}}(X_t)\,dt\right)^{1)},\quad s<0,\ i, j\geqq 1,$$

σ_0 being the first passage time to the set $\{0\}$. Applying Dynkin's formula to the Markov time σ_0,

$$R_{i0}(s)=\mu_1\widetilde{R}_{i1}(s)R_{00}(s)\quad i\geqq 1,$$
$$R_{ij}(s)=\widetilde{R}_{ij}(s)+\mu_1\widetilde{R}_{i1}(s)R_{0j}(s)\quad i, j\geqq 1.$$

(18) is an immediate consequence of these equalities.

Define the polynomials $\boldsymbol{P}_n(s)$ $n\geqq 0$ associated with $-\widetilde{A}=-(a_{ij};\ i, j\geqq 1)$ as follows.

$$\begin{aligned}
&\boldsymbol{P}_0(s)=0,\\
&\boldsymbol{P}_1(s)=1,\\
&s\boldsymbol{P}_1(s)=(\mu_1+\lambda_1)\boldsymbol{P}_1(s)-\lambda_1\boldsymbol{P}_2(s),\\
&s\boldsymbol{P}_n(s)=-\mu_n\boldsymbol{P}_{n-1}(s)+(\mu_n+\lambda_n)\boldsymbol{P}_n(s)-\lambda_n\boldsymbol{P}_{n+1}(s),\quad n=2, 3, \cdots.
\end{aligned}\tag{19}$$

If we consider $\widetilde{A}$, $\widetilde{R}(s)$, $\boldsymbol{P}_k(s)$, $k=0, 1, 2, \cdots$, instead of A, $R(s)$, $\boldsymbol{Q}_k(s)$, $k=0, 1, 2, \cdots$, (17) becomes

$$\det(\widetilde{R}(s))=\frac{1}{\lambda_1\cdots\lambda_{N-1}}\frac{1}{\boldsymbol{P}_N(s)-\mu_N\boldsymbol{P}_{N-1}(s)R^0_{NN}(s)}. \tag{20}$$

(17), (18) and (20) imply

Corollary.

$$R_{00}(s)=\frac{1}{\lambda_0}\frac{\boldsymbol{P}_N(s)-\mu_N\boldsymbol{P}_{N-1}(s)R^0_{NN}(s)}{\boldsymbol{Q}_N(s)-\mu_N\boldsymbol{Q}_{N-1}(s)R^0_{NN}(s)},\quad s<0. \tag{21}$$

Remark. Denote by χ_E the indicator function of a subset E of S. Consider the function of the sample functions of $\boldsymbol{X}$,

1) $\boldsymbol{E}_.$ denotes the expectation associated with $\boldsymbol{P}_.$-measure.

$\varphi_t=\int_0^t \chi_{\{0,1,\cdots,N-1\}}(X_s)ds$, and its inverse function $\tau_t=\sup\{t' : \varphi_{t'}\leqq t\}$. Then, it holds that $R_{ij}(s)=\int_0^{+\infty} P_{ij}^s(t)dt$, where $P_{ij}^s(t)=\boldsymbol{E}_i(e^{s\tau_t}\chi_{\{j\}}(X_{\tau_t}))$, $i, j<N$, $s<0$. Lemma 2 means that $\mathfrak{G}-U(s)$ is the generator of the process on $\{0, 1, \cdots, N-1\}$ with the transition function $P_{ij}^s(t)$.

§3. Proof of theorems

By (8), (9) and the definition of $\psi(x)$, we have $R_{00}(s)=\int_0^{+\infty}\frac{d\psi(x)}{x-s}$, $s<0$. We have also, $R_{NN}^0(s)=\int_0^{+\infty}\frac{d\alpha(x)}{x-s}$, $s<0$.

We can extend $R_{00}(s)$ and $R_{NN}^0(s)$ onto the whole complex plane minus the spectrum of ψ and that of α respectively, by analytic continuation. Let $\mathfrak{C}$ be the complex plane and $\mathfrak{S}$ be the spectrum of α, which is a compact set lying on the non-negative real line.

Proof of theorem 1. We have to prove that $R_{00}(s)$ is regular on $\mathfrak{C}-\mathfrak{S}$ except on its simple poles lying on $[0, +\infty)-\mathfrak{S}$, and that these poles are zeros of the denominator of the right hand side of (21). By virtue of the formula (21), we have only to show that the denominator and the numerator of the right hand side of (21) have no common zero on $\mathfrak{C}-\mathfrak{S}$.

This is true, since the definition of the polynomials $\boldsymbol{Q}_n(s)$ and $\boldsymbol{P}_n(s)$ leads us to the following equality,

$$\boldsymbol{Q}_N(s)\boldsymbol{P}_{N-1}(s)-\boldsymbol{P}_N(s)\boldsymbol{Q}_{N-1}(s)=-\prod_{k=1}^{N-1}\frac{\mu_k}{\lambda_k}, \quad N>1. \tag{22}$$

Proof of theorem 2. Let us fix c and d arbitrarily such that $a<c<d<b$. On account of the Stieltjes inversion formula.

$$\frac{\psi(d+)+\psi(d-)}{2}-\frac{\psi(c+)+\psi(c-)}{2}=\frac{1}{\pi}\lim_{\eta\downarrow 0}\int_c^d \mathfrak{Im}\, R_{00}(\xi+i\eta)\,d\xi,$$

it is sufficient to show that, $\mathfrak{Im}\, R_{00}(\xi+i\eta)$ is uniformly bounded in $\xi\in[c, d]$ and $\eta\in(0, \varepsilon)$, with some positive ε, and that, if $\xi\in[c, d]$,

(23) $$\lim_{\eta\downarrow 0}\frac{1}{\pi}\Im m\, R_{00}(\xi+i\eta)$$
$$=\frac{1}{\Pi_N}\frac{\alpha'(\xi)}{\left(Q_N(\xi)-\mu_N Q_{N-1}(\xi)\int^*_{[0,+\infty)}\frac{d\alpha(s)}{s-\xi}\right)^2+\mu_N^2\pi^2 Q_{N-1}(\xi)^2\alpha'(\xi)^2}.$$

1° First of all, we note that $\Re e\, R^0_{NN}(\xi+i\eta)$ is bounded in $\eta>0$, $\xi\in[c,d]$. In fact, for $\xi\in[c,d]$,

$$\Re e\, R^0_{NN}(\xi+i\eta)=\int_a^b\frac{x-\xi}{(x-\xi)^2+\eta^2}\alpha'(x)dx+O(1).$$

$O(1)$ is bounded in $\eta>0$, $\xi\in[c,d]$. Supposing that the function $\alpha'(x)$ vanishes identically on $(-\infty,+\infty)-(a,b)$, we have

$$\int_a^b\frac{x-\xi}{(x-\xi)^2+\eta^2}\alpha'(x)dx=\int_0^{+\infty}\frac{x}{x^2+\eta^2}(\alpha'(\xi+x)-\alpha'(\xi-x))dx.$$

By virtue of our assumption of the Hölder continuity of $\alpha'(x)$, we can choose positive constants K, β and δ such that, for any $\xi\in[c,d]$ and $|x|<\delta$,

(24) $$|\alpha'(\xi+x)-\alpha'(\xi-x)|<K|x|^\beta.$$

Therefore,

(25) $$\left|\int_a^b\frac{x-\xi}{(x-\xi)^2+\eta^2}\alpha'(x)dx\right|<K\int_0^\delta x^{\beta-1}dx+\frac{2}{\delta}\int_a^b\alpha'(x)dx=M<+\infty.$$

M is independent of $\xi\in[c,d]$ and $\eta>0$.

2° Using the above estimate (24) again, we have[1)]

(26) $$\lim_{\eta\downarrow 0}\Re e\, R^0_{NN}(\xi+i\eta)=\lim_{\varepsilon\downarrow 0}\int_{[0,+\infty)\cap\{|x-\xi|>\varepsilon\}}\frac{d\alpha(x)}{x-\xi},\quad \xi\in[c,d].$$

Moreover, since $\alpha'(x)$ is continuous in (a,b),

(27) $$\Im m\, R^0_{NN}(\xi+i\eta)\underset{\eta\downarrow 0}{\longrightarrow}\pi\alpha'(\xi),\quad \text{uniformly in } \xi\in[c,d].$$

3° From (21), we have, for $s=\xi+i\eta$,

(28) $$\Im m\, R_{00}(\xi+i\eta)=\frac{1}{\lambda_0}\frac{\Pi(s)+O(\eta)}{I(s)+O(\eta)},$$

1) cf. [9].

where,

$$I(s)=(\mathfrak{Re}\boldsymbol{Q}_N(s)-\mu_N\mathfrak{Re}\boldsymbol{Q}_{N-1}(s)\mathfrak{Re}R^0_{NN}(s))^2$$
$$+\mu_N^2(\mathfrak{Re}\boldsymbol{Q}_{N-1}(s)\mathfrak{Im}R^0_{NN}(s))^2,$$
$$II(s)=\mu_N(\mathfrak{Re}\boldsymbol{P}_N(s)\mathfrak{Re}\boldsymbol{Q}_{N-1}(s)-\mathfrak{Re}\boldsymbol{Q}_N(s)\mathfrak{Re}\boldsymbol{P}_{N-1}(s))\mathfrak{Im}R^0_{NN}(s).$$

Because of the boundedness of $\mathfrak{Re}R^0_{NN}(s)$ and $\mathfrak{Im}R^0_{NN}(s)$, $\dfrac{O(\eta)}{\eta}$ is bounded in $\eta>0$ and $\xi\in[c,d]$.

Now, (23) is an immediate consequence of (28), (27), (26) and (22).

Finally, since $II(\xi+i\eta)$ converges uniformly in $\xi\in[c,d]$ as η tends to zero, the uniform boundedness of $\mathfrak{Im}R_{00}(\xi+i\eta)$ will be assured by the next assertion.

4°. There exist strictly positive constants p and ε, and, for all $\xi\in[c,d]$, $\eta<\varepsilon$, it holds that,

$$|I(\xi+i\eta)|>p. \tag{29}$$

In order to prove (29), let $(c<)\xi_0<\xi_1<\cdots<\xi_k(<d)$ be roots of $\boldsymbol{Q}_{N-1}(\xi)=0$ belonging to the interval $[c,d]$. (Without loss of the generality, we may assume that c and d are no roots of $\boldsymbol{Q}_{N-1}(\xi)=0$). Since the zeros of $\boldsymbol{Q}_N(\xi)$ seperate those of $\boldsymbol{Q}_{N-1}(\xi)$, $\boldsymbol{Q}_N(\xi_i)\neq 0$, $i=0,1,2,\cdots,k$, and we can choose $\delta'>0$, $\varepsilon'>0$ and $p'>0$ such that, for all $\xi\in\bigcup_{i=0}^{k}(\xi_i-\delta',\xi_i+\delta')$, $\eta<\varepsilon'$,

$$I(\xi+i\eta)\geqq(\mathfrak{Re}\boldsymbol{Q}_N(\xi+i\eta)-\mu_N\mathfrak{Re}\boldsymbol{Q}_{N-1}(\xi+i\eta)\mathfrak{Re}R^0_{NN}(\xi+i\eta))^2$$
$$\geqq(|\mathfrak{Re}\boldsymbol{Q}_N(\xi+i\eta)|-\mu_N M'|\mathfrak{Re}\boldsymbol{Q}_{N-1}(\xi+i\eta)|)^2>p',$$

where M' is a suitable constant by 1°. On the other hand, the positivity of $\alpha'(\xi)$ and the uniform convergence of (27) assure the existance of $\varepsilon''>0$ and $p''>0$ such that $\mathfrak{Im}R^0_{NN}(\xi+i\eta)>p''$, for all $\xi\in[c,d]$ and $\eta<\varepsilon''$. Thus,

$$I(\xi+i\eta)\geqq\mu_N^2(\mathfrak{Re}\boldsymbol{Q}_{N-1}(\xi+i\eta))^2(\mathfrak{Im}R^0_{NN}(\xi+i\eta))^2$$
$$\geqq\mu_N^2(\mathfrak{Re}\boldsymbol{Q}_{N-1}(\xi+i\eta))^2p''^2,\quad \xi\in[c,d],\ \eta<\varepsilon'',$$

and, if we take $\varepsilon>0$ sufficiently small $(\varepsilon'>\varepsilon,\ \varepsilon''>\varepsilon)$,
we have for all $\xi\in[c,d]-\bigcup_{i=0}^{k}(\xi_i-\delta',\xi_i+\delta')$ and $\eta<\varepsilon$,

$I(\xi+i\eta)>p'''$, with some constant $p'''>0$.

Take $p=\min(p', p''')$. The proof of (29) is completed.

References

[1] F. V. Atkinson, Discrete and continuous boundary problems, Academic press (1964).

[2] S. Karlin and J. McGregor, The differential equations of birth-and-death processes, and the Stieltjes moment problem, Trans. Amer. Math. Soc., 8Ș (1957), 489-546.

[3] ——————, The classification of birth and death processes, Trans. Amer. Math. Soc. 86 (1957), 366-400.

[4] ——————, Many server queueing processes with Poisson input and exponential service times, Pacific J. Math., 8 (1958) 87-118.

[5] ——————, Random walks, Illinois J. Math., 3 (1959) 66-81.

[6] T. Kotera, Localized vibration and random walk, Prog. Theor. Phys. Suppl., 23 (1962), 141-156.

[7] E. W. Montroll, Theory of the vibration of simple cubic lattices with nearest neighbor interactions, Proceedings of the Third Berkeley Simposium on Statistics and Probability, vol 3 (1956), 209-246.

[8] J. Neveu, Une generalisation des processus á accroissements positifs independents, Abh. Math. Sem. Univ. Hamburg., (1961), 36-61.

[9] E. C. Titchmarsh, Theory of Fourier integrals, Oxford (1948).