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Non-embeddability of lens spaces mod 3

By

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§1. Introduction

In [2] and [3], W. S. Massey obtained some results on non-embeddability of projective spaces. The purpose of this note is to prove some non-embeddability theorems of lens spaces mod 3 by making use of Massey's technique. Here we shall use reduced power operations mod 3 $\mathcal{P}_3^i$ and Pontrjagin classes mod 3 in stead of squaring operations Sq^i and Stiefel-Whitney classes.

Let S^{2n+1} be the unit $(2n+1)$-sphere. A point of S^{2n+1} is represented by a sequence $(z_0, z_1, \cdots, z_n)$, where z_i $(i=0, 1, \cdots, n)$ are complex numbers with $\sum_{i=0}^{n} |z_i|^2=1$. Let p be an integer >1 and γ be the rotation of S^{2n+1} defined by

$$\gamma(z_0, z_1, \cdots, z_n) = (e^{2\pi i/p}\cdot z_0, e^{2\pi i/p}\cdot z_1, \cdots, e^{2\pi i/p}\cdot z_n).$$

Let Γ denote the topological transformation group of S^{2n+1} of order p generated by γ. Then the lens space mod p is defined to be the orbit space:

$$L^n(p) = S^{2n+1}/\Gamma .$$

Let M^n be an n-dimensional differentiable manifold and R^k be the k-dimensional Euclidean space. By $M^n \subset R^k$ (respectively $M^n \not\subset R^k$) we mean that M^n is embeddable (respectively non-embeddable) differentiably in R^k.

In §2 we shall recall some properties of lens spaces and in §3 we shall state a mod p analogy of the theorem of W. S. Massey. Our main theorems, which will be proved in §4, are the followings.

Theorem 2. *Let m be a positive integer and let $n=3^m$. Then $L^n(3)\not\subset R^{3n+1}$.*

Theorem 3. *Let l and k be integers such that $l\geqq k\geqq 0$ and $l\geqq 1$, and let $n=3^l+3^k$. Then $L^n(3)\not\subset R^{3n}$.*

Recently, T. Kambe has obtained in [1] the following non-embeddability theorem of lens spaces as an application of the calculation of K-rings of $L^n(p)$.

Theorem (Kambe). *Let p be an odd prime. Then $L^n(p)\not\subset R^{2n+2L(n,p)+1}$, where $L(n,p)$ is the integer defined by*

$$L(n,p) = \max\left\{i\leqq[n/2]\,\middle|\,\binom{n+i}{i}\not\equiv 0 \text{ and } p^{1+[(n-2i)/(p-1)]}\right\}$$

From this theorem, we have $L^n(3)\not\subset R^{3n}$ if $n=3^m$ and $L^n(3)\not\subset R^{3n-1}$ if $n=3^l+3^k$.

§ 2. Lens spaces

In the following, we assume that p is an odd prime. It is known that the integral cohomology groups of $L^n(p)$ is given by

$$H^i(L^n(p);Z) = \begin{cases} Z_p, & i = 2,4,\cdots,2n\,, \\ Z\,, & i = 0,\ 2n+1\,, \\ 0\,, & \text{for other } i\,. \end{cases}$$

The cohomology algebra over Z_p of $L^n(p)$ is summarized in the following proposition (cf. for example, [4], p. 68, Corollary 5.3).

Proposition 1. *$H^*(L^n(p);Z_p)$ is the tensor product of the exterior algebra on a generator $y\in H^1(L^n(p);Z_p)$ and the truncated polynomial algebra on a generator $x\in H^2(L^n(p);Z_p)$ with relations $y^2=0$, $\Delta y=-x$ and $x^{n+1}=0$, where*

$$\Delta: H^q(L^n(p);Z_p)\to H^{q+1}(L^n(p);Z_p)$$

is the Bockstein coboundary operator associated with the exact coefficient sequence:

$$0\to Z_p\to Z_{p^2}\to Z_p\to 0\,.$$

Let $q: S^{2n+1} \to L^n(p)$ be a natural projection. We represent a point of the complex n-dimensional projective space CP^n as a sequence $[z_0, z_1, \cdots, z_n]$, where z_i $(i=0, 1, \cdots, n)$ are complex numbers. We can define a map

$$\pi: L^n(p) \to CP^n$$

by $\pi(q(z_0, z_1, \cdots, z_n)) = [z_0, z_1, \cdots, z_n]$. Then π is a projection of a locally trivial fibre space $(L^n(p), \pi, CP^n)$ with fibre S^1. From the Gysin's exact sequence of this fibre space we can see that the induced homomorphism $\pi^*: H^{2i}(CP^n; Z_p) \to H^{2i}(L^n(p); Z_p)$ $(2i < 2n+1)$ is an epimorphism, and that $\pi^* z^i = x^i$, where $z \in H^2(CP^n; Z_p)$ is a generator of the truncated polynomial algebra $H^*(CP^n; Z_p)$ such that $z^{n+1}=0$.

Let $\tau(L^n(p))$ and $\tau(CP^n)$ denote tangent bundles of $L^n(p)$ and CP^n respectively, and let $\pi^! \tau(CP^n)$ denote the bundle induced from $\tau(CP^n)$ by the map π. Then, as is well known,

$$\tau(L^n(p)) = \pi^! \tau(CP^n) \oplus \alpha_\pi .$$

where α_π is the bundle along the fibre and $\oplus$ denotes the Whitney sum. Since $\pi_1(L^n(p)) = Z_p$ and the dimension of α_π is 1, we have $\alpha_\pi = 1$, the 1-dimensional trivial bundle over $L^n(p)$. Hence, we have

$$\tau(L^n(p)) = \pi^! \tau(CP^n) \oplus 1 .$$

It is well known that the total Pontrjagin class mod p of CP^n is determined by the equation:

$$p(CP^n) = (1+z^2)^{n+1} .$$

Since the cohomology groups of $L^n(p)$ have no elements of order 2, we have the following propositions.

Proposition 2. *The total Pontrjagin class mod p of $L^n(p)$ is given by the equation:*

$$p(L^n(p)) = (1+x^2)^{n+1} .$$

Proposition 3. *The dual total Pontrjagin class mod p of $L^n(p)$ is given by the equation:*

$$\bar{p}(L^n(p)) = (1+x^2)^{-n-1} = \sum_{i=0}^{[n/2]}(-1)^i\binom{n+i}{i}x^{2i}.$$

In order to calculate the binomial coefficients mod p we shall use the following theorem of Lucas.

Proposition 4. *Let $a=\sum_i a_i p^i$ and $b=\sum_i b_i p^i$ be p-adic expansions of a and b respectively, where p is a prime and $0\leqq a_i$, $b_i<p$. Then*

$$\binom{a}{b} \equiv \prod_i \binom{a_i}{b_i} \qquad (\mathrm{mod}\ p).$$

§ 3. Invariants of a sphere bundle

Let B be a compact connected CW-complex, and let (E, p, B, S^{k-1}) be a $(k-1)$-sphere bundle over B whose characteristic class, $W_k \in H^k(B; Z)$, is zero. Then its Gysin sequence breakes up into pieces of length 3, as follows:

$$0 \to H^q(B;G) \xrightarrow{p^*} H^q(E;G) \xrightarrow{\psi} H^{q-k+1}(B;G) \to 0,$$

where G is a coefficient ring ($G=Z_2$, in non-orientable case). The map ψ satisfies the following property ([2], Lemma 1).

Proposition 5. *If $x\in H^q(B;G)$ and $y\in H^r(E;G)$, then*

$$\psi(y\cdot p^*x) = (-1)^{kq}(\psi y)\cdot x.$$

Let $a\in H^{k-1}(E;G)$ be an element such that $\psi(a)=1$. Then the following direct sum decomposition holds ([2], § 8).

$$H^q(E;G) = p^*H^q(B;G) + a\cdot p^*H^{q-k+1}(B;G).$$

Thus, given any element $u\in H^q(E;G)$, there exist unique elements $u_1\in H^q(B;G)$ and $u_2\in H^{q-k+1}(B;G)$ such that

$$u = p^*u_1 + a\cdot p^*u_2.$$

Now, let $G=Z_p$ and let

$$\mathcal{P}^i = \mathcal{P}^i_p : H^q(E;Z_p) \to H^{q+2i(p-1)}(E;Z_p)$$

be the i-th Steenrod reduced power operation. Then there are unique elements $\alpha_i \in H^{k-1+2i(p-1)}(B; Z_p)$ and $\beta_i \in H^{2i(p-1)}(B; Z_p)$ such that

$$\mathcal{P}^i a = p^*\alpha_i + a \cdot p^*\beta_i .$$

If a' is another element of $H^{k-1}(E; Z_p)$ such that $\psi(a')=1$, then by exactness of the Gysin sequence there is an element $b \in H^{k-1}(B; Z_p)$ such that $p^*(b) = a' - a$. Analogously we have unique elements α_i' and β_i' such that

$$\mathcal{P}^i a' = p^*\alpha_i' + a' \cdot p^*\beta_i' .$$

Then, it is easily shown that

$$\alpha_i' = \alpha_i - b\beta_i + \mathcal{P}^i b ,$$
$$\beta_i' = \beta_i .$$

Thus, β_i is an invariant of the given fibre bundle.

When the coefficient is Z_2, it is known that β_i is the i-th Stiefel-Whitney class w_i of the bundle (cf. [2], Theorem III).

Let $\bar{E}$ be the total space of the k-dimensional cell bundle associated with the $(k-1)$-sphere bundle (E, p, B, S^{k-1}) and let $\bar{p}: \bar{E} \to B$ be the projection. Let

$$\phi : H^j(B; Z_p) \to H^{j+k}(\bar{E}, E; Z_p)$$

be the Thom isomorphism. Then the characteristic class $Q_i \in H^{2i(p-1)}(B; Z_p)$ is defined by

$$Q_i = \phi^{-1}\mathcal{P}^i U .$$

where U is the Thom class, that is, the image of $1 \in H^0(B; Z_p)$ under the isomorphism ϕ. Then we have

Theorem 1. $\beta_i = Q_i$.

Proof. By Proposition 5,

$$\psi(\mathcal{P}^i a) = \psi(p^*\alpha_i + a \cdot p^*\beta_i) = \psi(a \cdot p^*\beta_i) = (-1)^{2i(p-1)k}\beta_i = \beta_i .$$

Let $\delta^* : H^j(E; Z_p) \to H^{j+1}(\bar{E}, E; Z_p)$ denote the coboundary homomorphism. Then, it is known that $\delta^* = \phi\psi$. Now, we have

$$\psi(\mathcal{P}^i a) = \phi^{-1}\delta^*\mathcal{P}^i a = \phi^{-1}\mathcal{P}^i\delta^* a = \phi^{-1}\mathcal{P}^i\phi(1) = Q_i .$$

Therefore, $\beta_i = Q_i$.

By the theorem of Wu, $Q_i=p_i$ in case $p=3$. Hence, we have the following

Corollary. *If $p=3$, then $\beta_i=p_i$, the i-th Pontrjagin class mod 3 of the vector bundle associated with the sphere bundle (E, p, B).*

§ 4. Proof of the theorems

Let M^n be a compact, connected, differentiable manifold of dimension n which is embedded differentiably in $(n+k)$-sphere S^{n+k}. Let (E, p, M^n, S^{k-1}) be a normal sphere bundle associated with the embedding, where E is the boundary of an open tubular neighborhood N of M^n in S^{n+k} and $p: E\to M^n$ is the projection. Consider the inclusion map $j: E\to V=S^{n+k}-N$ and the induced homomorphism $j^*: H^q(V; G)\to H^q(E; G)$, where G is a coefficient ring. We write the image of $j^*=A^q$ and $\sum_q A^q=A^*$. Then the following proposition holds (cf. [2], [3]).

Proposition 6. (1) *A^* is a subring of $H^*(E; G)$.*
(2) *A^* is closed under cohomology operations.*
(3) *$H^q(E; G)=A^q+p^*H^q(M^n; G)$ $(0<q<n+k-1$, direct sum).*
(4) *$A^q=0$ $(q\geqq n+k-1)$.*

Now, we are ready to prove our main theorems.

Proof of Theorem 2. Let $n=3^m$ and $m\geqq 1$. Suppose that $L^n(3)$ can be embedded differentiably in R^{3n+1}. We may assume that $L^n(3)$ is embedded differentiably in S^{3n+1}. Let $(E, p, L^n(3), S^{n-1})$ be a normal sphere bundle associated with the embedding. Then there exists a subset $A^*=\sum_q A^q$ of $H^*(E; Z_3)$ which satisfies the following conditions.
(1) A^* is a subring of $H^*(E; Z_3)$.
(2) A^* is closed under cohomology operations.
(3) $H^q(E; Z_3)=A^q+p^*H^q(L^n(3); Z_3)$ $(0<q<3n$, direct sum).
(4) $A^q=0$ $(q\geqq 3n)$.

Since the characteristic class W_n of the normal bundle vanishes, we have the exact sequence

$$0\to H^q(L^n(3); Z_3)\xrightarrow{p^*} H^q(E; Z_3)\xrightarrow{\psi} H^{q-n+1}(L^n(3); Z_3)\to 0.$$

Let $a\in H^{n-1}(E;Z_3)$ be an element such that $\psi(a)=1\in H^0(L^n(3);Z_3)$. Then we have the direct sum decomposition

(5) $H^q(E;Z_3)=p^*H^q(L^n(3);Z_3)+a\cdot p^*H^{q-n+1}(L^n(3);Z_3)$.

From (3) and (5),

(6) $A^q\approx H^{q-n+1}(L^n(3);Z_3)\approx Z_3 \quad (n-1\leqq q\leqq 3n-1)$.

Let

$$\mathcal{P}^i=\mathcal{P}^i_3: H^{n-1}(E;Z_3)\to H^{n-1+4i}(E;Z_3)$$

be the i-th Steenrod reduced power operation. Then there exists $\alpha_i\in H^{n-1+4i}(L^n(3);Z_3)$ such that

$$\mathcal{P}^i a=p^*\alpha_i+a\cdot p^*\bar{p}_i,$$

where $\bar{p}_i$ is the i-th dual Pontrjagin class mod 3 of $L^n(3)$. Let $x\in H^2(L^n(3);Z_3)$ be a generator. Then, $x^{n+1}=0$ and by Propositions 3 and 4,

$$\bar{p}_1=(-1)\binom{n+1}{1}x^2=-x^2,$$

$$\bar{p}_{(n-1)/2}=(-1)^{(n-1)/2}\binom{n+(n-1)/2}{(n-1)/2}x^{n-1}=(-1)^{(n-1)/2}x^{n-1}.$$

I. First, suppose that $m>1$. We have

$$\mathcal{P}^1 a=p^*\alpha_1-a\cdot p^*x^2$$

and, since $\alpha_{(n-1)/2}\in H^{3n-3}(L^n(3);Z_3)=0$,

$$a^3=\mathcal{P}^{(n-1)/2}a=(-1)^{(n-1)/2}a\cdot p^*x^{n-1}.$$

We may assume that $a\in A^{n-1}$. Then, by (1), $a^2\in A^{2n-2}$. From (5), there exist elements $\alpha\in H^{2n-2}(L^n(3);Z_3)$ and $\beta\in H^{n-1}(L^n(3);Z_3)$ such that

$$a^2=p^*\alpha+a\cdot p^*\beta.$$

Then, $a^3=a\cdot p^*(\alpha+\beta^2)$. Therefore, the equation $\alpha+\beta^2=(-1)^{(n-1)/2}x^{n-1}$ holds.

(i) If m is odd (i.e., $(n-1)/2$ is odd), then $\alpha+\beta^2=-x^{n-1}$. Note that $\alpha=0$ or $\pm x^{n-1}$. If $\alpha=0$, then $\beta^2=-x^{n-1}$. This is impossible. If $\alpha=-x^{n-1}$, then $\beta=0$, and hence, $a^2=-p^*x^{n-1}\in p^*H^{2n-2}(L^n(3);Z_3)$. This is inconsistent with the fact that

$a^2 \in A^{2n-2}$ and the direct sum decomposition (3). Therefore, we must have $\alpha = x^{n-1}$. Then, $\beta = \pm x^{(n-1)/2}$.

Assume that $\beta = x^{(n-1)/2}$. Then

$$a^2 = p^* x^{n-1} + a \cdot p^* x^{(n-1)/2}.$$

From (6),

$$A^{2n-2} = \{0, \pm(p^* x^{n-1} + a \cdot p^* x^{(n-1)/2}\}.$$

Since $A^{n+2} \cdot A^{2n-2} \subset A^{3n}$, by (1), and $A^{3n} = 0$, by (4), we obtain

$$A^{n+2} = \{0, \pm(p^*(yx^{(n+1)/2} + a \cdot p^*(yx))\},$$

where $y \in H^1(L^n(3); Z_3)$ is a generator with $\Delta y = -x$. Note that $\Delta x = 0$ is obvious and that $\Delta a = 0$ can be easily proved. $\Delta A^{n+2} \subset A^{n+3}$ shows that

$$A^{n+3} = \{0, \pm(p^* x^{(n+3)/2} + a \cdot p^* x^2)\}.$$

On the other hand, $\mathcal{P}^1 a = p^* \alpha_1 - a \cdot p^* x^2 \in A^{n+3}$. Hence, we have $\mathcal{P}^1 a = -p^* x^{(n+3)/2} - a \cdot p^* x^2$. Therefore,

$$\mathcal{P}^1(p^*(yx^{(n+1)/2}) + a \cdot p^*(yx)) = p^*(yx^{(n+5)/2})^{1)} \in p^* H^{n+6}(L^n(3); Z_3).$$

While, $\mathcal{P}^1 A^{n+2} \subset A^{n+6}$ by (2). This is a contradiction.

In case $\beta = -x^{(n-1)/2}$, similarly we have a contradiction.

(ii) If m is even (i.e., $(n-1)/2$ is even), then $\alpha + \beta^2 = x^{n-1}$. Then as in case (i), we must have $\alpha = 0$ and $\beta = \pm x^{(n-1)/2}$, that is $a^2 = \pm a \cdot p^* x^{(n-1)/2}$. In this case we also have a contradiction by a similar argument.

II. Next, assume that $m = 1$. Then, we have

$$a^3 = \mathcal{P}^1 a = p^* \alpha_1 - a \cdot p^* x^2,$$

where $a \in A^2$ and $\alpha_1 \in H^6(L^3(3); Z_3)$. Let $a^2 = p^* \alpha + a \cdot p^* \beta$, where $\alpha \in H^4(L^3(3); Z_3)$ and $\beta \in H^2(L^3(3); Z_3)$. Then, $a^3 = p^*(\alpha\beta) + a \cdot p^*(\alpha + \beta^2)$, and hence,

$$\alpha + \beta^2 = -x^2 \quad \text{and} \quad \alpha\beta = \alpha_1.$$

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1) $\mathcal{P}^r x^i = \binom{i}{r} x^{i+2r}$.

Therefore, $\alpha=x^2$ and $\beta=\pm x$. Assume that $\beta=x$. Then, $\alpha_1=x^3$, $a^2=p^*x^2+a\cdot p^*x$ and $a^3=p^*x^3-a\cdot p^*x^2$. Thus, we have

$$A^4 = \{0, \pm(p^*x^2+a\cdot p^*x)\} \quad \text{and} \quad A^6 = \{0, \pm(p^*x^3-a\cdot p^*x^2)\} .$$

$\Delta A^5 \subset A^6$ shows that

$$A^5 = \{0, \pm(p^*(yx^2)-a\cdot p^*(yx))\} .$$

Now, we have

$$(p^*x^2+a\cdot p^*x)\cdot(p^*(yx^2)-a\cdot p^*(yx)) = -a\cdot p^*(yx^3) \neq 0$$

in $H^9(H;Z_3)$. But, $A^4\cdot A^5\subset A^9$. by (1), and $A^9=0$, by (4). This is impossible.

In case $\beta=-x$, we have a contradition similarly.

Thus Theorem 2 is proved.

Proof of Theorem 3. Let $n=3^l+3^k$, $l\geqq k\geqq 0$ and $l\geqq 1$. Suppose that $L^n(3)$ is embedded in R^{3n}. We may assume that $L^n(3)$ is embedded in S^{3n}. Let $(E, p, L^n(3), S^{n-2})$ be a normal sphere bundle associated with the embedding. Then there is a subring $A^*=\sum A^q$ of $H^*(E;Z_3)$ which is closed under cohomology operations and satisfies the following conditions:

$$H^q(E;Z_3) = A^q+p^*H^q(L^n(3);Z_3) \qquad (0<q<3n-1).$$
$$A^q = 0 \qquad (q\geqq 3n-1).$$

Let $a\in H^{n-2}(E;Z_3)$ be an element such that $\psi(a)=1$, then

$$H^q(E;Z_3) = p^*H^q(L^n(3);Z_3)+a\cdot p^*H^{q-n+2}(L^n(3);Z_3),$$

and so,

$$A^q \approx Z_3 \qquad (n-2\leqq q\leqq 3n-2).$$

We may assume that $a\in A^{n-2}$.

Let $x\in H^2(L^n(3);Z_3)$ be a generator. Then $x^{n+1}=0$ and

$$\bar{p}_1 = (-1)\binom{n+1}{1}x^2 = -\varepsilon x^2, \quad \text{where} \quad \varepsilon = \begin{cases} 1 & \text{if } k>0, \\ -1 & \text{if } k=0, \end{cases}$$

$$\bar{p}_{(n-2)/2} = (-1)^{(n-2)/2}\binom{n+(n-2)/2}{(n-2)/2}x^{n-2} = \begin{cases}(-1)^{n/2}x^{n-2} & \text{if } l>k\geqq 0, \\ x^{n-2} & \text{if } l=k>0. \end{cases}$$

I. First, assume that $l>1$. We have

$$\mathcal{P}^1 a = p^*\alpha_1 - \varepsilon a\cdot p^* x^2,$$

and, since $l>1$,

$$a^3 = \mathcal{P}^{(n-2)/2} a = \begin{cases} a\cdot p^* x^{n-2} & \text{if } l>k\geqq 0 \text{ and } n/2 \text{ is even; or } l=k, \\ -a\cdot p^* x^{n-2} & \text{if } l>k\geqq 0 \text{ and } n/2 \text{ is odd.} \end{cases}$$

(i) If $l+k$ is odd (i.e., $n/2$ is even), or if $l=k$, then

$$a^3 = a\cdot p^* x^{n-2}.$$

Let $a^2 = p^*\alpha + a\cdot p^*\beta$, where $\alpha \in H^{2n-4}(L^n(3); Z_3)$ and $\beta \in H^{n-2}(L^n(3); Z_3)$. Then, $a^3 = a\cdot p^*(\alpha+\beta^2)$. Hence, $\alpha+\beta^2 = x^{n-2}$. Therefore, $\alpha=0$ and $\beta = \pm x^{(n-2)/2}$. Assume that $\beta = x^{(n-2)/2}$. Then, we have

$$a^2 = a\cdot p^* x^{(n-2)/2}.$$

Now,

$$\mathcal{P}^1 a^2 = \mathcal{P}^1(a\cdot p^* x^{(n-2)/2}) = p^*(\alpha_1\cdot x^{(n-2)/2}) + \varepsilon a\cdot p^* x^{(n+2)/2}.$$

On the other hand,

$$-a\cdot \mathcal{P}^1 a = -a\cdot p^*\alpha_1 + \varepsilon a\cdot p^* x^{(n+2)/2}.$$

Since $\mathcal{P}^1 a^2 = -a\cdot\mathcal{P}^1 a \in A^{2n}$, the above two equations imply that $\alpha_1 = 0$. Thus, we have

$$\mathcal{P}^1 a = -\varepsilon a\cdot p^* x^2 \text{ and } \mathcal{P}^1 a^2 = \varepsilon a\cdot p^* x^{(n+2)/2}.$$

Hence,

$$A^{n-2} = \{0, \pm a\},\ A^{n+2} = \{0, \pm a\cdot p^* x^2\} \quad \text{and}$$
$$A^{2n} = \{0, \pm a\cdot p^* x^{(n+2)/2}\}.$$

From the fact that $A^{n-1}\cdot A^{2n} \subset A^{3n-1}$ and $A^{3n-1}=0$, we see

$$A^{n-1} = \{0, \pm(p^*(y x^{(n-2)/2}) - a\cdot p^* y)\}.$$

Since $\Delta a = 0$, $\Delta A^{n-1} \subset A^n$ shows that

$$A^n = \{0, \pm(p^* x^{n/2} - a\cdot p^* x)\}.$$

Now, we have

$$(p^*x^{n/2}-a\cdot p^*x)^2 = p^*x^n-a\cdot p^*x^{(n+2)/2}\notin A^{2n}.$$

But, this is impossible, because $(A^n)^2\subset A^{2n}$.

In case $\beta=-x^{(n-2)/2}$, we have a contradiction similarly.

(ii) If $l+k$ is even (i. e., $n/2$ is odd) and $l>k$, then,

$$a^2 = -a\cdot p^*x^{n-2}.$$

As in case (i), we also have a contradiction.

Thus, we have completed the proof when $l>1$.

II. Next, assume that $l=k=1$. Then,

$$a^3 = \mathcal{P}^2a = p^*\alpha_2+a\cdot p^*x^4,$$

where $a\in A^4$ aud $\alpha_2\in H^{12}(L^6(3);Z_3)$. Let $a^2=p^*\alpha+a\cdot p^*\beta$, where $\alpha\in H^8(L^6(3);Z_3)$ and $\beta\in H^4(L^6(3);Z_3)$. Then, $a^3=p^*(\alpha\beta)+a\cdot p^*(\alpha+\beta^2)$, and hence, $\alpha+\beta^2=x^4$ and $\alpha\beta=\alpha^2$. Therefore, $\alpha=0$ and $\beta=\pm x^2$, and so $\alpha_2=0$. Suppose that $\beta=x^2$. Then,

$$a^2 = a\cdot p^*x^2 \quad \text{and} \quad a^3=a\cdot p^*x^4.$$

Thus, we have

$$A^{12} = \{0, \pm a\cdot p^*x^4\}.$$

$\Delta A^{11}\subset A^{12}$ shows that

$$A^{11} = \{0, \pm a\cdot p^*(yx^3)\}.$$

Since $A^5\cdot A^{12}\subset A^{17}$ and $A^{17}=0$, we obtain

$$A^5 = \{0, \pm(p^*(yx^2)-a\cdot p^*y)\}.$$

$\Delta A^5\subset A^6$ implies that

$$A^6 = \{0, \pm(p^*x^3-a\cdot p^*x)\}.$$

Now,

$$(p^*(yx^2)-a\cdot p^*y)\cdot(p^*x^3-a\cdot p^*x) = p^*(yx^5)-a\cdot p^*(px^3)\notin A^{11}.$$

But, this is impossible, because $A^5\cdot A^6\subset A^{11}$.

In case $\beta=-x^2$, we have a contradiction similarly.

III. In the end, assume that $l=1$ and $k=0$. Then,

$$a^3 = \mathcal{P}^1 a = p^*\alpha_1 + a\cdot p^*x^2\,,$$

where $a\in A^2$ and $\alpha_1\in H^6(L^4(3)\,;Z_3)$. Let $a^2 = p^*\alpha + a\cdot p^*\beta$, where $\alpha\in H^4(L^4(3);Z_3)$ and $\beta\in H^2(L^4(3)\,;Z_3)$. Then, $a^3=p^*(\alpha\beta)+a\cdot p^*(\alpha+\beta^2)$. Thus, $\alpha+\beta^2=x^2$ and $\alpha\beta=\alpha_1$. Therefore, $\alpha=0$ and $\beta=\pm x$, and so $\alpha_1=0$. Suppose that $\beta=x$. Then,

$$a^2 = a\cdot p^*x,\ a^3 = a\cdot p^*x^2 \quad\text{and}\quad a^4 = a\cdot p^*x^3\,.$$

Hence,

$$A^4 = \{0,\ \pm a\cdot p^*x\} \quad\text{and}\quad A^8 = \{0,\ \pm a\cdot p^*x^3\}\,.$$

Since $\Delta A^3\subset A^4$, we have

$$A^3 = \{0,\ \pm a\cdot p^*y\}\,.$$

Now,

$$(a\cdot p^*y)\cdot(a\cdot p^*x^3) = a\cdot p^*(yx^4)\neq 0 \quad\text{in}\quad H^{11}(E\,;Z_3).$$

However, this is a contradiction, because $A^3\cdot A^8\subset A^{11}$ and $A^{11}=0$. If $\beta=-x$, similarly we have a contradiction.

Thus the proof of Theorem 3 is completed.

§5. Remark

Let M^n be a compact, orientable, differentiable manifold embedded differentiably in R^m, $m>n$. Then, as is well known, the Euler class of the normal bundle to M^n vanishes. Making use of the fact, we obtain the followings:

1) If $n=3^m$, then $L^n(3)\not\subset R^{3n}$.
2) If $n=3^l+3^k$, then $L^n(3)\not\subset R^{3n-1}$.

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