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A note on stunted lens space

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The purpose of the present note is to show some results on stunted lens spaces analogous to those on stunted projective spaces [5], [4], [3]. Throughout the note a prime p will always be odd.

Let S^{2n+1} be the unit $(2n+1)$-sphere, each point of which is represented by a sequence $(c_0, c_1, \cdots, c_n)$ of complex numbers c_i with $\sum|c_i|^2=1$. $S^1=U(1)$ is a group operating on S^{2n+1} by the formula $c.(c_0, \cdots, c_n)=(c.c_0, \cdots, c.c_n)$. Let p be an odd prime and let Z_p be a subgroup of S^1 generated by $e^{2\pi i/p}$. Then

$$L^n(p)=S^{2n+1}/Z_p \quad \text{and} \quad CP(n)=S^{2n+1}/S^1$$

are the $(2n+1)$-dimensional lens space and the complex projective space of complex n-dimension respectively. Let $\{c_0, \cdots, c_n\}\in L^n(p)$ denote the class of $(c_0, \cdots, c_n)\in S^{2n+1}$. The space $L^k(p)$, $k\leq n$, is naturally imbedded in $L^n(p)$ by identifying $\{c_0, \cdots, c_k\}$ with $\{c_0, \cdots, c_k, 0, \cdots, 0\}$. Denote $L_0^k(p)=\{\{c_0, \cdots, c_k\}\in L^k(p)\mid c_k:$ real, $c_k\geq 0\}$. Then $L^k(p)-L_0^k(p)$ and $L_0^k(p)-L^{k-1}(p)$, $k\leq n$, are $(2k+1)$- and $2k$-cells which make $L^n(p)$ a finite CW-complex.

For the class $\alpha\in KO(X)$ of a real s-vector bundle over a finite CW-complex X, X^α will denote the associated Thom complex, i.e., a mapping cone of an $(s-1)$-sphere bundle p: $E\rightarrow X$ associated with α. A cellular decomposition $X=\sum e_i^{n_i}$ of X gives naturally a cellular decomposition $X^\alpha=e^0+\sum e_i^{s+n_i}$ of X^α.

Denote by $\xi(\in K(CP(n)))$ the canonical line bundle (or its class) of $CP(n)$, by $r(\xi)(\in KO(CP(n)))$ the real restriction of ξ.

Let

$$\pi: L^n(p) \longrightarrow CP(n)$$

be the natural map and $\pi^* r(\xi) = r(\pi^* \xi)$ $(\in KO(L^n(p)))$ the induced bundle of $r(\xi)$.

Theorem 1. *There exists a cellular homeomorphism between the stunted lens space $L^n(p)/L^{n-k-1}(p)$ and the Thom complex $(L^k(p))^{(n-k)\pi^* r(\xi)}$.*

It follows directly

Corollary. *We have the following cellular homeomorphisms:*

$$L_0^n(p)/L^{n-k-1}(p) \approx (L_0^k(p))^{(n-k)\pi^* r(\xi)},$$
$$L_0^n(p)/L_0^{n-k}(p) \approx (L_0^k(p))^{(n-k)\pi^* r(\xi)}/S^{2n-2k},$$
$$L^n(p)/L_0^{n-k}(p) \approx (L^k(p))^{(n-k)\pi^* r(\xi)}/S^{2n-2k},$$

in the first two homeomorphisms, π stands for the restriction of the natural map π: $L^k(p) \to CP(k)$ on $L_0^k(p)$.

Let $J(X)$ be the J-group of X and J: $KO(X) \to J(X)$ the projection (J-homomorphism).

Theorem 2. *$J(L_0^n(p)) \approx Z + \widetilde{J}(L_0^n(p))$ and $\widetilde{J}(L_0^n(p))$ is a cyclic group of order $p^{\left[\frac{n}{p-1}\right]}$ generated by $J(\pi^* r(\xi) - 2)$. $J(L^n(p)) \approx J(L_0^n(p))$ if $n \not\equiv 0$ (mod 4), $\approx J(L_0^n(p)) + Z_2$ if $n \equiv 0$ (mod 4) and the order of $J(\pi^* r(\xi) - 2)$ in $J(L^n(p))$ is $p^{\left[\frac{n}{p-1}\right]}$.*

The following two theorems are easy consequences of the above theorems and the section 2 of [5].

Theorem 3. *If $m \geq n$ and $m \equiv n$ (mod $p^{\left[\frac{k}{p-1}\right]}$) then $L^m(p)/L^{m-k-1}(p)$ has the same stable homotopy type of $L^n(p)/L^{n-k-1}(p)$. Same is true for $L_0^m(p)/L^{m-k-1}(p)$, $L_0^m(p)/L_0^{m-k}(p)$ and $L^m(p)/L_0^{m-k}(p)$.*

Theorem 4. *$L^n(p)/L^{n-k-1}(p)$ (resp. $L_0^n(p)/L^{n-k-1}(p)$, $L_0^n(p)/L_0^{n-k}(p)$, $L^n(p)/L_0^{n-k}(p)$) is stable homotopy equivalent to $L^k(p)^+$ (resp. $L_0^k(p)^+$, $L_0^k(p)$, $L^k(p)$) if and only if $n \equiv k$ (mod $p^{\left[\frac{k}{p-1}\right]}$), where X^+ denotes the disjoint union of X and a point.*

Proof of Theorem 1. Denote by T_{n+1} the center of $U(n+1)$. T_{n+1} consists of diagonal matrices of same diagonal elements and is isomorphic to $U(1)$. Let $T_{n+1}(p)$ be the subgroup of T_{n+1} corresponding to Z_p. Let $L(n, k)$ and $C(n, k)$ be quotient spaces of $U(n-k)\times U(k+1)\,(\subset U(n+1))$ indicated in the following diagrams, where the maps are all natural projections.

$$\begin{array}{ccc}
S^{2n-2k-1}\times S^{2k+1}=\dfrac{U(n-k)\times U(k+1)}{U(n-k-1)\times 1\times U(k)\times 1} & \xrightarrow{p_0} & \dfrac{U(n-k)\times U(k+1)}{U(n-k)\times U(k)\times 1}=S^{2k+1} \\
\downarrow \bar{\pi}_0 & & \downarrow \pi_0 \\
L(n, k)=\dfrac{U(n-k)\times U(k+1)}{(U(n-k-1)\times 1\times U(k)\times 1).T_{n+1}(p)} & \xrightarrow{p} & \dfrac{U(n-k)\times U(k+1)}{U(n-k)\times U(k)\times Z_p}=L^k(p) \\
\downarrow \bar{\pi} & & \downarrow \pi \\
C(n, k)=\dfrac{U(n-k)\times U(k+1)}{(U(n-k-1)\times 1\times U(k)\times 1).T_{n+1}} & \xrightarrow{p'} & \dfrac{U(n-k)\times U(k+1)}{U(n-k)\times U(k)\times U(1)}=CP(k).
\end{array}$$

In the diagram p_0, p and p' are $(2n-2k-1)$-sphere bundles of structure group $U(n-k)$ $(=U(n-k)\times 1)$. Moreover $\bar{\pi}_0$ and $\bar{\pi}$ are bundle maps since the operation of $U(n-k)$ on $S^{2n-2k-1}\times S^{2k+1}$, $L(n, k)$ and $C(n, k)$ commutes with $\bar{\pi}_0$ and $\bar{\pi}$.

Consider the case $n=k+1$. Then p_0 defines a homeomorphism $\bar{p}_0$: $C(k+1, k)\approx S^{2k+1}$ which makes the bundle p': $C(k+1, k)\to CP(k)$ equivalent to the canonical $U(1)$-bundle $\pi\circ\pi_0$: $S^{2k+1}\to CP(k)$ associated with ξ. For general $n>k$, it is verified directly that the bundle p' is associated with $(n-k)$-fold Whitney sum $\xi\oplus\cdots\oplus\xi=(n-k)\xi$ of ξ. It follows that the bundle p is associated with $(n-k)\pi^*\xi$, or with $(n-k)\pi^*r(\xi)$ under the real restriction. The correspondence

$$(A, B, t)\to(\cos(\pi t/2)A,\ \sin(\pi t/2)B),$$
$$A\in U(n-k),\ B\in U(k+1),\ t\in I=[0, 1],$$

defines a map f: $L(n, k)\times I\to L^n(p)$ such that $f(L(n, k)\times 0)\subset L^{n-k-1}(p)$, $f(x, 1)=i'(p(x))$ for $x\in L(n, k)$ and for an imbedding i': $L^k(p)\to L^n(p)$ given by $i'\{c_0, \cdots, c_k\}=\{0, \cdots, 0, c_0, \cdots, c_k\}$ and f maps $L(n, k)\times(0, 1)$ homeomorphically onto $L^n(p)-L^{n-k-1}(p)-i'L^k(p)$. Thus f defines a homeomorphism of the mapping cone

$(L^k(p))^{(n-k)\pi^*r(\xi)}$ of p onto $L^n(p)/L^{n-k-1}(p)$.

Obviously f is cellular with respect to the given cellular decompositions.

Proof of Theorem 2. In [6], one of the authors has proved the following results (p: odd prime):

(1) *$K(L_0^n(p))$ is a ring generated by $\sigma=\pi^*\xi-1$ with a system of relations*: $(1+\sigma)^p=1$ *and* $\sigma^{n+1}=0$.

(2) *$K(L_0^n(p))$ is the direct sum of cyclic groups generated by $1, \sigma, \cdots, \sigma^{p-1}$. Let $n=r(p-1)+s$, $0\leq s<p-1$, then the order of σ^i is p^{r+1} if $0<i\leq s$ and p^r if $s<i\leq p-1$. Thus the order of $\widetilde{K}(L_0^n(p))$ is p^n.*

(3) *The real restriction $r: \widetilde{K}(L_0^n(p))\rightarrow\widetilde{KO}(L_0^n(p))$ is an epimorphism and the element $\bar{\sigma}=r(\sigma)=\pi^*r(\xi)-2$ generates multiplicatively the ring $KO(L_0^n(p))$.*

First consider the kernel of r in (3). Let $c: KO\rightarrow K$ and $t: K\rightarrow K$ be the complexification and the conjugation, then $1+t=c\circ r$ and $r=r\circ t$. By use of (1) we have $t(1+\sigma)=\pi^*t(\xi)=\pi^*\xi^{-1}=(1+\sigma)^{-1}=(1+\sigma)^{p-1}$ and $t((1+\sigma)^j)=(1+\sigma)^{p-j}$. Thus $r((1+\sigma)^j-(1+\sigma)^{p-j})=0$. Conversely, assume $r(y)=0$ for some $y\in K(L_0^n(p))$. Obviously $y\in\widetilde{K}(L_0^n(p))$. By (2), $x=y/2$ exists. Then $y+t(y)=c(r(y))=0$ and $y=y/2+y/2=y/2-t(y/2)=x-t(x)$.

By (1) x is a linear combination of $(1+\sigma)^j$, $0\leq j\leq p-1$. Thus y is a linear combination of $(1+\sigma)^j-(1+\sigma)^{p-j}$. We have obtained

(4). *The kernel of $r: K(L_0^n(p))\rightarrow KO(L_0^n(p))$ is generated* (additively) *by* $(1+\sigma)^j-(1+\sigma)^{p-j}$, $0<j<p$.

Adams has defined groups $J'(X)=J''(X)$ [3: Theorem (1.1)] for finite CW-complexes X. $J''(X)$ is defined by

$$J''(X)=KO(X)/\sum_k(\bigcap_e k^e(\Psi^k-1)KO(X))$$

and gives a lower bound of $J(X)$, i.e., the following diagram (5) is commutative:

$$\begin{array}{ccc} KO(X) & \xrightarrow{J} & J(X) \\ & \searrow J'' \quad \rho \swarrow & \\ & J''(X)=J'(X), & \end{array} \tag{5}$$

where J'' is the natural projection and ρ is an epimorphism.

We shall compute the groups $J''(L_0^n(p))$. Since Ψ^k commutes with the real restriction $r:K(L_0^n(p))\to KO(L_0^n(p))$ [4: Lemma A.2], the kernel of the epimorphism

$$J''\circ r\colon \widetilde{K}(L_0^n(p))\longrightarrow \widetilde{J}''(L_0^n(p))$$

is generated by the elements of (4) and $\bigcap_e k^e(\Psi^k-1)K(L_0^n(p))$. Since $\pi^*\xi=1+\sigma$ is a line bundle and since Ψ^k is a ring homomorphism, we have

$$\Psi^k(1+\sigma)=(1+\sigma)^k \text{ and } \Psi^k((1+\sigma)^j)=(1+\sigma)^{kj}.$$

By (2), the order of $K(L_0^n(p))$ is p^n. It follows that $\bigcap k^e(\Psi^k-1)K(L_0^n(p))=0$ if $k\equiv 0 \pmod p$ and $\bigcap k^e(\Psi^k-1)K(L_0^n(p))=(\Psi^k-1)K(L_0^n(p))$ is generated by $(1+\sigma)^{kj}-(1+\sigma)^j$ if $k\not\equiv 0 \pmod p$. Thus the kernel of the epimorphism $J''\circ r$ is generated by

$$(1+\sigma)^j-(1+\sigma), \quad 1<j<p, \tag{*}$$

since $(1+\sigma)^{j+pi}=(1+\sigma)^j$ by (1). Adams has also proved [1: Theorem (1.3)]

(6) *Assume that $x\in KO(X)$ is a linear combination of* $O(1)$- *and* $O(2)$*-bundles. Then, for each k, there is an integer $e>0$ such that $J(k^e(\Psi^k-1)x)=0$.*

This is true for $x=r(1+\sigma)$ since $r(1+\sigma)=\pi^*r(\xi)$ is an $O(2)$-bundle. That is $J(k^e(\Psi^k-1)r(1+\sigma))=(J\circ r)(k^e\Psi^k-1)(1+\sigma))=0$. For $k\not\equiv 0 \pmod p$ we have seen that this implies

$$(J\circ r)((\Psi^k-1)(1+\sigma))=(J\circ r)((1+\sigma)^j-(1+\sigma))=0.$$

Thus the elements (*) vanish under $J\circ r$. From the commutativity of the diagram (5) we obtain

(7) *The elements* (*) *generates the kernel of the epimorphism* $J\circ r\colon \tilde{K}(L_0^n(p))\to\tilde{J}(K_0^n(p))$ *and* $J(L_0^n(p))=J''(L_0^n(p))$.

As linear combinations of (*), we have $(1+\sigma)^k-(1+\sigma)^{k-1}=\sigma(1+\sigma)^{k-1}$, $1<k<p$; $\sigma(1+\sigma)^{k-1}-\sigma(1+\sigma)^{k-2}=\sigma^2(1+\sigma)^{k-2}$, $2<k<p$; and so on. Thus the kernel of $J\circ r$ contains

$$(**)\qquad \sigma^{k-1}(1+\sigma)=\sigma^{k-1}+\sigma^k,\quad 1<k<p.$$

Conversely, the elements (*) are linear combinations of (**). Thus the elements (**) generates the kernel of $J\circ r$. Then it follows from (2) and (7)

Proposition 1. *$J(L_0^n(p))$ is the direct sum of two cyclic groups generated by $1=J(1)$ and $J(r(\sigma))=-J(r(\sigma^2))=\cdots=-J(r(\sigma^{p-1}))$ and the order of $J(r(\sigma))=J(\pi^*r(\xi)-2)$ coincides with that of σ^{p-1}, i.e.,* $p^{\left[\frac{n}{p-1}\right]}$.

Next consider the following exact and commutative diagram (see [6]):

$$\begin{array}{ccccccc}
 & & 0\longrightarrow & K(L^n(p)) & \xrightarrow{i^*} & K(L_0^n(p)) & \longrightarrow 0\\
 & & & \downarrow r & & \downarrow r & \\
0\longrightarrow & \tilde{KO}(S^{2n+1}) & \xrightarrow{\pi^*} & KO(L^n(p)) & \xrightarrow{i^*} & KO(L_0^n(p)) & \longrightarrow 0,
\end{array}$$

where $\tilde{KO}(S^{2n+1})\approx Z_2$ if $n\equiv 0 \pmod 4$ and $=0$ if $n\not\equiv 0 \pmod 4$. The orders of $\tilde{KO}(S^{2n+1})$ and $\tilde{K}(L^n(p))$ are relatively prime. It follows that $KO(L^n(p))$ is the direct sum of $r(K(L^n(p))$ and $\pi^*\tilde{KO}(S^{2n+1})$ and i^* maps $r(K(L_u(p))$ isomorphically onto $K(L_0^n(p))$. Moreover, the operator Ψ^k-1 splits by the naturality. Thus $J''(L^n(p))\approx J''(L_0^n(p))+\tilde{J}''(S^{2n+1})$ and the kernel of $J\circ r\colon K(L^n(p))\to J(L^n(p))$ contains the elements (**). It is known [2: Example (3.5)] $\tilde{KO}(S^{2n+1})=\tilde{J}''(S^{2n+1})$ hence $=\tilde{J}(S^{2n+1})$. Then it follows

$$J(L^n(p))=J''(L^n(p))$$

and

Proposition 2. *$J(L^n(p)) \approx J(L_0^n(p)) + \widetilde{KO}(S^{2n+1})$ and the order of $J(r(\sigma)) = J(\pi^* r(\xi) - 2) \in J(L^n(p))$ is $p^{\left[\frac{n}{p-1}\right]}$.*

These two propositions establish the proof of Theorem 2.

Proof of Theorem 3 *and Theorem* 4. By Theorem 2, if $m \equiv n$ (mod $p^{\left[\frac{k}{p-1}\right]}$) then $J(2(m-n)) = J(2(m-n) + (m-n)(\pi^* r(\xi) - 2)) = J((m-n)\pi^* r(\xi))$ and $J((n-k)\pi^* r(\xi) + 2(m-n)) = J((m-k)\pi^* r(\xi))$ in $J(L^k(p))$. By Lemma (2.4) and Proposition (2.6) of [5], $(L^k(p))^{(n-k)\pi^* r(\xi)}$ and $(L^k(p))^{(m-k)\pi^* r(\xi)}$ have the same stable homotopy type. Then Theorem 3 follows from Theorem 1. Similarly, Theorem 4 is proved by use of Proposition (2.9) of [5].

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