J. Math. Kyoto Univ.
6-2 (1967) 187-293

A duality theorem for locally compact groups

By

Nobuhiko TATSUUMA

(Received and Communicated by Prof. Yoshizawa, on December 3, 1966)

Summary

The purpose of the present paper is to prove a dual relation between locally compact groups G and the set Ω (resp. $\widehat{G}$) of equivalence classes of (resp. all irreducible) unitary representations of G. This duality may be considered as an extension of Pontrjagin duality for abelian groups and Tannaka duality for compact groups.

In such a duality theorem, G is characterized as the "dual group" of Ω (resp. $\widehat{G}$), that is, as the set of all "birepresentations" which are operator fields over Ω (resp. $\widehat{G}$) commuting with the operation of Kronecker product. "Birepresentation" is a generalization of a character over the dual group in abelian case. And the initial topology of G coincides with the "weak topology" on the set of operator fields over Ω.

The duality between G and Ω is called the weak duality, and the one between G and $\widehat{G}$ is called the strong duality. The first one is proved for general locally compact groups, but the strong duality is proved under the type I restriction for G.

The results are strengthened for special groups.

§0. Introduction.

1. In the theory of representations of locally compact groups, the dual object, that is, the set of all equivalence classes of unitary irreducible representations of the given group, plays an important role. And many investigations have been done about *topological* structure or *Borel* structure of this object.

From the name "dual object", naturally, there arises a question whether the initial group is characterized by its dual object in some canonical way, or in other words, whether two different groups have dual objects of different structure. But the fact that any separable compact group with infinite elements has discrete dual with countably infinite elements shows that if we consider only topological or Borel structure on the dual object, then it cannot characterize the initial group.

However there exists *algebraic* structure, that is the so-called Kronecker product of representations. Considering that the dual object having topological or Borel structure is furnished moreover by the structure defined by Kronecker product, we have some examples for which the above question is ansowered affirmatively.

2. In the case of a locally compact abelian group A, the well-known duality of L. Pontrjagin [18] is valid. This duality is stated as follows. The dual space $\hat{A}$ of A can be identified with the set of all continuous unitary characters on A, with the locally compact topology defined by uniform convergence on any compact set, and with the Borel structure generated by this topology. The Kronecker product of elements of $\hat{A}$ corresponds to the ordinary product of characters as functions. With these structures, $\hat{A}$ becomes a locally compact abelian group.

Now consider the dual group $\hat{\hat{A}}$ of locally compact abelian group $\hat{A}$. That is, $\hat{\hat{A}}$ is the set of all functions $\tilde{x}$ over $\hat{A}$ such that

(1) $|\tilde{x}(\chi)|=1$, for any χ in $\hat{A}$ (unitarity),

(2) $\tilde{x}(\chi_1)\tilde{x}(\chi_2)=\tilde{x}(\chi_1\chi_2)$, for any χ_1, χ_2 in $\hat{A}$ (commutativity with product operation),

(3) $\tilde{x}$ is continuous, or more loosely, Borel measurable on $\hat{A}$ (measurability).

Consider an element a in A as a function $\tilde{a}$ in $\hat{\hat{A}}$ with the equality

$$\tilde{a}(\chi)=\chi(a), \qquad \text{for any } \chi \text{ in } \hat{A}.$$

Then we can define a *homomorphism* φ of A *into* $\hat{\hat{A}}$.

The duality theorem asserts that φ is an *isomorphism onto* $\hat{\hat{A}}$.

3. For a compact group K, T. Tannaka [19] gave a similar duality theorem. In this case we attach discrete structure to $\hat{K}$. Generally an element in $\hat{K}$ is not one-dimensional, so Kronecker product $\rho\otimes\sigma$ of a pair of elements ρ and σ in $\hat{K}$ is decomposed into a direct sum $\tau_1\oplus\cdots\oplus\tau_m$ of finite elements in $\hat{K}$.

As the dual group $\hat{\hat{K}}$ of $\hat{K}$, we consider the set of all operator (matrix) fields $\boldsymbol{T}=\{T(\rho)\}$ over $\hat{K}$ such that

(1′) $T(\rho)$ is a unitary matrix on the space of representation ρ,

(2′) $T(\rho)\otimes T(\sigma)=T(\tau_1)\oplus\cdots\oplus T(\tau_m)$,

for any pair (ρ, σ) in $\hat{K}$ and the above decomposition formula. Define the group structure into $\hat{\hat{K}}$ defined by componentwise product

$$\boldsymbol{T}_1\boldsymbol{T}_2=\{T_1(\rho)\,T_2(\rho)\}, \qquad \text{for } \boldsymbol{T}_j=\{T_j(\rho)\},$$

and the weakest topology among those which make all matrix elements $\{\langle T(\rho)u, v\rangle\}_{\rho,u,v}$ continuous. (We call such topology the "weak topology".) It is easy to see that $\hat{\hat{K}}$ is a topological group and the matrix field $\boldsymbol{W}_k=\{W_k(\rho)\}$, of which $W_k(\rho)$ is the matrix of representation ρ, gives an element in $\hat{\hat{K}}$. Therefore we can define a *homomorphism* φ of K *into* $\hat{\hat{K}}$ by assigning k in K to $\boldsymbol{W}_k$.

Tannaka duality theorem says that φ is an *isomorphism onto* $\hat{\hat{K}}$, as in the case of abelian groups.

4. The third example is given by the real unimodular group G of second order [20]. This duality is stated in quite the same form as in compact case. In the case of G, the Kronecker product $\rho\otimes\sigma$ of elements of $\hat{G}$ is decomposed into a non discrete but continuous direct sum of irreducible representations, in general: $\rho\otimes\sigma\sim\int\omega d\nu_{\rho,\sigma}(\omega)$.

Let this equivalence relation be given by an isometry U of the space of $\rho\otimes\sigma$ onto the space of the right hand side.

Take as the dual $\hat{\hat{G}}$ of $\hat{G}$ the set of all operator fields $\boldsymbol{T}=\{T(\rho)\}$ over $\hat{G}$ such that

(1″) $T(\rho)$ is a unitary operator over the space of representation ρ,

(2″) $U(T(\rho)\otimes T(\sigma))U^{-1}=\int T(\omega)d\nu_{\rho,\sigma}(\omega)$,

for any pair (ρ,σ) in $\hat{G}\times\hat{G}$ and any irreducible decomposition of $\rho\otimes\sigma$. By the same group structure and topology as in the compact case, $\hat{\hat{G}}$ becomes a topological group. And a *homomorphism* φ of G *into* $\hat{\hat{G}}$ is defined by considering the field of representation operators $\boldsymbol{U}_g=\{U_g(\rho)\}$ as an element of $\hat{\hat{G}}$.

The main result of the previous paper [20] asserts that φ is an *isomorphism onto* $\hat{\hat{G}}$.

5. From the above three examples, we may make a conjecture that the following theorem holds for some wide class of locally compact groups G.

Consider the set $\hat{\hat{G}}$ of all operator fields $\boldsymbol{T}=\{T(\rho)\}$ over the dual $\hat{G}$ of G such that

(i) $T(\rho)$ is a unitary operator over the space of representation ρ,

(ii) for any irreducible decomposition of the Kronecker product $\rho\otimes\sigma\sim\int\omega d\nu_{\rho,\sigma}(\omega)$ of any two elements ρ, σ of $\hat{G}$, which are related by an isometry U of the space of $\rho\otimes\sigma$ onto the space of right hand side, the following equation is valid

$$U(T(\rho)\otimes T(\sigma))U^{-1}=\int T(\omega)d\nu_{\rho,\sigma}(\omega).$$

(iii) $\{T(\rho)\}$ satisfies *some* measurability condition with respect to the Borel structure on $\hat{G}$.

We shall call such an operator field *birepresentation*. The condition (i) will be weakened later.

We give a group structure in $\hat{\hat{G}}$ by introducing componentwise product

$$\boldsymbol{T}_1\boldsymbol{T}_2=\{T_1(\rho)\,T_2(\rho)\},$$

and consider *certain* topology which is compatible with this group structure.

Then a *homomorphism* φ of G *into* $\hat{\hat{G}}$ is defined by the correspodence of g to a birepresentation $\boldsymbol{U}_g=\{U_g(\rho)\}$.

Our main theorem is stated as follows.

Theorem. *φ is an isomorphism onto $\hat{\hat{G}}$. In other words, for any given birepresentation $\boldsymbol{T}$, there exists a unique g in G such that $\boldsymbol{T}=\boldsymbol{U}_g$, and the initial topology of G corresponds to the above given topology of $\hat{\hat{G}}$ by φ.*

Although our main purpose is to find a family of groups for which the above mentioned duality theorem holds, the above definition of $\hat{\hat{G}}$ contains the ambiguous condition (iii) and the ambiguous topology, as readers may notice.

The last two examples do not need the measurability condition (iii). But in the case of non-compact abelian groups, if we omit this condition then there exist non-measurable unitary characters over the dual group $\hat{A}$ and it gives examples which satisfy (i) and (ii) but are not in A. The first problem is to make clear the reason why such difference arises, and to seek adequate condition for measurability.

Analogously for the last two examples, the good topology in $\hat{\hat{G}}$ is just the "weak topology". But for non-compact abelian case, its "weak topology" is equal to the simple convergence topology of characters over $\hat{A}$, and strictly weaker than the initial topology of A. It is the second problem to set up a generally applicable topology.

The third problem is how we can loosen the unitary condition (i). C. Chevalley [2] shows that in the case of compact Lie groups the family of matrix fields satisfying the conditions (1′) and (2′) except unitarity of $T(\rho)$ gives the complexification of K. Moreover the condition of reality $T(\rho)=\overline{T(\rho)}$ yields the unitarity. This circumstance may be comparable to the existence of non-unitary characters over an abelian group.

Therefore it seems that the unitarity condition is necessary for this theorem. But as shown in the following for the case of non-compact connected simple Lie groups with finite centre this condition can be replaced by a weaker condition of boundedness (§5. proposition 5.2). This fact is based on a property that the *main part* of $\widehat{G}$ is finitely generated for these groups.

6. The first step of solving these questions is to prove a duality theorem of somewhat weaker type which we state as follows (§2. proposition 2.1).

Let G be a locally compact group. We consider the set of all equivalence classes Ω of unitary (in general, not irreducible) representations of G, dimensions of which are lower than a sufficiently large cardinal number equal to, for instance, the square of the dimension of $L^2(G)$. For brevity of notations, we attach a representative $\omega=\{U_g(\omega), \mathfrak{H}(\omega)\}$ to each element of Ω. In Ω, as usual, the operation "$\oplus$" (direct sum) and "$\otimes$" (Kronecker product) are defined. It is shown later that the set of multiples of the regular representation R of G is an analogue of an ideal with respect to these operations.

As the dual $\widehat{\Omega}$ of Ω, take the set of all operator fields $\boldsymbol{T}=\{T(\omega)\}$ over Ω such that

(i) $T(\omega)$ is a closed operator on $\mathfrak{H}(\omega)$, and $T(R)$ is a non-zero bounded operator on $L^2(G)$,

(ii) $U_1(T(\omega_1)\oplus T(\omega_2))U_1^{-1}=T(\omega_3)$,

where $\omega_1\oplus\omega_2$ and ω_3 are connected by the isometric operator U_1 of $\mathfrak{H}(\omega_1)\oplus\mathfrak{H}(\omega_2)$ onto $\mathfrak{H}(\omega_3)$,

(iii) $U_2(T(\omega_1)\otimes T(\omega_2))U_2^{-1}=T(\omega_4)$,

where $\omega_1\otimes\omega_2$ and ω_4 are connected by the isometric operator U_2 of $\mathfrak{H}(\omega_1)\otimes\mathfrak{H}(\omega_2)$ onto $\mathfrak{H}(\omega_4)$.

We shall call such an operator field a *weak birepresentation.* Being equipped with the componentwise product and the weak topology as above, $\widehat{\Omega}$ becomes a topological group, and the map φ: $g(\in G) \to \boldsymbol{U}_g \equiv \{U_g(\omega)\}$ gives a *homomorphism* of G *into* $\widehat{\Omega}$.

In this form, we can prove generally that φ is an *isomorphism onto* $\widehat{\Omega}$ as a topological group.*) This fact may be called weak duality.

The proof of this weak duality runs as follows.

The Kronecker product of two right regular representations R of G is decomposed into a discrete direct sum of R with multiplicity equal to dim $L^2(G)$:

$$R \otimes R \sim \sum_{\alpha} \oplus R_\alpha\,, \quad (R_\alpha \sim R).$$

There are many equivalence relations A connecting the representations of both sides. Denote the left regular representation of G by $L = \{L_g, L^2(G)\}$. Let T be a non-zero bounded operator on $L^2(G)$ such that

a) $TL_g = L_g T$, for any g *in* G,

b) for any equivalence relation A,
$A(Tf_1 \otimes Tf_2) = \{Th_\alpha\}_\alpha\,, \qquad$ if $A(f_1 \otimes f_2) = \{h_\alpha\}_\alpha\,.$

Call such an operator T to be *admissible.* It is easy to see that the component $T(R)$ on R of any weak birepresentation is admissible and $\boldsymbol{T} = \{T(\omega)\}$ is uniquely determined by $T(R)$. So $\widehat{\Omega}$ is imbedded in the space $\widehat{\Omega}_0$ of all admissible operators in $L^2(G)$. If we introduce the weak topology of operators in $\widehat{\Omega}_0$, this imbedding is continuous. To prove the weak duality it is sufficient to show that $\widehat{\Omega}_0$ is isomorphic to G. This is done in §2.

7. Now we shall return to the duality theorem in the stong form. The weak duality assures that for any given birepresentation $\boldsymbol{T}$ over $\widehat{G}$, if we can define non-zero bounded operator $T(R)$ on $L^2(G)$ satisfying a) and b), which corresponds to $\boldsymbol{T}$, then by con-

*) The definition of an adequate topology is given in J. Ernest's paper [4].

sidering the topology which is induced by the weak topology of $\widehat{\Omega}_0$, the duality theorem is proved for $\hat{\hat{G}}$.

The meaning of "correspond to" is as follows. Let $R=\int_{\Omega(R)}\omega d\nu_R(\omega)$ be an irreducible decomposition of the regular representation, then the equation

$$R_g=\int_{\Omega(R)}U_g(\omega)d\nu_R(\omega)$$

holds for every g in G. Therefore, if the theorem can be proved, the corresponding admissible operator $T(R)$ must be defined by

$$T(R)=\int_{\Omega(R)}T(\omega)d\nu_R(\omega).$$

But this integral defines a non-zero bounded operator, if and only if

α) $T(\omega)$ is ν_R-measureable; that is, for any $\{v(\omega)\}$ in the space of $\int_{\Omega(R)}\omega d\nu_R(\omega)$, $\{T(\omega)v(\omega)\}$ is also in the same space,

β) $\{\|T(\omega)\|\}$ is ν_R-essentially bounded,

γ) $\{T(\omega)\}$ is not zero on a ν_R-positive set.

8. In this view point, the difference between abelian and compact cases becomes clear.

The dual object of a compact group has discrete structure, so we do not need any assumption of measurability, and weak convergence in $\hat{\hat{K}}$ gives rise to weak convergence of $T(R)$ immediately.

The conditions β) and γ) should be given in both cases, and for this it is sufficient to assume the unitarity of $T(R)$ (condition (i)).

Then what happens in the case of $SL(2,\boldsymbol{R})$? To make clear this situation, we attend to the condition (ii) of birepresentation. In order that the formula

$$U(T(\rho)\otimes T(\sigma))U^{-1}=\int T(\omega)d\nu_{\rho,\sigma}(\omega)$$

is possible, the right hand side must exist and define a bounded operator. This assumption hides the $\nu_{\rho,\sigma}$-measurability of $\{T(\omega)\}$. Extending this consideration to repeated product, we conclude that if R is contained as a subrepresentation of a discrete direct sum of

Kronecker products of irreducible representations taken from a countable family (shortly, R is countably generated), then α) follows from the condition (ii) of birepresentation.

For some class of semi-direct product groups, R has such a property. (§7. proposition 7.1)

Moreover, from the equality

$$\| T(\rho) \otimes T(\sigma) \| = \| T(\rho) \| \cdot \| T(\sigma) \|,$$

the boundedness of operators $T(\rho)$, $T(\sigma)$ gives the essential boundedness of $T(\omega)$ over the carrier of $\nu_{\rho,\sigma}$. The consideration as above shows that if R is contained as a subrepresentation in a direct sum of Kronecker products of finite number of irreducible representations (shortly, R is finitely generated), then the boundedness of $T(\rho)$ on the basic finite irreducible representations leads us to β). And it is easy to show that the weak convergence of $T(R)$'s follows from the weak convergences of $T(\rho)$'s on the basic representations.

We shall show that R is finitely generated for $SL(2, \boldsymbol{R})$, and more generally, for non-compact connected simple Lie groups with finite centre.

9. The contents of this paper are as follows.

§1 is devoted to state the notions and elementary properties which are used in the following §'s.

In §2, we shall prove the weak duality theorem which holds for general locally compact groups.

This weak duality is extended to the case of homogeneous spaces of G by a compact subgroup, after N. Iwahori's paper [12], in §3.

The connection between weak and strong forms of duality theorem is discussed in §4.

In the case of connected semi-simple Lie groups with finite centre, we can prove a duality theorem under weaker conditions. In §5, we consider this fact, using the result of §6 which states a property of orbits space on such a group.

For some class of semi-direct product groups, which contains the motion group over m-dimensional Euclidean space, and the m-dimensional proper inhomogeneous Lorentz group, we can show that R is countably generated, which means that the orbit spaces are "well-behaved". So in §7, we prove a duality theorem in stronger form for such groups.

The author wishcs to thank Professor H. Yoshizawa for his kind advices.

Short summaries of the results of this paper have been published in [21] and [22].

Notations. Locally compact groups will be denoted by G, H, K, N etc., and their elements by corresponding g, h, k, n etc. respectively. We mean by *representation* of a group a unitary strongly continuous representation of the given group over some complex Hilbert space, through this paper. A representation $\omega = \{U_g(\omega), \mathfrak{H}(\omega)\}$ is given by unitary operators $U_g(\omega)$ on the Hilbert space $\mathfrak{H}(\omega)$. Elements of $\mathfrak{H}(\omega)$ are shown by u, v, w etc. $I(\omega)$ (or I) shows the identity operator over $\mathfrak{H}(\omega)$. Representations of a given group are shown by Greek alphabet such as ω, ρ, σ, τ etc. The restriction of a representation ω of G to a subgroup H of G is denoted by $\omega|_H$. We denote by 1 the trivial representation, whose operators are the identity operator over one-dimensional complex vector space $\boldsymbol{C}$.

On G, we can define right Haar measure μ_r (or μ) and left Haar measure μ_l. Then the modulus of two Haar measures

$$\Delta_G(g) \text{ (or } \Delta(g)) \equiv (d\mu/d\mu_l)(g),$$

is a positive continuous function on G.

For a function f over a locally compact space M, $[f]$ means its carrier, and $\bar{f}$ means its complex conjugate. $C_\infty(M)$ is the set of all continuous bounded functions on M, and $C_0(M)$ is the set of elements with compact carrier in $C_\infty(M)$, moreover $C_0^+(M)$ consists of all elements which takes real non-negative value in $C_0(M)$. For given measure ν on M, $L_\nu^2(M)$ means the Hilbert space of all ν-square-summable functions on M. When M is a group G, we denote $L_\mu^2(G)$ by $L^2(G)$ shortly.

The right regular representation $R=\{R_g, L^2(G)\}$ of G is the representation on $L^2(G)$, operators of which are given by

$$(R_{g_0}f)(g)\equiv f(gg_0), \qquad \text{for any } f \text{ in } L^2(G),$$

and the left regular representation $L=\{L_g, L^2(G)\}$ is defined on the same space $L^2(G)$ by

$$(L_{g_0}f)(g)\equiv(\Delta(g_0))^{1/2}f(g_0^{-1}g), \qquad \text{for any } f \text{ in } L^2(G).$$

If two μ-measurable sets E and F in G are different only by a set of μ-measure zero, then we denote $E\sim F$.

By χ_E we denote the characteristic function of E, that is, the function which is equal to 1 on E and to zero on the outside of E.

$\boldsymbol{C}$, $\boldsymbol{R}$ mean the fields of all complex or of all real numbers, respectively.

For a set of bounded operators $\mathfrak{A}=\{A\}$ on $\mathfrak{H}$, the set of all bounded operators which commute with each element of $\mathfrak{A}$, is a weakly closed $*$-ring; we denote this ring by $\mathfrak{A}'$ or $\{A\}'$.

§1. Preliminaries.

1. When a locally compact group G and its representation ω are given, we call a bounded operator A in $\mathfrak{H}(\omega)$ *G-invariant*, if

$$(1.1) \qquad AU_g(\omega)=U_g(\omega)A, \qquad \text{for any } g \text{ in } G,$$

and a subset V in $\mathfrak{H}(\omega)$ *G-invariant*, when V is invariant as a set by $U_g(\omega)$ for any g in G. When moreover V is a closed subspace, it means that the projection P_V with the range V is G-invariant.

A representation ω of G is called *irreducible* when there exist no proper non-trivial closed subspace of $\mathfrak{H}(\omega)$ which is G-invariant.

Two representations ω_1 and ω_2 of G are *unitary equivalent* (or shortly *equivalent*) by U, when U is an isometric map of $\mathfrak{H}(\omega_1)$ onto $\mathfrak{H}(\omega_2)$ such that

$$UU_g(\omega_1) = U_g(\omega_2)\,U, \qquad \text{for any } g \text{ in } G. \tag{1.2}$$

We denote this relation by $\omega_1 \overset{U}{\sim} \omega_2$ or simple by $\omega_1 \sim \omega_2$. The unitary equivalence gives an equivalence relation on the set of representations of G.

2. We refer the readers to Dixmier's book [3] for the definition and elementary properties of Kronecker product of finite number of Hilbert spaces and operators on these spaces. And we shall remark now only the following:

Lemma 1.1. *For non-zero vectors u in $\mathfrak{H}(\omega_1)$ and v in $\mathfrak{H}(\omega_2)$,*

$$u \otimes v \neq 0, \qquad \text{in } \mathfrak{H}(\omega_1) \otimes \mathfrak{H}(\omega_2). \tag{1.3}$$

Therefore for two vectors u_1, u_2 in $\mathfrak{H}(\omega_1)$, if there exists a non-zero vector v in $\mathfrak{H}(\omega_1)$ such that

$$u_1 \otimes v = u_2 \otimes v, \tag{1.4}$$

then $$u_1 = u_2.$$

(The proof is trivial.)

Now we shall turn to the definition of Kronecker product of representations.

Definition 1.1. *If a finite set of locally compact groups $\{G_j\}_{1 \leqq j \leqq n}$ and representations $\omega_j = \{U_{g_j}(\omega_j), \mathfrak{H}_j\}$ of each G_j are given, then the family of unitary operators $\{U_{g_1}(\omega_1) \otimes U_{g_2}(\omega_2) \otimes \cdots \otimes U_{g_n}(\omega_n)\}$ on $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_n$ gives a representation of the product group $G_1 \times G_2 \times \cdots \times G_n$. We call this representation the* outer Kronecker product *of $\{\omega_j\}$ and denote by $\omega_1 \widehat{\otimes} \omega_2 \widehat{\otimes} \cdots \widehat{\otimes} \omega_n$.*

Lemma 1.2. *The outer Kronecker product $\omega_1 \widehat{\otimes} \omega_2 \widehat{\otimes} \cdots \widehat{\otimes} \omega_n$ is irreducible if and only if each ω_j is irreducible.*

Proof. It is enough to show it when $n=2$. And obviously $H \otimes \mathfrak{H}(\omega_2)$ is a non-zero closed proper G-invariant subspace of $\mathfrak{H}(\omega_1) \otimes \mathfrak{H}(\omega_2)$ for any non-zero closed proper G-invariant subspace H of $\mathfrak{H}(\omega_1)$. This shows "only if" part.

Conversely, let ω_1, ω_2 be irreducible and A be a bounded operator on $\mathfrak{H}(\omega_1)\otimes\mathfrak{H}(\omega_2)$ commuting with all $U_{g_1}(\omega_1)\otimes U_{g_2}(\omega_2)$. Take a complete orthonormal system $\{v_j\}$ in $\mathfrak{H}(\omega_2)$; then the vector $A(u\otimes v)$ is expanded uniquely as

$$A(u\otimes v)=\sum_j u_j\otimes v_j\,. \tag{1.5}$$

For a fixed v, the map $A_j(v)$ on $\mathfrak{H}(\omega_1)$, defined by

$$A_j(v)u=u_j \tag{1.6}$$

is a bounded operator commuting with $\{U_{g_1}(\omega_1)\}$. Hence from the irreducibility of ω_1, for some scalar $c_j(v)$,

$$A_j(v)=c_j(v)I(\omega_1), \tag{1.7}$$

$$A(u\otimes v)=\sum_j c_j(v)u\otimes v_j=u\otimes\sum_j c_j(v)v_j\,. \tag{1.8}$$

Next put A_0 a bounded operator on $\mathfrak{H}(\omega_2)$ defined by

$$A_0v=\sum_j c_j(v)v_j\,. \tag{1.9}$$

Then A_0 commutes with $\{U_{g_2}(\omega_2)\}$. That is, A_0 is a scalar operator. Thus $A=I(\omega_1)\otimes A_0$ is a scalar operator. This gives the irreducibility of $\omega_1\widehat{\otimes}\omega_2$.

Definition 1.2. *In the product group* $G^n\equiv\overbrace{G\times\cdots\times G}^{n}$ *of same locally compact group G with multiplicity n, the set* $\widetilde{G}_n$ *of all diagonal elements* $\{(\overbrace{g,\cdots,g}^{n}):g\in G\}$ *is a closed subgroup in* G^n *which is isomorphic to G with the induced topology and by the map* π_n: $g\to(g,\cdots,g)$. *We call* $\widetilde{G}_n$ *the diagonal group in* G^n.

Definition 1.3. *In the definition1. 1, if all* G_j*'s are equal to same group G, we can consider* $(\omega_1\widehat{\otimes}\omega_2\widehat{\otimes}\cdots\widehat{\otimes}\omega_n)\,|_{\widetilde{G}_n}$ *the restriction of* $\omega_1\widehat{\otimes}\omega_2\widehat{\otimes}\cdots\widehat{\otimes}\omega_n$ *to the diagonal group* $\widetilde{G}_n$ *in* G^n, *a sa representation of G by the map* π_n. *This representation of G is called the* inner Kronecker product (*or shortly* Kronecker product) *of* $\{\omega_j\}$, *and is denoted by* $\omega_1\otimes\omega_2\otimes\cdots\otimes\omega_n$.

Lemma 1.3. *For given bounded operators* T_j *on* $\mathfrak{H}_j$, (1.10) $$\|T_1\otimes\cdots\otimes T_n\|=\|T_1\|\times\cdots\times\|T_n\|.$$

Proof. The proof of inequality $\|T_1\otimes\cdots\otimes T_n\|\leqq\|T_1\|\times\cdots\times\|T_n\|$, is given in Dixmier's book [3] (p. 23).

Conversely for any $\varepsilon>0$, there exists v_j in $\mathfrak{H}_j$ such that $\|T_jv_j\|>(\|T_j\|-\varepsilon)\|v_j\|$. Then

(1.11) $$\begin{aligned}\|T_1v_1\otimes\cdots\otimes T_nv_n\| &\geqq \prod_j(\|T_j\|-\varepsilon)\|v_j\| \\ &= \prod_j(\|T_j\|-\varepsilon)\|v_1\otimes\cdots\otimes v_n\|.\end{aligned}$$

That is, the converse inequality is deduced from

(1.12) $$\|T_1\otimes\cdots\otimes T_n\|\geqq\prod_j(\|T_j\|-\varepsilon).$$

F. J. Murray and J. von Neumann [17] studied rings of operators on Hilbert spaces, and defined "factor" as W^*-ring with 1-dimensional centre. By their classification, there are three types of factors, so-called of type I, II, III.

Applying the theory to the rings of operators on representation space $\mathfrak{H}(\omega)$ generated by the operators $U_g(\omega)$ for some representation ω of G, we get following definitions.

Definition 1.4. *A representation ω of G is called a* factor representation *when the W^*-ring generated by $\{U_g(\omega)\}$ is a factor.*

Definition 1.5. *A group G such that any factor representation of G is of type I, is called a* group of type I.

3. In this section, we assume that all Hilbert spaces which we shall consider are separable and all groups G are separable too.

The author refers J. Dixmier's book [3] for the definitions and simple properties of direct integrals of Hilbert spaces etc. Let $\{X, \mathfrak{B}, \nu\}$ be a measure space, and let for any x in X, a Hilbert space $H(x)$ be attached, the direct integral of $\{H(x)\}$ with respect to $\{X, \mathfrak{B}, \nu\}$ is denoted by $\int_X H(x)d\nu(x)$. Direct integral of operators, and of representations are denoted in analogous way.

The followings are well-known results of classical theory.

Lemma 1. 4. (*F. I. Mautner*, [*15*]). *For any representation* ω *of a separable locally compact group* G, *there exist a measure space* $(X, \mathfrak{B}, \nu)$ *and a family* $\{\omega(x)\}$ *of irreducible representations of* G *which is associated to each* x *in* X, *and*

$$\omega \sim \int_X \omega(x) d\nu(x). \tag{1.13}$$

Such an integral form of the given representation ω is called the *irreducible decomposition* of ω.

Lemma 1. 5. *For any representation* ω *of* G, *the irreducible decomposition of the* W^*-*ring which is generated by* $\{U_g(\omega)\} \cup \{U_g(\omega)\}'$ *is unique up to unitary equivalence. And its almost all components are factors.*

This decomposition of ω is called the *central decomposition* of ω.

Lemma 1. 6. *If* G *is a type I group, then for any representation* ω, *its irreducible decomposition is unique up to unitary equivalence.*

Lemma 1. 7. *For type I group* G, *operators of almost all components* ρ *in the central decomposition of the given representation are of the form*

$$U_g(\rho) \equiv I_{n(\rho)} \otimes U_g(\omega(\rho)), \tag{1.14}$$

where $\omega(\rho)$ *is an irreducible representation and* $I_{n(\rho)}$ *is the identity operator in some* $n(\rho)$-*dimenfional Hilbert space* $\mathfrak{H}'_\rho$.

Proof. In fact, operators of representation of type I factor are of the form of (1. 14).

Lemma 1. 8. *For a direct integral*

$$\omega \equiv \int_X \omega(x) d\nu(x), \tag{1.15}$$

if ν-*almost all components* $\omega(x)$ *are equal to the same representation* ω_0, *then*

$$\omega \sim \sum \oplus \omega_0 \tag{1.16}$$

with the multiplicity which is equal to dim $L^2_\nu(X)$.

Proof. It is considerable that all the spaces $\mathfrak{H}(\omega(x))$ are same to $\mathfrak{H}(\omega_0)$. Let $\{\varphi_\alpha\}$ be a complete orthonormal system of $L^2_\nu(X)$, and consider for any element

$$v\equiv\{v(x)\}, \qquad \text{in } \mathfrak{H}(\omega), \tag{1.17}$$

$$\tilde{v}_\alpha\equiv\int_X v(x)\varphi_\alpha(x)d\nu(x). \tag{1.18}$$

Then it is easy to see that the map $\varphi;\ v\rightarrow\{\tilde{v}_\alpha\}$ gives the equivalence relation of (1.16).

Lemma 1.9. *When a direct integral*

$$\omega\equiv\int_X\omega(x)d\nu(x) \tag{1.19}$$

is given, then there exists an irreducible decomposition

$$\omega\sim\int_X\left\{\int_{Y_x}\omega(x,y)d\nu_x(y)\right\}d\nu(x)=\int_{\tilde{X}}\omega(\tilde{x})d\tilde{\nu}(\tilde{x}) \tag{1.20}$$

such that, for ν-almost all x,

$$\omega(x)\sim\int_{Y_x}\omega(x,y)d\nu_x(y) \tag{1.21}$$

gives an irreducible decomposition of $\omega(x)$.

Proof. Let $\mathfrak{A}$ be the abelian ring generated by the diagonal elements of the direct integral (1.19). (That is, the map: $f(x)\rightarrow c(x)f(x)$ in $\mathfrak{H}(\omega)$ where $c(x)$ is a ν-essentially bounded function on X). Consider a maximal abelian subring $\tilde{\mathfrak{A}}$ of $\{U_g(\omega)\}'$ which contains $\mathfrak{A}$. Then the irreducible decomposition of ω with respect to $\tilde{\mathfrak{A}}$ is given as follows:

$$\omega\sim\int_{\tilde{X}}\omega(\tilde{x})d\tilde{\nu}(\tilde{x}). \tag{1.22}$$

But because of separability of $\mathfrak{H}$ and G, $\mathfrak{A}$ contained in $\tilde{\mathfrak{A}}$ separates the measure space $\{\tilde{X},\tilde{\nu}\}$ as a double integral form $\{\{(x,y)\}: x\in X, y\in Y_x, \tilde{\nu}=\nu\times\nu_x\}$. And it is easy to see that the both hand sides of (1.21) are equivalent for almost all x, so (1.20) gives an irreducible decomposition.

Lemma 1.10. *For given two direct integrals*

$$\omega_j \equiv \int_{X_j} \omega(x)\,d\nu_j(x), \qquad (j=1,2), \tag{1.23}$$

we have

$$\omega_1 \otimes \omega_2 \sim \int_{X_1 \times X_2} \omega(x) \otimes \omega(y)\,d\nu_1(x)\,d\nu_2(y). \tag{1.24}$$

Proof. Consider the linear map φ generated by $u \otimes v \to \{u(x) \otimes v(y)\}$ from $\mathfrak{H}(\omega_1 \otimes \omega_2)$ to the space of representation of right hand side of (1.24), where $\{u(x)\}$, $\{v(y)\}$ are corresponding vector to u, v in the integral (1.23).

The map φ gives the unitary equivalence of (1.24).

Corollary. *If the irreducible decompositions of representations* ω_j $(j=1,2)$ *of type I group G are given by* $\omega_j \sim \int_{\Omega_j} \omega(x)\,d\nu_j(x)$ *respectively, then the irreducible decomposition of* $\omega_1 \otimes \omega_2$ *has a form*

$$\omega_1 \otimes \omega_2 \sim \int_{\Omega_1 \times \Omega_2} d\nu_1(x)\,d\nu_2(y) \left\{ \int_{\Omega(x,y)} \omega(w;\, x, y)\,d\nu_{x,y}(w) \right\}, \tag{1.25}$$

where

$$\omega(x) \otimes \omega(y) \sim \int_{\Omega(x,y)} \omega(w;\, x, y)\,d\nu_{x,y}(w), \tag{1.26}$$

is a form of irreducible decomposition of $\omega(x) \otimes \omega(y)$.

Proof. From lemma 1.10,

$$\omega_1 \otimes \omega_2 \sim \int_{\Omega_1 \times \Omega_2} \omega(x) \otimes \omega(y)\,d\nu_1(x)\,d\nu_2(y). \tag{1.27}$$

Apply the result of lemma 1.9, then uniqueness of irreducible decomposition leads to (1.25).

Hereafter we assume that G is type I group.

Now consider the set $\widehat{G}$ of all equivalence classes of irreducible representations of G. From each element of $\widehat{G}$, we choose a representative ω.

Let $\boldsymbol{T} = \{T(\omega)\}$ be an operator field on $\widehat{G}$ in which $T(\omega)$ is a bounded operator on $\mathfrak{H}(\omega)$, for any irreducible representation ω_0, $\boldsymbol{T}$

is extendable uniquely by

$$T(\omega_0)=UT(\omega)U^{-1}, \tag{1.28}$$

where ω is the representative of equivalence class containing ω_0 and U is the equivalence relation connecting ω_0 to ω.

Let ω_1 be given representation of G and let its irreducible decomposition is given by

$$\omega_1 \overset{U}{\sim} \int_{\Omega(\omega_1)} \omega(\tau)\,d\nu_1(\tau). \tag{1.29}$$

And let vectors u, v in $\mathfrak{H}(\omega_1)$ correspond to $\{u(\omega(\tau))\}$, $\{v(\omega(\tau))\}$ respectively in this decomposition, by U.

Definition 1.6. ***T*** *is called* integrable *on ω_1 when for any vectors u, v in $\mathfrak{H}(\omega_1)$, the function*

$$\langle T(\omega)u(\omega), v(\omega)\rangle_{\mathfrak{H}(\omega)} \tag{1.30}$$

is ν_1-measurable over $\Omega(\omega_1)$.

It is easy to see that integrability does not depened on the way of irreducible decomposition.

Lemma 1.11. *If* ***T*** *is integrable on ω_1, then $\|T(\omega)\|_{\mathfrak{H}(\omega)}$ is ν_1-measurable function over $\Omega(\omega_1)$.*

Proof. Let $\{v_j\}$ be a dense countable set in $\mathfrak{H}(\omega_1)$; then, for ν_1-almost all ω,

$$\|T(\omega)\|_{\mathfrak{H}(\omega)}=\sup_{j,k}\chi_{[v_j]\cap[v_k]}\frac{\langle T(\omega)v_j(\omega), v_k(\omega)\rangle}{\|v_j(\omega)\|\|v_k(\omega)\|}. \tag{1.31}$$

That is, $\|T(\omega)\|_{\mathfrak{H}(\omega)}$ is ν_1-measurable.

Definition 1.7. *If* ***T*** *is integrable on ω_1, and*

$$\nu_1\text{-}\operatorname*{ess\text{-}sup}_{\omega}\|T(\omega)\|_{\mathfrak{H}(\omega)}<+\infty, \tag{1.32}$$

then ***T*** *is called bounded on ω_1.*

And if

$$\nu_1\text{-}\operatorname*{ess\text{-}sup}_{\omega}\|T(\omega)\|_{\mathfrak{H}(\omega)}\neq 0, \tag{1.33}$$

then ***T*** *is called* non-zero *on ω_1.*

Lemma 1.12. *If* $\boldsymbol{T}$ *is integrable on* ω_1, *then,*

$$T(\omega_1) \equiv \int_{\Omega(\omega_1)} T(\omega(\tau)) d\nu_1(\tau) \tag{1.34}$$

is defined as a closed operator on $\mathfrak{H}(\omega_1)$ *with dense domain.*

When G is type I, $T(\omega_1)$ *does not depend on the form of irreducible decompostion.*

Moreover, if $\boldsymbol{T}$ *is bounded on* ω_1, *then* $T(\omega_1)$ *is bounded and*

$$\nu_1\text{-}\underset{\omega}{\text{ess-sup}} \| T(\omega) \|_{\mathfrak{H}(\omega)} = \| T(\omega_1) \|. \tag{1.35}$$

Especially, if $\boldsymbol{T}$ *is non-zero on* ω_1, *then* $T(\omega_1) \neq 0$.

Proof. Divide $\Omega(\omega_1)$ into a disjoint sum of measurable subsets,

$$\Omega(\omega_1) = \Omega_1 + \Omega_2 + \cdots, \tag{1.36}$$

where

$$\Omega_n = \{\omega;\ n-1 \leqq \| T(\omega) \| < n\}. \tag{1.37}$$

Evidently $\mathfrak{H}(\omega_1)$ is the direct sum of closed subspaces

$$H_n = \{v;\ [v(\omega)] \subset \Omega_n\}, \tag{1.38}$$

and the norm of $T(\omega_1)|_{H_n}$ is smaller than n. This shows that $T(\omega_1)$ is a closed operator with a dense domain.

Uniqueness of $T(\omega_1)$ follows from the uniqueness of decomposition.

Next, since

$$\begin{aligned} \| T(\omega_1) v \|^2 &= \int \| T(\omega(\tau)) \cdot v(\omega(\tau)) \|^2_{\mathfrak{H}(\omega(\tau))} d\nu_1(\tau) \\ &\leqq \int \| T(\omega(\tau)) \|^2 \| v(\omega(\tau)) \|^2_{\mathfrak{H}(\omega(\tau))} d\nu_1(\tau), \end{aligned} \tag{1.39}$$

the right hand side of (1.35) is smaller than the left one. But for any given $\varepsilon > 0$, we can select adequate non-zero $\{v(\omega)\}$ such that

$$\begin{aligned} &\int \| T(\omega(\tau)) v(\omega(\tau)) \|^2_{\mathfrak{H}(\omega(\tau))} d\nu_1(\tau) \\ &\geqq \left((\nu_1\text{-}\underset{\omega}{\text{ess-sup}} \| T(\omega) \|_{\mathfrak{H}(\omega)}) - \varepsilon \right) \int \| v(\omega(\tau)) \|^2 d\nu_1(\tau). \end{aligned} \tag{1.40}$$

This shows the converse inequality.

Lemma 1.13. *If* $\boldsymbol{T}$ *is integrable on* ω_1, *then for any subrepresentation* ω_2 *of* ω_1, $\boldsymbol{T}$ *is integrable on* ω_2 *too.*

Proof. The irreducible decomposition is given by the decomposition (1.29) of ω_1 and a ν_1-measurable subset $\Omega(\omega_2)$ of $\Omega(\omega_1)$, as

$$\omega_2 \sim \int_{\Omega(\omega_2)} \omega(\tau)\, d\nu_1(\tau). \tag{1.41}$$

The integrablility of $\boldsymbol{T}$ on ω_2 follows immediately from ν_1-measurability of functions

$$\chi_{\Omega(\omega_2)}(\omega) \langle T(\omega) u(\omega),\, v(\omega) \rangle_{\mathfrak{H}(\omega)} . \tag{1.42}$$

4. Now we shall give a simple sketch of the theory of induced representations developed by G. W. Mackey [13].

If a locally compact group G and its closed subgroup K are given, the homogeneous space $M = K \diagdown G$ of left K-cosets is a locally compact space over which G operates as a transitive transformation group. Denote, by π the canonical map from G to M which transfers an element g of G to the left K-coset containing g.

For a measure ν over M and g in G, we can define a measure ν_g by $\nu_g(E) \equiv \nu(Eg)$ for all measurable set E on M. ν is called *quasi-invariant* if and only if ν_g and ν have same null sets, that is, mutually absolutely continuous.

Lemma 1.14. *There exists a quasi-invariant measure* ν *on* M *always. And any two quasi-invariant measures are mutually absolutely continuous. For such a* ν, *a measurable set* E *on* M *is* ν*-null set if and only if* $\pi^{-1}(E)$ *is* μ*-null set, where* μ *is a Haar measure on* G.

We refer to Mackey's paper [13] for the proof of this lemma.

Now we shall assume G is separable, and a representation $\tau = \{W_k, H\}$ of K on a separable Hilbert space H is given. Denote by $\mathfrak{H}$, the set of all H-valued function f on G such that,

(i) $\langle f(g), v \rangle$ is a Borel function of g for any v in H,

(ii) $f(kg) = W_k(f(g))$, for any k in K and any g in G,

(iii) $\|f\|^2 \equiv \int_M \langle f(\pi^{-1}(x)), f(\pi^{-1}(x)) \rangle_H \, d\nu(x) < +\infty$, where $\langle f(g), f(g)\rangle$ is considered as a Borel function on M from the conditions (i) and (ii).

It is easy to verify, $\mathfrak{H}$ becomes a Hilbert space and the family of operators $\{U_g\}$ on $\mathfrak{H}$ which are defined as

$$(U_g f)(g_0) \equiv (d\nu(\pi(g_0)g)/d\nu(\pi(g_0)))^{1/2} f(g_0 g),$$

gives a representation of G on $\mathfrak{H}$.

Lemma 1.15. *$\{U_g, \mathfrak{H}\}$ is independent of the selection of ν within unitary equivalence.*

Proof. Let ν_1, ν_2 be two quasi-invariant measures on M, and let $\rho(g) \equiv (d\nu_1(\pi(g))/d\nu_2(\pi(g)))$ be the modulus of these measures based on lemma 1.14. Then the map $f(g) \to (\rho(g))^{1/2} f(g)$ gives the unitary equivalence between the spaces defined from ν_1, and from ν_2.

Definition 1.18. *The representation $\{U_g, \mathfrak{H}\}$ is called as the representation of G* induced *by the representation of K, and we shall use the notation, $\underset{K\to G}{Ind}\tau$, for this representation.*

The following lemmata which we state without proof are obtained by G. W. Mackey [13], and will be used in the following sections.

Lemma 1.16. *If $\tau_0 \equiv \sum_j \oplus \tau_j$ then*

$$\underset{K\to G}{Ind}\tau_0 \sim \sum_j \oplus \underset{K\to G}{Ind}\tau_j. \tag{1.43}$$

Lemma 1.17. *Let K_1, K_2 be closed subgroups of G and $K_1 \subseteq K_2$, and let τ be a representation of K_1, then*

$$\underset{K_1\to G}{Ind}\tau \sim \underset{K_2\to G}{Ind}\{\underset{K_1\to K_2}{Ind}\tau\}. \tag{1.44}$$

Lemma 1.18. *Denote by $\{e\}$ the closed subgroup consisted of only the unit element e of G, and by R the regular representation of G, then*

$$R \sim \underset{\{e\}\to G}{Ind} I. \tag{1.45}$$

Corollary. *Let R_K be there gular representation of a closed subgroup K of G, then*

(1.46) $$R \sim \underset{K\to G}{Ind}\, R_K .$$

Now consider a finite family of groups G_j and representations $\tau_j = \{W_{k_j}^j, H_j\}$ of closed subgroups K_j of each G_j, then for their outer Kronecker product, the following is valid.

Lemma 1.19.

(1.47) $$(\underset{K_1\to G_1}{Ind}\,\tau_1)\hat{\otimes}(\underset{K_2\to G_2}{Ind}\,\tau_2)\hat{\otimes}\cdots\hat{\otimes}(\underset{K_n\to G_n}{Ind}\,\tau_n)$$
$$\sim \underset{K_1\times K_2\times\cdots\times K_n\to G_1\times G_2\times\cdots\times G_n}{Ind}\,\tau_1\hat{\otimes}\tau_2\hat{\otimes}\cdots\hat{\otimes}\tau_n .$$

And the equivalence relation between both sides is generated by the correspondence of vectors

(1.48) $$f_1\otimes f_2\otimes\cdots\otimes f_n \to f_1(g_1)\otimes f_2(g_2)\otimes\cdots\otimes f_n(g_n),$$

where f_j is a vector in $\mathfrak{H}^j$ and the right hand side is regarded as a $(H_1\otimes H_2\otimes\cdots\otimes H_n)$-valued function on $G_1\times G_2\times\cdots\times G_n$.

Definition 1.9. *Two closed subgroups K_1 and K_2 in G are* regularly related *when the space $\mathfrak{D}$ of double K_1-K_2 cosets in G is countably separated except μ-null set. That is, there exists a sequence $E_0, E_1, E_2, \cdots$ of double K_1-K_2 cosetwise measurable subsets such that $\mu(E_0)=0$, and any double K_1-K_2 coset not in E_0 is representable as the intersection of E_j's which contain this coset.*

Lemma 1.20. *(J. Glimm[8]). For the case of $E_0=\phi$, regularly relatedness of K_1 and K_2 is equivalent to the property that the space $\mathfrak{D}$ is T_0-space.*

Lemma 1.21. *Let ν be a finite measure in G with the same null sets as the Haar measure μ of G. For any measurable set F in $\mathfrak{D}$, let F_0 be the subset of G which is the union of double K_1-K_2 cosets belonging to F and put $\nu_0(F)\equiv\nu(F_0)$, then ν_0 gives a measure on $\mathfrak{D}$.*

We shall call such a measure ν_0 *admissible.*

Lemma 1.22. *Let $\tau=\{W_k, H\}$ be a representation of K_1 and suppose that K_1 and K_2 are closed subgroups of G which are regularly* *related. For each g in G, consider the subgroup $K_2 \cap g^{-1}K_1 g \equiv K_2(g)$ of K_2, and the representation, $\omega(g) \equiv \underset{K_2(g)\to K_2}{Ind}\{W_{gk_2g^{-1}}, H\}$ of K_2. Then $\omega(g)$ is determined to within equivalence by the double coset $D \equiv K_1 g K_2$, and we may write $\omega(g) = \omega(D)$. Finally $(\underset{K_1\to G}{Ind}\,\tau)|_{K_2}$ is a direct integral over $\mathfrak{D}$, with respect to any admissible measure ν_0 in $\mathfrak{D}$, of the representations $\omega(D)$. And the equivalence relation of this decomposition is given by the correspondence of vectors*

$$f(\equiv f(g)) \to f_g(k_2) \equiv f(gk_2), \tag{1.49}$$

where f is in the space of $\underset{K_1\to G}{Ind}\,\tau$ given as a function on G, and $f_g(k_2)$ is regarded as a function on K_2 and as the component of f on the representation $\omega(D) \equiv \omega(g)$, $(g \in D)$, in $\int \omega(D)\, d\nu_0(D)$.

Combining the lemmata 1.19 and 1.22, we get the followings which is useful later.

Lemma 1.23. *Let K_j be closed subgroups of G and let $\tau_j = \{W^j_{k_j}, H^j\}$ be a representation of each K_j. Now we assume the subgroups $\widetilde{K} \equiv K_1 \times \cdots \times K_n$ and the diagonal group $\widetilde{G}_n$ are regularly related in G^n. For each $\tilde{g} \equiv (g_1, \cdots, g_n)$ consider the representations $\gamma \to W^j_{g_j \gamma g_j^{-1}}$ of the subgroup $\Gamma(\tilde{g}) \equiv g_1^{-1}K_1 g_1 \cap \cdots \cap g_n^{-1}K_n g_n$ of G. Let us denote their Kronecker product by $\tau(\{\tau_j\} : \tilde{g})$ and put*

$$\underset{\Gamma(\tilde{g})\to G}{Ind}\ \tau(\{\tau_j\} : \tilde{g}) \equiv \omega(\{\tau_j\} : \tilde{g}). \tag{1.50}$$

Then $\omega(\{\tau_j\} : \tilde{g})$ is determined to within unitary equivalence by the double coset $D \equiv \widetilde{K}\tilde{g}\widetilde{G}_n$ in G^n, and we may write $\omega(\{\tau_j\} : \tilde{g})$ by $\omega(D)$. Finally

$$\underset{K_1\to G}{Ind}\,\tau_1 \otimes \underset{K_2\to G}{Ind}\,\tau_2 \otimes \cdots \otimes \underset{K_n\to G}{Ind}\,\tau_n \sim \int_D \omega(D)\, d\nu(D), \tag{1.51}$$

where D is the set of double $\widetilde{K} - \widetilde{G}_n$ cosets and ν is any admissible measure in $\mathfrak{D}$. The equivalence relation of this relation is given by

$$f_1 \otimes f_2 \otimes \cdots \otimes f_n \to f_1(g_1 g) \otimes f_2(g_2 g) \otimes \cdots \otimes f_n(g_n g), \tag{1.52}$$

where the right hand side is regarded as a function of g.

5. The regular representation R of G is in special position in the set of representations. The followings state this situations.

Lemma 1. 24. *For any representation ω and the right regular representation R of G, $\omega\otimes R$ is unitary equivalent to the multiple of R with multiplicity $(\dim \omega)$. And the map which gives this equivalence, is generated by*

$$v\otimes f\rightarrow\{\langle U_g v, \varphi_\alpha\rangle f(g)\}_\alpha, \quad \text{for any } v \text{ in } \mathfrak{H}(\omega),\ f \text{ in } L^2(G), \tag{1.53}$$

where $\{\varphi_\alpha\}$ is any fixed complete orthonormal system in $\mathfrak{H}(\omega)$ and the right hand side is considered as a vector in $\sum_\alpha\oplus L^2(G)$ with α-th component $\langle U_g v, \varphi_\alpha\rangle f(g)$.

Proof. We consider the map generated by (1. 53). Obviously this map is isometric onto, and by this map,

$$\sum_j U_{g_0}v_j\otimes R_{g_0}f_j\rightarrow\{\sum_j\langle U_{gg_0}v_j, \varphi_\alpha\rangle f_j(gg_0)\}_\alpha. \tag{1.54}$$

This gives the operation corresponding to $\sum_\alpha\oplus R$.

Remark. This lemma is deduced as a special case of lemma 1. 23 in the case of G and $\mathfrak{H}(\omega)$ being separable. And the general case was given by J. M. G. Fell [5].

Lemma 1. 25.

$$R\otimes R\sim\sum_\alpha\oplus R_\alpha, \quad (R_\alpha\sim R), \tag{1.55}$$

with multiplicity of $\dim L^2(G)$. And the equivalence relation is generated by

$$f_1\otimes f_2\rightarrow\left\{\int f_1(g_1 g)\overline{\varphi_\alpha(g_1)}d\mu(g_1)\cdot f_2(g)\right\}_\alpha, \quad \text{for any } f_1,\ f_2 \text{ in } L^2(G), \tag{1.56}$$

where $\Phi=\{\varphi_\alpha\}$ is an arbitrary fixed complete orthonormal system in $L^2(G)$.

Proof. This lemma is an immediate corollary to lemma 1. 24, when $\omega\sim R$.

Notation. *We denote by* $A(\Phi)$, *the map from* $L^2(G)\otimes L^2(G)$ *onto* $\sum_{\alpha}\oplus L^2(G)$, *which is given in lemma 1.25 (1.56), for fixed complete orthonormal system* $\Phi=\{\varphi_\alpha\}$ *in* $L^2(G)$.

Corollary. *Let* $L\equiv\{L_g, L^2(G)\}$ *is the left regular representation of* G, *then*

$$L\otimes L\sim\sum_{\alpha}\oplus L_\alpha, \qquad (L_\alpha\sim L), \tag{1.57}$$

with multiplicity of $\dim L^2(G)$. *And the equivalence relation is generated by*

$$\begin{aligned} f_1\otimes f_2 &\to \{\langle L_g^{-1}f_1, \varphi_\alpha\rangle f_2(g)\}_\alpha \\ &= \left\{\int f_1(gg_1)(\Delta(g))^{1/2}\overline{\varphi_\alpha(g_1)}d\mu(g_1)f_2(g)\right\}_\alpha \end{aligned} \tag{1.58}$$

for f_1, f_2 *in* $L^2(G)$ *and a complete orthonormal system* $\{\varphi_\alpha\}$ *in* $L^2(G)$.

Proof. Since the left regular representation L is equivalent to right regular representation by the equivalence relation,

$$f(g)\to(\Delta(g))^{-1/2}f(g^{-1}), \tag{1.59}$$

so the result follows from lemma 1. 25, immediately.

§2. Duality theorem in a weak form.

1. Let Ω be the set of all equivalence classes of representations of G, dimensions of which are lower than a certain fixed sufficiently large cardinal number (for instance the square of the dimension of $L^2(G)$). We attach a representative ω to each class in Ω. Hereafter we deal Ω as the set of ω's.

Definition 2. 1. *An operator field* $\boldsymbol{T}=\{T(\omega)\}$ *over* Ω *is called a* weak birepresentation *when*

(i) $T(\omega)$ *is a closed operator with dense domain in* $\mathfrak{H}(\omega)$, *and* $T(R)$ *is a non-zero bounded operator on* $L^2(G)$,

(ii) *if* $\omega_1\oplus\omega_2\overset{U_1}{\sim}\omega_3$, *then*

$$U_1(T(\omega_1)\oplus T(\omega_2))U_1^{-1}\subseteqq T(\omega_3), \tag{2.1}$$

(iii) *if* $\omega_1 \otimes \omega_2 \overset{U_2}{\sim} \omega_4$, *then*

$$U_2(T(\omega_1) \otimes T(\omega_2)) U_2^{-1} \subseteqq T(\omega_4). \tag{2.2}$$

Denote the totality of weak birepresentations by $\widehat{\Omega}$, then a product operation in $\widehat{\Omega}$ is introduced by the definition

$$\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = \{T_1(\omega) \cdot T_2(\omega)\}, \qquad \text{for } \boldsymbol{T}_j = \{T_j(\omega)\} \quad (j=1, 2). \tag{2.3}$$

As shown later all weak birepresentation is invertible, so with this product and identity $\boldsymbol{I} = \{I(\omega)$; identity operator over $\mathfrak{H}(\omega)\}$, $\widehat{\Omega}$ becomes a group.

For given g in G, the operator field $\boldsymbol{U}_g \equiv \{U_g(\omega)\}$ over Ω is a weak birepresentation. And it is easy to see that the map φ: $g \rightarrow \boldsymbol{U}_g$ gives an algebraic *homomorphism* of G *into* $\widehat{\Omega}$. We shall show the following.

Proposition 2.1. *φ is an (algebraic)* isomorphism *of G* onto *$\widehat{\Omega}$. That is the same, for any given weak birepresentation $\boldsymbol{T}$, there exists unique g in G such that*

$$\boldsymbol{U}_g = \boldsymbol{T}. \tag{2.4}$$

On the other hand, we can define topology τ on $\widehat{\Omega}$, as the weakest topology which makes all matrix elements $\langle T(\omega)u, v\rangle$ (ω and u, v in $\mathfrak{H}(\omega)$ are fixed) continuous.

Definition 2.2. *τ is called the* weak topology *on $\widehat{\Omega}$.*

Hereafter we consider $\widehat{\Omega}$ with the weak topology; then,

Proposition 2.2. *φ is a bicontinuous map.*

Connecting the above two propositions, we get

Theorem 1. *G is isomorphic to $\widehat{\Omega}$ as a topological group, and the isomorphism is given by φ.*

In the following section 2 and 3, we shall prove the proposition 2.1, and the section 4 is devoted to show proposition 2.2, some examples are given in the section 5.

2. Let $\boldsymbol{T}=\{T(\omega)\}$ be a given weak birepresentation.

Lemma 2. 1.

$$T(1)=I(1). \tag{2.5}$$

Proof. Put $\omega_1=\omega_4=R$, $\omega_2=1$, and $U_2=I(R)$, in (2. 2), then we get the result.

Lemma 2. 2.

$$T(\omega)\in\{U_g(\omega): g\in G\}''. \tag{2.6}$$

($T(\omega)$ commutes with any element of $\{U_g(\omega): g\in G\}'$.)

Proof. Again in (2. 2) put $\omega_1=\omega_4=\omega$, $\omega_2=1$, and let U_2 be any unitary operator in $\{U_g(\omega): g\in G\}'$ over $\mathfrak{H}(\omega)$, then the result follows from lemma 2. 1.

From lemma 2. 2, if we define for any representation ω of G,

$$T(\omega)\equiv U^{-1}T(\omega')U, \tag{2.7}$$

where ω' is the representative in Ω belonging to the same equivalence class of ω, then $T(\omega)$ is uniquely defined and the relations (ii) and (iii) are true for this extendedly defined $T(\omega)$. Only for simplicity of notions, we shall use this extension of $\boldsymbol{T}$ under the same notations.

Lemma 2. 3. *Let A be any set of parameters α. If*

$$\sum_{\alpha\in A}\oplus\omega_\alpha\overset{U}{\sim}\omega, \tag{2.8}$$

and $T(\omega_\alpha)$'s are all bounded operators, then

$$U(\sum_{\alpha\in A}\oplus T(\omega_\alpha))U^{-1}=T(\omega). \tag{2.9}$$

That is, the relation (ii) *is extendable to infinite direct sum.*

Proof. When A is a finite set, (2. 9) follows from (2. 1) easily.

Let A be infinite, and put $T_\alpha\equiv U(T(\omega_\alpha))U^{-1}$, and $\mathfrak{H}_\alpha\equiv U\mathfrak{H}(\omega_\alpha)$, $P_\alpha\equiv$(the projection of $\mathfrak{H}(\omega)$, image of which is $\mathfrak{H}_\alpha$), then it is sufficient to show,

$$\sum_{\alpha\in A}\oplus T_\alpha=T(\omega)\quad(\equiv T). \tag{2.10}$$

At first the left hand side, $\widetilde{T}\equiv\sum_{\alpha\in A}\oplus T_\alpha$ is closed. In fact, let $\{v^j\}$ be a sequence in $\mathfrak{H}(\omega)$, such that $v=\lim v^j$, and $v_0=\lim\widetilde{T}v^j$. Since $P_\alpha v=\lim P_\alpha v^j$ and T_α is bounded,

$$T_\alpha P_\alpha v=\lim_j T_\alpha P_\alpha v^j=\lim_j P_\alpha \widetilde{T}v^j=P_\alpha \lim_j \widetilde{T}v^j=P_\alpha v_0. \tag{2.11}$$

This assures that $\widetilde{T}v=\sum_\alpha T_\alpha P_\alpha v$ exists and is equal to v_0. That is, v is in $\mathfrak{D}(\widetilde{T})$ and $v_0=\widetilde{T}v$ therefore, $\widetilde{T}$ is closed.

While for any given finite set F in A, put $\mathfrak{H}_F=\sum_{\alpha\in F}\oplus\mathfrak{H}_\alpha$ (a subspace of $\mathfrak{H}(\omega)$), and ω_F the restriction of ω onto $\mathfrak{H}_F$. Then $\sum_{\alpha\in F}\oplus\omega_\alpha\overset{U_F}{\sim}\omega_F$ where U_F is a restriction of U onto $\sum_{\alpha\in F}\oplus\mathfrak{H}(\omega_\alpha)$. Thus from the finiteness of F, $\sum_{\alpha\in F}\oplus T_\alpha=T_F$, in which T_F is the restriction of T on $\mathfrak{H}_F$ and zero on $\mathfrak{H}_F{}^\perp$. Then for any v in $\mathfrak{H}_F$,

$$\widetilde{T}v=\sum_{\alpha\in A}T_\alpha v=\sum_{\alpha\in F}T_\alpha v=T_F v=Tv.$$

But the union of $\mathfrak{H}_F$'s, in which F are finite subsets in A, is dense in $\mathfrak{H}(\omega)$, and T is closed, so T must be equal to $\widetilde{T}$, as two closed operators which coincides on a dense set.

Lemma 2.4. *For two weak birepresentations* $\boldsymbol{T}_1$ *and* $\boldsymbol{T}_2$, *if*

$$T_1(R)=T_2(R), \tag{2.12}$$

then $\boldsymbol{T}_1=\boldsymbol{T}_2$, *that is the same,* $T_1(\omega)=T_2(\omega)$ *for any* ω *in* Ω.

Proof. For (2.2) and lemmata 1.24 and 2.3, $T_j(\omega)v\otimes T_j(R)f$, $(j=1,2)$, correspond to the same vector $\{T_1(R)(\langle U.v,\varphi_\alpha\rangle f)(g)\}_\alpha=\{T_2(R)(\langle U.v,\varphi_\alpha\rangle f)(g)\}_\alpha$ in $\sum_\alpha\oplus L^2(G)$ by (1.53). That is, $T_1(\omega)v\otimes T_1(R)f=T_2(\omega)v\otimes T_2(R)f$, for any v in $\mathfrak{H}(\omega)$, f in $L^2(G)$. Since $T_1(R)=T_2(R)\neq 0$ by the assumption, there exists a non-zero vector f in $L^2(G)$ such that $T_1(R)f=T_2(R)f\neq 0$. Therefore, by lemma 1.1, $T_1(\omega)v=T_2(\omega)v$ for any v in $\mathfrak{H}(\omega)$. That is, $T_1(\omega)=T_2(\omega)$ for any ω.

Lemma 2.4 asserts that for given weak birepresentation $\boldsymbol{T}$, if there exists an element g in G such that $T(R)=R_g$, then $\boldsymbol{T}=\boldsymbol{U}_g$. While the regular representation separates any two elements in G, so such a g must be unique, and in this case proposition 2.1 is proved. So we have to determine the form of operator $T(R)$.

Lemma 2.5. *For any weak birepresentation* $\boldsymbol{T}$, $T(R)$ *(denote by* T*) satisfies followings.*

(i) T *is a non-zero bounded operator on* $L^2(G)$,

(ii) *for any* g *in* G,

$$TL_g = L_g T, \tag{2.13}$$

(iii) *for any complete orthonormal system* Φ *in* $L^2(G)$, *if* $A(\Phi)(f_1 \otimes f_2) = \{h_\alpha\}_\alpha$, *then*

$$A(\Phi)(Tf_1 \otimes Tf_2) = \{Th_\alpha\}_\alpha . \tag{2.14}$$

Proof. (i) and (iii) follow from the definition of weak birepresentation and lemma 2.3, immediately. From $L_g \in \{R_{g_1}: g_1 \in G\}'$ for any g in G, lemma 2.2 results (ii).

Lemma 2.6. *From the conditions* (i) *and* (iii) *of lemma 2.5, it results,*

$$\| T \| = 1. \tag{2.15}$$

Proof. (iii) asserts that $T \otimes T$ corresponds to a multiple of T, so $\| T \|^2 = \| T \otimes T \| = \| T \|$ (see, lemma 1.3). The assumption (i) of "$T \neq 0$", leads to (2.16).

Remark. The same argument as lemma 2.6, applying to the relation between $T(\omega) \otimes T(R)$ and multiple of $T(R)$, results

$$\| T(\omega) \| \leqq 1, \quad \text{for any } \omega \text{ in } \Omega. \tag{2.16}$$

Definition 2.3. *An operator* T *over* $L^2(G)$ *satisfying the conditions* (i)$\sim$(iii) *of lemma 2.5, is called an* admissible *operator.*

3. As a consequence of previous section, for proving the pro position 2.1, it is sufficient to show arbitrary given admissible operator T is the form of $R_{g(T)}$ for some $g(T)$ in G.

Lemma 2.7. *Under the assumption of conditions* (i) *and* (ii) *in lemma 2.5, the condition* (iii) *is equivalent to the following* (iii)$'$,

(iii)$'$ *for any h in $C_0(G)$ and f in $L^2(G)$,*

$$T(h\cdot f)(g)=(Th)(g)\cdot(Tf)(g), \qquad a.e.\mu, \tag{2.17}$$

where "·" shows ordinary product of functions.

Proof. (iii)$\rightarrow$(iii)$'$. We use only properties of boundness of T and (iii). Because of $h, f, h\cdot f$ are all in $L^2(G)$ and T is bounded, both sides in (2.17) are determined as functions of g except on a μ-null set.

By the lemma 1.25, the followings are valid,

$$\{Th_\alpha\}_\alpha=\left\{T\left[\int h(g_1\cdot)\overline{\varphi_\alpha(g_1)}d\mu(g_1)\cdot f\right](g)\right\}_\alpha, \tag{2.18}$$

$$A(\Phi)(Th\otimes Tf)=\left\{\int(Th)(g_1g)\overline{\varphi_\alpha(g_1)}d\mu(g_1)(Tf)(g)\right\}_\alpha. \tag{2.19}$$

The condition (iii) asserts the coincidence of each components of the right hand sides in (2.18) and (2.19) as functions in $L^2(G)$, that is, for any α,

$$\begin{aligned}\int(Th)(g_1g)\overline{\varphi_\alpha(g_1)}d\mu(g_1)\cdot(Tf)(g)\\ =T\left[\int h(g_1\cdot)\overline{\varphi_\alpha(g_1)}d\mu(g_1)\cdot f\right](g), \qquad a.e.\mu.\end{aligned} \tag{2.20}$$

But from arbitrariness of complete orthonormal system Φ in $L^2(G)$, (2.20) is true even if we replace φ_α by any function φ in $L^2(G)$. So

$$\begin{aligned}\int(Th)(g_1g)\overline{\varphi(g_1)}d\mu(g_1)\cdot(Tf)(g)\\ =T\left[\int h(g_1\cdot)\overline{\varphi(g_1)}d\mu(g_1)\cdot f\right](g), \qquad a.e.\mu.\end{aligned} \tag{2.21}$$

Now consider a filter base of functions on $C_0(G)$ which tends to the Dirac's measure δ on e of G. That is, for any neighborhood V of e, let $F_V=\{\varphi_V\}$ be the set of non-negative functions in $C_0(G)$ such that $[\varphi_V]\subset V$ and $\int\varphi_V(g)d\mu(g)=1$, then $\mathfrak{F}=\{F_V\}_V$ generates a filter base which converges to δ.

In the left side of (2.21), Th is in $L^2(G)$ and $(Th)(g_1g)$ is equal to $\Delta(g_1)^{-1/2}(L_{g_1}Th)(g)$, so it is near to Th in $L^2(G)$ if g_1 is sufficiently near to e in G. The convexity of norm assures that

$$\int (Th)(g_1g)\overline{\varphi_V(g_1)}d\mu(g_1)=\int \Delta(g_1)^{-1/2}(L_{g_1}Th)(g)\overline{\varphi_V(g_1)}d\mu(g_1)$$

(denote by $(Th)_V(g)$) is near to Th in $L^2(G)$ for sufficiently small neighborhood V of e. This shows that $\{(Th)_V\}_V$ converge to (Th) in $L^2(G)$.

On the other hand, in the right side, $\int h(g_1g)\overline{\varphi_V(g_1)}d\mu(g_1)$ (write by $h_V(g)$) converges to $h(g)$ uniformly, then $h_V\cdot f$ converges to $h\cdot f$ in $L^2(G)$. Using the boundedness of T, the convergence of right sides to $T(hf)$ in $L^2(G)$ follows.

Since $L^2(G)$ is a metric space, we can select a sequence $\{V(j)\}$ of neighborhoods of e, such that both sequences $\{(Th)_{V(j)}\}$ and $\{T(h_{V(j)}\cdot f)\}$ converge to Th and $T(h\cdot f)$ in $L^2(G)$ respectively. If it is necessary, taking a subsequence, the functions $\{(Th)_{V(j)}(g)\}$ and $\{T(h_{V(j)}\cdot f)(g)\}$ converge to $(Th)(g)$ and $T(h\cdot f)(g)$ almost everywhere in μ, satisfying (2.21). So in the limit, we get (2.17).

(iii)$\to$(iii)$'$. It is sufficient to show (2.21), for any h, φ, f in $L^2(G)$. At first, if h and φ are in $C_0(G)$, from the conditions (i) (ii) (iii)$'$,

$$\begin{aligned}&T\left[\int h(g_1\cdot)\overline{\varphi(g_1)}d\mu(g_1)\cdot f\right]\\ &=T\left(\int h(g_1\cdot)\overline{\varphi(g_1)}d\mu(g_1)\right)(g)\cdot(Tf)(g)\\ &=\int (Th)(g_1g)\overline{\varphi(g_1)}d\mu(g_1)\cdot(Tf)(g), \qquad \text{in } L^2(G).\end{aligned}$$

That is, (2.21) holds in this case. While $C_0(G)$ is dense in $L^2(G)$ and from

$$\left|\int f_1(g_1g)\overline{f_2(g_1)}d\mu(g_1)\right|=|\langle R_gf_1, f_2\rangle|\leqq\|f_1\|_2\cdot\|f_2\|_2,$$

$$\|k\cdot f\|_2\leqq\|k\|_\infty\|f\|_2,$$

the above equation is easily extendable to the case in which all h, φ, f are any elements of $L^2(G)$.

Hereafter in this §, let T be a given admissible operator.

Lemma 2.8. *Let E be a G_δ-compact set in G, then there exists a measurable set $T(E)$, such that*

$$T(\chi_E)=\chi_{T(E)}, \quad \text{in } L^2(G). \tag{2.22}$$

Proof. There exists ψ_E in $C_0(G)$ such that $0\leq\psi_E(g)\leq 1$, and $E=\{g: \psi_E(g)=1\}$ (see, P. R. Halmos [11]). Obviously ψ_E^n converges to χ_E in $L^2(G)$ when n tends to ∞. From the boundedness of T, $(T(\psi_E))^n\equiv T(\psi_E^n)\to T(\chi_E)$, $(n\to\infty)$, in $L^2(G)$. If it is necessary, select a subsequence $(T(\psi_E))^n\to T(\chi_E)$, $(n\to\infty)$ *a.e.μ*. Put ψ_E^n as h in lemma 2.7, and take the limits,

$$T(\chi_E\cdot f)=T(\chi_E)\cdot T(f), \quad a.e.\mu. \tag{2.23}$$

Especially $\quad T(\chi_E)=T(\chi_E\cdot\chi_E)=T(\chi_E)\cdot T(\chi_E), \quad a.e.\mu.$

Thus $T(\chi_E)$ must take only values 0 or 1 except over null-set. Put $T(E)=\{g: (T\chi_E)(g)=1\}$, then

$$T(\chi_E)=\chi_{T(E)}, \quad \text{in } L^2(G).$$

Lemma 2.9. *For any G_δ-compact sets E and F in G,*

$$\mu(T(E))\leq\mu(E), \tag{2.24}$$

$$T(E\cap F)\sim T(E)\cap T(F), \tag{2.25}$$

$$T(gE)\sim gT(E), \quad \textit{for any } g \textit{ in } G. \tag{2.26}$$

Proof. From $\|T\|=1$, $\|T\chi_E\|\leq\|\chi_E\|$ follows, so that,

$$\begin{aligned}\mu(E)&=\int\chi_E(g)d\mu(g)=\|\chi_E\|^2\geq\|T\chi_E\|^2=\|\chi_{T(E)}\|^2\\&=\int\chi_{T(E)}(g)d\mu(g)=\mu(T(E)).\end{aligned}$$

Next from (2.22) and (2.23)

$$\begin{aligned}\chi_{T(E\cap F)}&=T(\chi_{E\cap F})=T(\chi_E\cdot\chi_F)=T(\chi_E)\cdot T(\chi_F)\\&=\chi_{T(E)}\cdot\chi_{T(F)}=\chi_{T(E)\cap T(F)}.\end{aligned}$$

This leads to (2.25) immediately.

Lastly we use the condition (ii) of lemma 2.5,

$$\chi_{T(gE)}=T(\chi_{gE})=T(\varDelta(g)^{-1/2}L_g\chi_E)=(\varDelta(g))^{-1/2}L_g(T(\chi_E))$$
$$=(\varDelta(g))^{-1/2}L_g\chi_{T(E)}=\chi_{gT(E)}\,.$$

This shows, (2. 26) is true.

Lemma 2. 10. *T gives an isometry on $L^2(G)$.*

Proof *). Now we consider a set function $\mu_T^0(E)\equiv\mu(T(E))$ on the class $\mathfrak{C}$ of all G_δ-compact set in G_0, which is an open closed subgroup of G generated by a compact neighborhood of e. It is easy to show that μ_T^0 is σ-finite and countably additive on $\mathfrak{C}$, by extension theorem (for instance see P. R. Halmos [11] p. 54 *ThA*), there exists a Bair measure μ_T over G_0 which coincides to μ_T^0 on $\mathfrak{C}$. This measure is uniquely extendable onto whole G by (2. 26), we shall denote this extended measure on G by μ_T too. (2. 24) asserts the absolutely continuity of μ_T with respect to μ and from (2. 26) it is follows that μ_T must be coincide with $c_T\mu$ for some constant c_T. I.e.

$$\|T\chi_E\|^2=\int\chi_{T(E)}(g)\,d\mu(g)=\mu(T(E))=\mu_T(E)=c_T\mu(E)=c_T\|\chi_E\|^2.$$

And it is easy to see, for any step function f on G,

(2. 27) $$\|Tf\|^2=c_T\|f\|^2.$$

But the space of all step functions is dense in $L^2(G)$, (2. 27) is true for all functions f in $L^2(G)$. And (2. 15) results $c_T=1$.

Corollary. *For any G_δ-compact set E,*

(2. 28) $$\mu(T(E))=\mu(E).$$

Proof. In the proof of lemma 2. 10, it is shown that $\mu_T=c_T\mu$, and $c_T=1$. So that (2. 28) is deduced.

Remark. From the relation (2. 22), and the linearity of bounded operator T, easily it is proved that T is real, i.e., for any real valued function f in $L^2(G)$, $T(f)$ is real valued too. Moreover, using

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*) This proof was given by Professor H. Yoshizawa. It is shorter than the author's original proof.

(2. 23), T is positive, that is, for any non negative real valued f in $L^2(G)$, $T(f)$ is non negative too.

Conversely if we presuppose that

i) T is a closed operator on $L^2(G)$ such that its domain $\mathfrak{D}(T)$ contains $C_0(G)$,

ii) $$TL_g \subseteq L_g T,$$

iii) for any h in $C_0(G)$ and f in $L^2(G)$,

$$T(hf) = T(h)\,T(f), \qquad a.e.\,\mu,$$

iv) T is real, (therefore combinating with iii), T is positive), then the boundedness of T follows.

In fact, simple argument starting from i) iii) and iv), as the theory about Radon measures, shows that T induces a regular measure μ_T on G. But from ii), $\Delta^{-1}\mu_T$ is left invariant, therefore by the uniqueness of invariant measure, $\mu_T = c_T\mu$ for some positive number c_T. This results (2. 27), i.e., $c_T^{-1}T$ is an isometry.

So we can take i)$\sim$iv) as a definition of admissible operator. Indeed, this is done in a formulation of Tannaka duality theorem.

Lemma 2. 11. (*N. Iwahori*). *For any h in $C_0(G)$,*

$$\|h\|_\infty = \|Th\|_\infty, \tag{2.29}$$

where $\|\ \|_\infty$ shows essentially superior norm.

Proof. Denote by $\|\ \|_p$ the norm in $L^p(G)$-space then,

$$\begin{aligned}\|Th\|_{2p}^{2p} &= \int |(Th)(g)|^{2p} d\mu(g) = \int |((Th)(g))^p|^2 d\mu(g)\\ &= \int |(T(h^p))(g)|^2 d\mu(g) = \|T(h^p)\|^2 = \|h^p\|^2\\ &= \int |(h(g))^p|^2 d\mu(g) = \|h\|_{2p}^{2p}.\end{aligned}$$

Taking the limits of $2p$-roots of both sides for $p\to\infty$, we get the result.

Lemma 2. 12. *For any G_δ-compact set E, there exists an element $g(E)$ in G such that*

$$T(E) \sim Eg(E). \tag{2.30}$$

Proof. Consider the function

$$\varphi(g) \equiv \int \chi_E(g_1)\chi_E(g_1 g)\,d\mu(g_1) = \mu(E \cap Eg^{-1}). \tag{2.31}$$

Then from (2.26),

$$\begin{aligned}(T\varphi)(g) &= \int \chi_E(g_1)\, T\chi_{g_1^{-1}E}(g)\,d\mu(g_1)\\ &= \int \chi_E(g_1)\chi_{g_1^{-1}T(E)}(g)\,d\mu(g_1)\\ &= \int \chi_E(g_1)\chi_{T(E)}(g_1 g)\,d\mu(g_1) = \mu(E\cap T(E)g^{-1}).\end{aligned} \tag{2.32}$$

Obviously φ is in $C_0^+(G)$ and $T\varphi$ is continuous, so we can apply lemma 2.11, and get,

$$\sup_{g\in G} |(T\varphi)(g)| = \|T\varphi\|_\infty = \|\varphi\|_\infty = \max|\varphi(g)| = \varphi(e) = \mu(E). \tag{2.33}$$

There are two posibilities.

(1) There exists $g(E)$ such that $(T\varphi)(g(E)) = \mu(E)$.

(2) There exists a diverging sequence $\{g_j\}$ such that $\{(T\varphi)(g_j)\}$ increase to $\mu(E)$.

But the second case is excluded. In fact, if (2) is true, then for sufficiently large N, $(T\varphi)(g_j) > (1/2)\mu(E)$, for any $j \geq N$. While the compactness of E assures the existence of j larger than N such that

$$Eg_j g_N^{-1} \cap E = \phi. \tag{2.34}$$

So

$$\begin{aligned}\mu(E) &= \mu(T(E)) = \mu(T(E)g_N^{-1})\\ &\geq \mu(E\cap T(E)g_N^{-1}) + \mu(Eg_j g_N^{-1} \cap T(E)g_N^{-1})\\ &= (T\varphi)(g_N) + \mu(E\cap T(E)g_j^{-1}) = (T\varphi)(g_N) + (T\varphi)(g_j)\\ &> (1/2)\mu(E) + (1/2)\mu(E) = \mu(E).\end{aligned} \tag{2.35}$$

That is contradiction.

For the only case (1), $\mu(E\cap T(E)\cdot g(E)^{-1}) = (T\varphi)(g(E)) = \mu(E) = \mu(T(E))$. This leads to the result immediately.

Lemma 2.13. *For a fundamental system* $\{E_\alpha\}$ *of* G_δ*-compact neighborhoods of e in G, the family* $\mathfrak{F}=\{F_\alpha=\{g_\beta: E_\beta\subseteq E_\alpha\}\}_\alpha$, *where* $g_\alpha\equiv g(E_\alpha)$ *satisfies* (2.30) *for each* E_α *respectively, constructs a base of Cauchy filter in the complete space G. Consequently, there exists unique limit point*

$$\lim g_\alpha=(g_T)^{-1}, \qquad \text{in } G. \tag{2.36}$$

Proof. Since μ is a Haar measure, any open set in G is not μ-measure zero. From (2.25), for any α and β.

$$\begin{aligned} 0\neq\mu(E_\alpha\cap E_\beta)&=\mu(T(E_\alpha\cap E_\beta))\\ &=\mu(T(E_\alpha)\cap T(E_\beta))=\mu(E_\alpha g_\alpha\cap E_\beta g_\beta). \end{aligned} \tag{2.37}$$

I.e.,

$$g_\alpha g_\beta^{-1}\in E_\alpha^{-1}E_\beta, \qquad \text{for any } \alpha \text{ and } \beta. \tag{2.38}$$

Because $\{E_\alpha\}$ is a fundamental system of neighborhoods of e, the family $\mathfrak{F}$ constructs a base of Cauchy filtre.

Lemma 2.14. *For any f in* $L^2(G)$,

$$T\cdot f=R_{g_T}f. \tag{2.39}$$

Proof. Let $\{E_\alpha\}$ be a fundamental system of neighborhoods as in lemma 2.13. Now, for any h in $C_0(G)$ define a function h_α by

$$h_\alpha(g)\equiv(1/\mu(E_\alpha))\int h(g_1^{-1})\chi_{E_\alpha}(g_1g)d\mu(g_1). \tag{2.40}$$

Then,

$$(Th_\alpha)(g)=(1/\mu(E_\alpha))\int h(g_1^{-1})(T\chi_{g_1^{-1}E_\alpha})(g)d\mu(g_1). \tag{2.41}$$

Substitute the following in (2.41),

$$\begin{aligned} (T\chi_{g_1^{-1}E_\alpha})(g)&=\chi_{T(g_1^{-1}E_\alpha)}(g)=\chi_{g_1^{-1}T(E_\alpha)}(g)=\chi_{g_1^{-1}E_\alpha\cdot g_\alpha}(g)\\ &=\chi_{E_\alpha}(g_1gg_\alpha^{-1}). \end{aligned} \tag{2.42}$$

So, we get,

$$\begin{aligned} (Th_\alpha)(g)&=(1/\mu(E_\alpha))\int h(g_1^{-1})\chi_{E_\alpha}(g_1gg_\alpha^{-1})d\mu(g_1)\\ &=(1/\mu(E_\alpha))\int h(gg_\alpha^{-1}g_1^{-1})\chi_{E_\alpha}(g_1)d\mu(g_1). \end{aligned} \tag{2.43}$$

Now take the limits of both sides for $E_\alpha \to \{e\}$. Since h_α converges to h in $L^2(G)$, Th_α converges to $R_{g_T}h$ in $L^2(G)$, so we get

$$(2.44) \qquad Th = R_{g_T}h, \qquad \text{in } L^2(G).$$

While both T and R_{g_T} are bounded and $C_0(G)$ is dense in $L^2(G)$, this shows (2.39) soon.

(2.39) gives the assertion of proposition 2.1.

4. Now we shall state the proof of proposition 2.2, following to the paper of J. Ernest [4]. Based on the proposition 2.1, which is just proved, we shall identify $\widehat{\Omega}$ to G in this section.

Lemma 2.15. *The weak topology τ on $\widehat{\Omega} = G$ is weaker than the initial topology τ_0 on G.*

Proof. Since any matrix element $\langle U_g(\omega)u, v\rangle$ is a continuous function on G with respect to τ_0, and τ is the weakest topology which makes all matrix elements continuous. Thus τ must be weaker than τ_0.

Lemma 2.16. *The topology τ' which makes matrix elements $\{\langle R_g h, h\rangle : (h \in C_0(G))\}$ continuous, is stronger than τ_0.*

Proof. It is sufficient to show that for any neighborhood V of e in G, there exists non-zero matrix element $\langle R_g h, h\rangle$, the carrier of which as a function on G is in V. Let U be a neighborhood of e such that $UU^{-1} \subset V$, and h be a function in $C_0^+(G)$, carrier of which is in U. Then the carrier of function

$$(2.45) \qquad \langle R_g h, h\rangle = \int h(g_1 g)\overline{h(g_1)}d\mu(g_1),$$

is in $UU^{-1} \subset V$.

It is evident that τ is stonger than τ', so combining lemmata 2.15 and 2.16, we get a proof of proposition 2.2.

5. In the definition of birepresentation, we took apparently unnatural condition (i), i.e., we assumed

a) $T(R)$ is bounded,

b) $T(R)\neq 0$.

But the following examples show that these conditions are necessary to prove the duality of this type. (cf. §0).

Example 1. (Necessity of condition a) I). Let G be $\boldsymbol{R}=\{x\}$ (additive group of real numbers), then $\hat{G}=\{\omega(\rho)\}$ can be identified to $\boldsymbol{R}=\{\rho\}$ too. Any element x of initial group G corresponds in one-to-one way to a unitary character $e^{ix\rho}$ on $\hat{G}$. For given representation ω of $\boldsymbol{R}$ with an irreducible decomposition

$$\omega\cong\int_{\boldsymbol{R}}[n(\rho)]\,\omega(\rho)d\nu(\rho),$$

where $n(\rho)$ gives the multiplicity of $\omega(\rho)$ in ω, $U_x(\omega)$ corresponds to the map multiplying $e^{ix\rho}$ on $\omega(\rho)$-components of vectors.

Now consider a non-unitary character $e^{iz\rho}$ ($z\in\boldsymbol{C}$, and $\mathfrak{J}_m z\neq 0$) and the map on $\mathfrak{H}(\omega)$ multiplying $e^{iz\rho}$ on $\omega(\rho)$-components of vectors, then easily to see, this map defines a non-zero non-bounded (in general) closed operator $T(\omega)$ with dense domain in $\mathfrak{H}(\omega)$, and the operator field $\boldsymbol{T}=\{T(\omega)\}$ satisfies the conditions of definition of birepresentation with only exception a).

Example 2. (Necessity of condition a) II). In the case of that G is a compact Lie group, C. Chevalley's result [2] shows that if we don't assume unitarity nor reality of $\boldsymbol{T}$, then operator fields given by such a method, corresponds in one-to-one way to the complexification of G. The remark cited after lemma 2. 10 means that this is the case in which $T(R)$ is not bounded. This results the necessity of condition a) also.

Remark. The above two examples tempt us to make a conjecture that if we take the set of all operator fields, components $T(\omega)$ of which are bounded on each $\mathfrak{H}(\omega)\,(\omega\in\hat{G})$, but not necessarily (uniformly) bounded on the regular representation R and satisfy (ii) and (iii), it will become a group corresponding to a complexification of G. However we shall show later (§5), for some semi-simple Lie groups, if $\boldsymbol{T}$ satisfies (ii) and (iii), the boundedness of $T(R)$ follows from the boundedness of $T(\omega)$ on $\mathfrak{H}(\omega)(\omega\in\widehat{G})$. Therefore, in such a case, the above mentioned set of operator fields does not be extended from the initial group.

Example 3. (Necessity of condition b) I). Let G be a separable locally compact group having no finite dimensional irreducible representation except trivial representation 1, (for instance, $SL(n, \boldsymbol{C})$, $SL(n, \boldsymbol{R})$ etc.). Then any Kronecker product $\omega_1\otimes\omega_2$ of irreducible representations ω_1 and ω_2 except $1\otimes 1$ has not subrepresentation 1. This assures that if we define an operator field $\boldsymbol{T}=\{T(\omega)\}$, such that, (i) $T(1)=1$, (ii) $T(\omega)=0$, for any $\omega\neq 1$ in $\widehat{G}$, (iii) $T(\omega)\overset{U}{\sim}\int T(\omega(\sigma))d\nu(\sigma)$ for general ω in Ω which has the irreducible decomposition $\omega\overset{U}{\sim}\int\omega(\sigma)d\nu(\sigma)$, then $\boldsymbol{T}$ gives an operator field satisfying (i)$\sim$(iii) except b) on Ω. Evidently $\boldsymbol{T}\neq 0$, but $T(\omega)$ $(\omega\neq 1)$ are non-unitary. Therefore such a $\boldsymbol{T}$ does not correspond to any element in G by φ.

Example 4. (Necessity of condition b) II). Example 3 deals very simple operator field, so there arises a question, that omitting such operator fields, can we find a group with operator field satisfying (i)$\sim$(iii) except b)? To answer this question, consider a locally compact semi-direct product G of a separable non-compact closed abelian normal subgroup N and a closed subgroup K, which is mentioned in the examples in §4 and §7.

It is easy to see that the set of all representations of type $\mathfrak{D}(\hat{e},\tau)$, which are considerable as representations of the factor group $K\cong N\backslash G$ (cf. §7), generates a "subring" of Ω, and the regular representation has no non-trivial component consisted of these representations. (Call such a component as $\mathfrak{D}(\hat{e},\tau)$-component). Moreover, any Kronecker product of two representations without non-trivial $\mathfrak{D}(\hat{e},\tau)$-components has no non-trivial $\mathfrak{D}(\hat{e},\tau)$-component. While any representation in Ω can be decomposed into the form, $\omega\overset{U}{\sim}\omega_1\oplus\int\mathfrak{D}(\hat{e},\tau)d\nu(\tau)$, where ω_1 has no non-trivial $\mathfrak{D}(\hat{e},\tau)$-component.

Now fix an element k in K and let $W_k(\tau)$ be the operator of representation τ of K. Define an operator field $\boldsymbol{T}=\{T(\omega)\}$ by $T(\omega)\overset{U}{\sim}0(\omega_1)\oplus\int W_k(\tau)d\nu(\tau)$, where $0(\omega_1)$ is zero operator on $\mathfrak{H}(\omega_1)$. Short arguments shows that $\boldsymbol{T}$ satisfies i) $\sim$ iii) except b) and $T(R)=0$.

Example 5. (Necessity of condition b) III). Analogous example is given for non-compact locally compact abelian groups too. Let Ω_0 be the set of discrete direct sum of elements of $\widehat{G}$ in Ω. Any representation ω is decomposable to the form $\omega_1\oplus\omega_2$ where ω_1 is in Ω_0 and ω_2 has no dirscrete irreducible component. Let ω_1 be represented as a sum $\sum_\alpha\oplus\omega_1^\alpha$ by irreducible components ω_1^α.

Fix an element g in G, and denote by $g(\omega)$, the character on $\widehat{G}$ corresponding to g. And define for above mentioned ω,

$$T(\omega)\equiv T(\omega_1)\oplus 0(\omega_2)\equiv\sum_\alpha\oplus g(\omega_1^\alpha)\oplus 0(\omega_2).$$

Then $\boldsymbol{T}=\{T(\omega)\}$ gives an example required.

Example 6. (Loosening of condition b) I). For a compact group G, since, as well-known, any representation of G is imbedded as a non-trivial subrepresentation in a multiple of regular representations, if $\boldsymbol{T}=\{T(\omega)\}$ satisfies i) $\sim$ iii) except b) and $T(R)=0$, then $\boldsymbol{T}\equiv 0$.

So that, for such a group, the only assumption $\boldsymbol{T}\not\equiv 0$ results automatically, $T(R)\neq 0$.

Example 7. (Loosening of condition b) II). When $G=SL(2, \boldsymbol{C})$, the situation is analogous to example 6. To explain this we quote the results of M. A. Naimark [16].

In this case, $\widehat{G}$ is separated into three parts as follows [7].

α) $\widehat{G}_0=\{$trivial representation, $1\}$,

β) $\widehat{G}_p=\{$principal series, $\mathfrak{S}_{m,\rho}\}$,

γ) $\widehat{G}_s=\{$supplementary series, $\mathfrak{D}_\nu$, $(0<\nu<2)\}$.

The extended Plancherel theorem [7] shows that the regular representation of G is decomposed as a continuous direct sum over $\widehat{G}_p$.

And the results of M. A. Naimark [16] asserts the following.

Lemma 2.17. 1) *Kronecker product* $\omega_1 \otimes \omega_2$ *of elements of* $\widehat{G}$ *is equivalent to a subrepresentation of* R *with the exceptional case of,*

(i) *at least, one of* ω_1 *and* ω_2 *is 1,*

(ii) $\omega_1 = \mathfrak{D}_{\nu_1}$, *and* $\omega_2 = \mathfrak{D}_{\nu_2}$, $(2 < \nu_1 + \nu_2)$.

2) *In the case of* (ii) $(\omega_1 \otimes \omega_2 = \mathfrak{D}_{\nu_1} \otimes \mathfrak{D}_{\nu_2})$, *if we remove the component of* $\omega_1 \otimes \omega_2$ *which is equivalent to* $\mathfrak{D}_{\nu_1+\nu_2-2}$, *then the remaining subrepresentation is equivalent to a subrepresentation of* R. (Proof is omitted).

We can obtain some results about exceptional case (ii).

Lemma 2.18. *Let* $2 < \nu_1 + \nu_2$, *and the projection onto* $\mathfrak{D}_{\nu_1+\nu_2-2}$-*component in* $\mathfrak{D}_{\nu_1} \otimes \mathfrak{D}_{\nu_2}$ *be* P_0. *Then for vectors* u *in* $\mathfrak{H}(\mathfrak{D}_{\nu_1})$, *and* v *in* $\mathfrak{H}(\mathfrak{D}_{\nu_2})$ *if*

$$(I - P_0)(u \otimes v) = 0, \tag{2.46}$$

then

$$u \otimes v = 0.$$

Proof. As shown in [16], $\mathfrak{H}(\mathfrak{D}_{\nu_1}) \otimes \mathfrak{H}(\mathfrak{D}_{\nu_2})$ is considered as a space of the completion of $C_0(\boldsymbol{C} \times \boldsymbol{C})$ with respect to the norm

$$\|f\| = \{\int |z_1 - z_1'|^{-2+\nu_1} |z_2 - z_2'|^{-2+\nu_2} \times f(z_1, z_2) \overline{f(z_1', z_2')} dz_1 dz_2 dz_1' dz_2'\}^{1/2}. \tag{2.47}$$

Consider a sequence $\{k_n(z_1, z_2) = k(z_1) h_n(z_1 - z_2)\}$ in $C_0(\boldsymbol{C} \times \boldsymbol{C})$, in which k is a fixed element in $C_0(\boldsymbol{C})$ and $\{h_n\}$ is a sequence which tends to the Dirac's measure δ on the origin 0 of $\boldsymbol{C}$. It is easy to see that if $\nu_1 + \nu_2 - 2 > 0$, then $\{k_n\}$ constructs a Cauchy sequence in $\mathfrak{H}(\mathfrak{D}_{\nu_1}) \otimes \mathfrak{H}(\mathfrak{D}_{\nu_2})$. Denote the limit point of this sequence by k^0, then its norm is given by

$$\|k^0\| = \left\{\int |z - z'|^{-4+\nu_1+\nu_2} k(z) \overline{k(z')} dz dz'\right\}^{1/2}, \tag{2.48}$$

and immediate calculation shows

$$(2.49)\qquad U_g(\mathfrak{D}_{\nu_1}\otimes\mathfrak{D}_{\nu_2})k^0=\left(|\beta z+\delta|^{-\nu_1-\nu_2}k\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\right)^0.$$

Thus, the map $k\to k^0$ gives a unitary equivalence between $\mathfrak{D}_{\nu_1+\nu_2-2}$ and a subrepresentation of $\mathfrak{D}_{\nu_1}\otimes\mathfrak{D}_{\nu_2}$ on the G-invariant space $\overline{\{k^0; k\in C_0(\boldsymbol{C})\}}$.

Now we consider the Fourier transform of f with respect to (z_1, z_2),

$$(2.50)\qquad \tilde{f}(w_1, w_2)\equiv\int_{\boldsymbol{C}^2} f(z_1, z_2)e^{i\Re e(z_1\bar{w}_1+z_2\bar{w}_2)}dz_1dz_2,$$

$$(2.51)\qquad \|f\|^2=\|\tilde{f}\|^2\equiv c_1\int_{\boldsymbol{C}^2}|w_1|^{-\nu_1}|w_2|^{-\nu_2}|\tilde{f}(w_1, w_2)|^2dw_1dw_2,$$

with normalizing constant $c_1(\neq 0)$.

Since $C_0(\boldsymbol{C}\times\boldsymbol{C})$ is dense in $\mathfrak{H}(\mathfrak{D}_{\nu_1})\otimes\mathfrak{H}(\mathfrak{D}_{\nu_2})$ the latter space is mapped into $L^2(\boldsymbol{C}^2, |w_1|^{-\nu_1}|w_2|^{-\nu_2}dw_1dw_2)$. Especially the image of a vector $u\otimes v$ is a product of functions $f_1(w_1)$, $f_2(w_2)$ which correspond to u, v respectively, so f_j is in $L^2(\boldsymbol{C}, |w|^{-\nu_j}dw)$.

While the image of k^0 in the subspace $\mathfrak{H}(\mathfrak{D}_{\nu_1+\nu_2-2})$ is the limit of

$$\begin{aligned}(2.52)\qquad (\tilde{k}_n)(w_1, w_2)&=\int_{\boldsymbol{C}^2}k(z_1)h_n(z_1-z_2)e^{i\Re e(z_1\bar{w}_1+z_2\bar{w}_2)}dz_1dz_2\\ &=\int_{\boldsymbol{C}^2}k(z_1)h_n(z_2)e^{i\Re e(z_1\bar{w}_1+(z_1-z_2)\bar{w}_2)}dz_1dz_2\\ &=\tilde{k}(w_1+w_2)\tilde{h}_n(-w_2).\end{aligned}$$

Because the limit of $\tilde{h}_n$ in $(n\to\infty)$ is 1, the image of vector in the space $\mathfrak{H}(\mathfrak{D}_{\nu_1+\nu_2-2})$ is a function of type of $f(w_1+w_2)$.

(2.46) shows that $u\otimes v$ is in $\mathfrak{H}(\mathfrak{D}_{\nu_1+\nu_2-2})$, so that the corresponding function in $L^2(\boldsymbol{C}^2, |w_1|^{-\nu_1}|w_2|^{-\nu_2}dw_1dw_2)$ is the form of $f(w_1+w_2)$ and $f_1(w_1)f_2(w_2)$ at same time. Therefore

$$(2.53)\qquad f(w_1+w_2)=f_1(w_1)f_2(w_2).$$

Substitute w_1+w_2+w in w_1+w_2 and consider integrals, for any φ in $C_0(\boldsymbol{C})$,

$$(2.54)\qquad \int f_1(w_1+w)\varphi(w_1)dw_1\cdot f_2(w_2)=\int f_1(w_1)\varphi(w_1)dw_1\cdot f_2(w_2+w).$$

This shows that f_2 is continuous. Analogously f_1 is continuous too. Moreover if one of $f_j(0)=0$, then $f(w)=f_1(0)f_2(w)=f_1(w)f_2(0)=0$, this is the result of lemma. So we assume that $u\otimes v\neq 0$, then $f_1(0)f_2(0)\neq 0$, and

$$f_2(w)=cf_1(w), \quad (c=(f_2(0)/f_1(0))\neq 0). \tag{2.55}$$

That is,

$$\tilde{f}(w_1+w_2)=f_1(w_1)\cdot cf_1(w_2)=f_1(w_1+w_2)\cdot cf_1(0). \tag{2.56}$$

This results that $(f_1(w)/f_1(0))$ must be a character of additive group $\boldsymbol{C}$.

But any character on $\boldsymbol{C}$ does not belong to $L^2(\boldsymbol{C}, |w|^{-\nu_1}dw)$. That is contradiction.

Let ω be a representation of $SL(2, \boldsymbol{C})$ without 1-components. Simple argument shows that $\omega\otimes\omega$ does not contain 1-components, so the irreducible decomposition of $\omega\otimes\omega$ is separated into two parts,

$$\omega\otimes\omega\sim\int_{X_p}\mathfrak{S}_{m\rho}\oplus\int_{X_s}\mathfrak{D}_\nu .$$

The first term is a subrepresentation of $\sum\oplus R$ by lemma 2.17. We put the projection onto this subrepresentation as P_1.

Corollary. *For any u, v in $\mathfrak{H}(\omega)$, where ω is as above, if*

$$P_1(u\otimes v)=0, \tag{2.57}$$

then

$$u\otimes v=0.$$

Proof. Let the irreducible decomposition of ω be

$$\omega\sim\int_A\omega(\tau)d\nu_0(\tau). \tag{2.58}$$

And let $u=\{u(\tau)\}$ be the correspondence of vectors in this decomposition, then

$$\omega\otimes\omega\sim\iint_{A\times A}\omega(\tau_1)\otimes\omega(\tau_2)d\nu_0(\tau_1)d\nu_0(\tau_2). \tag{2.59}$$

The G-invariant projection P_1 is induced by a field of G-invariant projections $\{P_1(\tau_1, \tau_2)\}$ on $A\times A$, such that the range of $[I-P_1(\tau_1, \tau_2)]$ is $\mathfrak{D}_\nu$-component (for some ν) in $\omega(\tau_1)\otimes\omega(\tau_2)$ for almost all (τ_1, τ_2). So that, (2.57) is equivalent to

$$(2.60) \qquad P_1(\tau_1, \tau_2)(u(\tau_1)\otimes v(\tau_2))=0, \qquad \text{for almost all } (\tau_1, \tau_2).$$

Using lemma 2.18, such a $u(\tau_1)\otimes v(\tau_2)$ must be zero for (τ_1, τ_2), on which (2.60) is valid. This leads us to the result.

Now we consider an operator field $\boldsymbol{T}=\{T(\omega)\}$ over Ω satisfying (i)$\sim$(iii) of definition 2.1 except b) ($T(R)\neq 0$).

Lemma 2.19. *Under the same assumption of* ω *as above,* "$T(\omega)=0$" *follows from* "$T(R)=0$".

Proof. If $T(\omega)\neq 0$, there exists a vector u in $\mathfrak{H}(\omega)$ such that $T(\omega)u\neq 0$. I.e.,

$$(2.61) \qquad T(\omega)u\otimes T(\omega)u\neq 0.$$

While by (iii),

$$(2.62) \qquad P_1(T(\omega)u\otimes T(\omega)u)=T\left(\int_{X_p}\mathfrak{S}_{m,p}\right)P_1(u\otimes u).$$

This vector must be zero by the assumption "$T(R)=0$", and (ii).

This contradicts to the above corollary.

This lemma 2.19 shows that the operator field given in example 3, is the only example of non-zero $\boldsymbol{T}$ satisfying "$T(R)=0$" and (i)$\sim$(iii) of definition 2.1 except b).

§3. Duality theorem on homogeneous spaces.

1. In [12] N. and N. Iwahori gave a formulation of an extension of Tannaka duality theorem for homogeneous spaces of compact groups. Here we shall prove similar results for a homogeneous space $X\equiv H\diagdown G$, in which H is a compact subgroup of a locally compact group G.

Let π be the canonical map of G onto X by which g in G corresponds to the coset Hg. Because of compactness of H, there is an invariant measure ν_0 on X which is unique besides equivalence (cf. A. Weil [23]). For simplicity we take a measure $\nu=\Delta\nu_0$ on X, in which $\Delta(g)$ is considered as a continuous function on X, because the modulus Δ of two Haar measures satisfies $\Delta(h)\equiv 1$ for any h in H.

Now we define a representation ω of G over X by

$$\text{(3.1)} \qquad \begin{cases} \mathfrak{H}(\omega)=L^2(\nu;\ X), \\ (U_g(\omega)f)(x)=(\Delta(g))^{1/2}f(x\cdot g), \qquad \text{for } f \text{ in } \mathfrak{H}(\omega). \end{cases}$$

Denote by N the normalizer of H in G, then we can construct a unitary representation of N on this space $\mathfrak{H}(\omega)$.

$$\text{(3.2)} \qquad (W_n f)(x)=f(n\cdot x), \qquad \text{for } f \text{ in } \mathfrak{H}(\omega),$$

where $n\cdot x$ shows the H-coset Hng, if $x=Hg$.

On the other hand, another formulation of these representations are given as in the following. Consider the left regular representation $L=\{L_g,\ L^2(\mu_l;\ G)\}$ of G. Put $\mathfrak{H}$ be the space of all H-invariant vectors, i.e. of all functions f in $L^2(\mu_l;\ G)$ such that

$$\text{(3.3)} \qquad f(hg)=f(g), \qquad \text{for any } h \text{ in } H.$$

Consider the right regular representation

$$\text{(3.4)} \qquad R'_g f(g_1)=(\Delta(g))^{1/2}f(g_1 g), \quad \text{for } f \text{ in } L^2(\mu_l;\ G),$$

at the same time. Then $\{L_n;\ n\in N\}$ and $\{R'_g;\ g\in G\}$ make $\mathfrak{H}$ invariant.

If we consider a function in $\mathfrak{H}$ as a function over $H\diagdown G$ in natural way, the restrictions of $\{L_n\}$ and $\{R'_g\}$ to $\mathfrak{H}$ give equivalent representations to $\{W_n\}$ and $\{U_g(\omega)\}$ respectively. We shall identify these representations.

Now consider $L\otimes L$. As in the case of right regular representation, this representation is decomposed into discrete direct sum of L for any complete orthonormal system $\Phi=\{\varphi_\alpha\}$ in $L^2(\mu_l, G)$, (see, corollary to lemma 1. 25)

$$\text{(3.5)} \qquad L\otimes L\sim\sum\oplus L_\alpha, \qquad (L_\alpha\sim L).$$

And the equivalence relation $A(\Phi)$ is given by

$$f_1 \otimes f_2 \to \{\langle L_g^{-1} f_1, \varphi_\alpha \rangle f_2(g)\}_\alpha . \tag{3.6}$$

Since a product of H-invariant vectors is also H-invariant, the image of $\mathfrak{H} \otimes \mathfrak{H}$ by $A(\Phi)$ is a subspace of $\sum \oplus \mathfrak{H}_\alpha$ in $\sum \oplus \mathfrak{H}(L_\alpha)$, where $\mathfrak{H}_\alpha$ is the subspace of all H-invariant vectors in $\mathfrak{H}(L_\alpha)$.

Let P_α be the projection in $\sum \oplus L^2(\mu_l, G)$ onto $\mathfrak{H}_\alpha$.

Lemma 3.1. *If φ_α is real non negative, then*

$$P_\alpha A(\Phi)(\mathfrak{H} \otimes \mathfrak{H}) = \mathfrak{H}_\alpha . \tag{3.7}$$

Proof. Because of arbitrariness of f_2, it is enough to show that for any g_0 in G, there exists a neighborhood U of g_0, and f_1 in $\mathfrak{H}$ such that

$$\langle L_{g^{-1}} f_1, \varphi_\alpha \rangle \neq 0, \qquad \text{for any } g \text{ in } U. \tag{3.8}$$

Indeed for any k in $C_0(H \backslash G)$ such that $[k] \subset HU$, then $k_0(g) \equiv (k(g) / \langle L_{g^{-1}} f_1, \varphi_\alpha \rangle)$ is also in $C_0(H \backslash G)$ so in $\mathfrak{H}$, and

$$P_\alpha A(\Phi)(f_1 \otimes k_0)(g) = \langle L_{g^{-1}} f_1, \varphi_\alpha \rangle k_0(g) = k(g) \tag{3.9}$$

is in $\mathfrak{H}_\alpha$. But when g_0 runs over G, such functions k spans $\mathfrak{H}_\alpha$.

Now we shall assume that there exists g_0 such that for any f_1 in $\mathfrak{H}$,

$$\begin{aligned} 0 = \langle L_{g_0^{-1}} f_1, \varphi_\alpha \rangle &= \int_G f_1(g_0 g) \overline{\varphi_\alpha(g)}\, d\mu_l(g) \\ &= \int_G f_1(g) \overline{\varphi_\alpha(g_0^{-1} g)}\, d\mu_l(g). \end{aligned} \tag{3.10}$$

Replacing f_1 by

$$\tilde{f}(g) = \int_H f(hg)\, d\mu_H(h), \quad (f \in C_0(G)), \tag{3.11}$$

we get the equation

$$\begin{aligned} 0 &= \int_G \tilde{f}(g) \overline{\varphi_\alpha(g_0^{-1} g)}\, d\mu_l(g) \\ &= \int_G d\mu_l(g) \left\{ \int_H f(hg)\, d\mu_H(h) \overline{\varphi_\alpha(g_0^{-1} g)} \right\} \end{aligned} \tag{3.12}$$

$$=\int_H d\mu_H(h)\left\{\int_G f(g)\overline{\varphi_\alpha(g_0^{-1}h^{-1}g)}d\mu_l(g)\right\}$$

$$=\int_G\left\{f(g)\int_H\overline{\varphi_\alpha(g_0^{-1}h^{-1}g)}d\mu_H(h)\right\}d\mu_l(g).$$

And this results

(3. 13) $$\int_H \varphi_\alpha(g_0^{-1}hg)d\mu_H(h)=0, \quad \text{for almost all } g.$$

But this is impossible.

For any n in N, the followings are evident,

(3. 14) $$W_n U_g(\omega)=U_g(\omega)W_n,$$

(3. 15) $$P_\alpha A(\Phi)(W_n\otimes W_n)A(\Phi)^{-1}=W_n.$$

Conversely we shall take as the definition of admissible operator over $\mathfrak{H}$ by these relations.

Definition 3. 1. *An operator T is called* admissible *operator when*

i) *T is a non-zero bounded operator over $\mathfrak{H}$,*

ii) *$TU_g(\omega)=U_g(\omega)T$, for any g in G,*

iii) *if $P_\alpha A(\Phi)(f_1\otimes f_2)=h_\alpha$, then $P_\alpha A(\Phi)(Tf_1\otimes Tf_2)=Th_\alpha$,*

for any f_1, f_2 in $\mathfrak{H}$ and any complete orthonormal system Φ in $L^2(\mu_l, G)$.

As stated in §2, the set $\tilde{N}$ of all admissible operators constructs a group, and the map φ: $n\to L_n$ is an algebraic homomorphism of N *into* $\tilde{N}$. Under this situation extended Tannaka's duality on the homogeneous space X is stated as follows.

Proposition 3.1. *φ is an (algebraic) homomorphism of N onto $\tilde{N}$, with kernel H. That is the same, for any given admissible operator T there exists n in N such that*

(3. 16) $$W_n=T.$$

Such a n is determined H-cosetwise, that is,

(3. 17) $$W_{n_1}=W_{n_2},$$

if and only if n_1, n_2 *belong to same H-coset in N. So* φ *induces an (algebraic) isomorphism* $\tilde{\varphi}$ *of* $H\backslash N$ *onto* $\widetilde{N}$.

Moreover let τ be the weakest topology on $\widetilde{N}$ which makes all matrix elements $\{\langle Tu, v\rangle, (\forall u, v\in\mathfrak{H})\}$ continuous, then,

Proposition 3. 2. $\tilde{\varphi}$ *is a bicontinuous map.*

Combining these propositions, we get the following.

Theorem 2. $H\backslash N$ *is isomorphic to* $\widetilde{N}$ *by* $\tilde{\varphi}$.

Remark. Obviously this theorem contains the results of §2 as a special case. But by the reason of simplicity of situations, we adopted this procedure.

2. Proofs of these propositions are completely analogous to the case in §2. The series of following lemmata is valid, for given admissible operator T too.

Lemma 3. 2.

$$\|T\|=1. \tag{3.18}$$

Proof. $\|T\|\leq 1$ follows from the condition iii) of definition 3. 1, and

$$A(\Phi)(\mathfrak{H}\otimes\mathfrak{H})\subset\sum_{\alpha}\oplus\mathfrak{H}_{\alpha}\,. \tag{3.19}$$

But the results of lemma 3. 1 and the condition iii) of definition 3. 1 give the contrary inequality.

Lemma 3. 3. T_1 *is an admissible operator, if and only if it satisfies* i) *and* ii) *of definition 3. 1, and*

(iii)′ *for any* h *in* $C_0(X)$ *and* f *in* $\mathfrak{H}$,

$$T_1(h\cdot f)(x)=(T_1 h)(x)\cdot(T_1 f)(x), \tag{3.20}$$

for almost all x *with respect to* ν.

Proof. Analogous to the proof of lemma 2. 7.

Lemma 3. 4. *For any* G_δ*-compact set* E *in* X, *there exists a measurable set* $T(E)$ *such that*

(3.21) $$T(\chi_E)=\chi_{T(E)}, \qquad in\ \mathfrak{H}.$$

Proof. Analogous to the proof of lemma 2.8.

Lemma 3.5.

(3.22) $$\nu(T(E))\leq\nu(E),$$

(3.23) $$T(E\cap F)\overset{\nu}{\sim}T(E)\cap T(F),$$

(3.24) $$T(Eg)\overset{\nu}{\sim}T(E)g, \qquad for\ any\ g\ in\ G,$$

for any G_δ-compact sets E and F in X.

Proof. Analogous to the proof of lemma 2.9.

Lemma 3.6. *T gives an isometry on $L^2(\nu, X)$.*

Proof. Using the uniqueness of G-invariant measure over X, the proof is given analogously as the proof of lemma 2.10.

Corollary.

(3.25) $$\nu(T(E))=\nu(E).$$

Proof. Analogous to the proof of corollary to lemma 2.10.

Lemma 3.7. *For any h in $C_0(X)$,*

(3.26) $$\|h\|_\infty=\|Th\|_\infty .$$

Proof. Analogous to the proof of lemma 2.11.

Since in this case, the related functions are all H-cosetwise, so we have to prove some additive lemma.

Lemma 3.8. *For arbitrary given neighborhood W of e in G, there exists a neighborhood V of e such that*

(3.27) $$HVV^{-1}H\subset HW.$$

Proof. For any neighborhood V_0 of e, a neighborhood V of e such that

(3.28) $$VV^{-1}\subset V_0$$

exists. Therefore, it is enough to show that the existence of V_0

which satisfies

$$HV_0H\subset HW. \tag{3.29}$$

Because of continuity of the map

$$(g_1, g_2)\rightarrow g_1g_2g_1^{-1} \tag{3.30}$$

at (g, e), there is a neighborhood $V(g)$ of e such that

$$V(g)\cdot g\cdot V(g)\cdot(V(g)g)^{-1}\subset W. \tag{3.31}$$

While H is compact, thus finite covering $\bigcup_j V(h_j)h_j$ of H can be selected. Put

$$V_0=\bigcap_j V(h_j), \tag{3.32}$$

then for any h in H, there is a $V(h_j)h_j$ containing h and

$$V(h_j)h_jV(h_j)h_j^{-1}V(h_j)^{-1}\subset W, \tag{3.33}$$

that is,

$$hV_0h^{-1}\subset W, \qquad \text{for any } h \text{ in } H. \tag{3.34}$$

Multiplying H from left, we get (3.29) immediately.

Lemma 3.9. *For any G_δ-compact set E in X, there exists an element $g(E)$ in G such that*

$$\pi^{-1}(T(E))\sim g(E)\pi^{-1}(E), \tag{3.35}$$

$$g(E)^{-1}Hg(E)\subset\pi^{-1}(E)\cdot(\pi^{-1}(E))^{-1}. \tag{3.36}$$

Moreover for any h in H, $g(E)h$ has the same properties.

Proof. Instead of φ in lemma 2.12 (2.31), we put

$$\begin{aligned}\varphi(\pi(g)) &= \int\chi_{\pi^{-1}(E)}(g_1)\chi_{\pi^{-1}(E)}(gg_1)d\mu_l(g_1)\\ &=\mu_l(\pi^{-1}(E)\cap g^{-1}\pi^{-1}(E)).\end{aligned} \tag{3.37}$$

Using the results of lemmata 3.4 and 3.5, we get,

$$(T\varphi)(\pi(g))=\mu_l(\pi^{-1}(E)\cap g^{-1}\pi^{-1}(T(E))). \tag{3.38}$$

The same arguments as in lemma 2.12 can be adopted, so it is obtained the existence of $g(E)$ in G such that

(3. 39) $$\pi^{-1}(T(E)) \sim g(E)\pi^{-1}(E).$$

But $\pi^{-1}(T(E))$ is a H-cosetwise set in G, so

(3. 40) $$hg(E)\pi^{-1}(E) \sim h\pi^{-1}(T(E)) \sim \pi^{-1}(T(E)) \sim g(E)\pi^{-1}(E),$$
for any h in H.

This results (3. 36) immediately.

Because $\pi^{-1}(E)$ is H-cosetwise set, so $g(E)h$ satisfies the same properties.

Lemma 3. 10. *For a fundamental system $\{W_\alpha\}$ of G_δ-compact neighborhoods of e in G, the family $\mathfrak{F} = \{F_\alpha = \{\pi(g_\beta^{-1}) : W_\beta \subseteq W_\alpha\}\}_\alpha$, where $g_\alpha \equiv g(\pi(W_\alpha))$ satisfies (3.35) for each $\pi(W_\alpha)$ respectively, constructs a base of Cauchy filter in the complete space $X = H\backslash G$. Consequently, there exists unique limit point*

(3. 41) $$\lim \pi(g_\alpha^{-1}) = \pi(g_T), \quad in\ X,$$

(3. 42) $$g_T \in N.$$

Proof. As the proof of lemma 2. 13, it is obtained that,

(3. 43) $$T(\pi(W_\alpha)) \cap T(\pi(W_\beta)) \neq \phi.$$

From (3. 35),

(3. 44) $$g_\alpha H W_\alpha \cap g_\beta H W_\beta \neq \phi, \qquad \text{for any } \alpha \text{ and } \beta.$$

On the other hand, from the result of lemma 3. 8, for any neighborhood W of e in G, there exists a W_α such that

(3. 45) $$HW_\alpha W_\alpha^{-1} H \subset HW,$$

so for any W_β, W_γ which are contained in W_α,

(3. 46) $$g_\beta^{-1} \subset HW_\beta W_\gamma^{-1} H g_\gamma^{-1} \subset HW_\alpha W_\alpha^{-1} H g_\gamma^{-1} \subset HW g_\gamma^{-1}.$$

This shows that $\mathfrak{F}$ constructs a base of Cauchy filtre. Let the limit of this filtre be $\pi(g_T)$, we can select $\{(g_\alpha')^{-1} = (h_\alpha g_\alpha)^{-1}, h_\alpha \in H\}$ which converges to g_T in G.

While from (3. 36) being applied to g_α',

(3. 47) $$(g_\alpha')^{-1} H (g_\alpha') \subset HW_\alpha W_\alpha^{-1} H \subset HW.$$

Taking the limit, we get

$$g_T H g_T^{-1} \subset H. \tag{3.48}$$

So (3.42) follows from (3.48) soon.

Hereafter we shall denote g_T by n_T for the reason of (3.42).

Lemma 3.11. *For any* k *in* $C_0(G)$,

$$k_\alpha(g) \equiv \int_G k(g_1 g)(\chi_{HW_\alpha}(g_1)/\mu_l(HW_\alpha))\,d\mu_l(g_1), \tag{3.49}$$

converges uniformly as $W_\alpha \to \{e\}$, *to*

$$\tilde{k}(\pi(g)) \equiv \int_H k(hg)\,d\mu_H(g). \tag{3.50}$$

Proof. The integral with Haar measure μ_l is represented as multiple integral as follows. (cf. A. Weil [23]).

$$\begin{aligned} \int_G f(g)\,d\mu_l(g) &= \int_G f(g)\Delta(g)\,d\mu(g) \\ &= \int_X d\nu_0(\pi(g))\left\{\int_H f(hg)\Delta(hg)\,d\mu_H(h)\right\} \\ &= \int_X \Delta(g)\,d\nu_0(\pi(g))\left\{\int_H f(hg)\,d\mu_H(h)\right\} \\ &= \int_X d\nu(\pi(g))\left\{\int_H f(hg)\,d\mu_H(h)\right\}. \end{aligned} \tag{3.51}$$

Especially,

$$\begin{aligned} \mu_l(HW_\alpha) &= \int_G \chi_{HW_\alpha}(g)\,d\mu_l(g) \\ &= \int_X \chi_{\pi(W_\alpha)}(\pi(g))\,d\nu(\pi(g)) = \nu(\pi(W_\alpha)). \end{aligned} \tag{3.52}$$

So that,

$$\begin{aligned} k_\alpha(g) &= \int_X d\nu(\pi(g_1))\left\{\int_H k(hg_1 g)(\chi_{HW_\alpha}(hg_1)/\mu_l(HW_\alpha))\,d\mu_H(h)\right\} \\ &= \int_X (\chi_{\pi(W_\alpha)}(\pi(g_1))/\nu(\pi(W_\alpha)))\tilde{k}(\pi(g_1 g))\,d\nu(\pi(g_1)). \end{aligned} \tag{3.53}$$

But $\tilde{k}$ is in $C_0(X)$, so the integral converges to $\tilde{k}(\pi(g))$ uniformly in g, as $W_\alpha \to \{e\}$.

Lemma 3. 12. *For any f in $\mathfrak{H}$,*

(3. 54) $$Tf = W_{n_T} f.$$

Proof. For any k in $C_0(X)$, put

(3. 55) $$k_\alpha(\pi(g)) = (1/\mu_l(HW_\alpha))\int_G k(\pi(g_1^{-1}))\chi_{HW_\alpha}(gg_1)d\mu_l(g_1).$$

Analogous arguments to the proof of lemma 2. 14 leads us to,

(3. 56)
$$(Tk_\alpha)(\pi(g)) = \int k(\pi(g_1^{-1}(g'_\alpha)^{-1}g))(\chi_{HW_\alpha}(g_1)/\mu_l(HW_\alpha))d\mu_l(g_1).$$

From the result of lemma 3. 11, the limit of right side is

(3. 57) $$\int k(\pi(h^{-1}n_T g))d\mu_H(h) = k(\pi(n_T g)) = (W_{n_T}k)(\pi(g)).$$

While $\{k_\alpha(\pi(g))\}$ converges to $k(\pi(g))$ uniformly having carriers in some fixed compact set. Therefore $\{Th_\alpha\}$ converges to Tk in $L^2(\nu;\ X)$. Consequently,

(3. 58) $$Tk = W_{n_T}k, \qquad \text{in } L^2(\nu;\ X).$$

This relation is easily extendable to (3. 54) on $\mathfrak{H}$.

The above lemma 3. 12 shows that φ is onto map, so we must show the kernel of φ is H.

Lemma 3. 13. *W_{n_1} is equal to W_{n_2}, if and only if n_1 and n_2 belong to same H-coset in N.*

Proof. The "if" part is evident, so we consider the converse. Taking $W_{n_1}^{-1}W_{n_2} = W_{n_1^{-1}n_2}$, it is enough to show that if $W_n = I$, then n is in H.

But the condition

(3. 59) $$k(nx) = (W_n k)(x) = k(x), \qquad \text{for any } k \text{ in } C_0(X),$$

results

(3. 60) $$nx = x,$$

that is,

$$nHg=Hg, \quad \text{for any } g \text{ in } G. \tag{3.61}$$

Especially,

$$nH=H. \tag{3.62}$$

This asserts that n is in H.

Thus we have proved proposition 3.1, so next we shall prove the topological assertion proposition 3.2. Because of continuity of L_n, the weak topology τ on $\widetilde{N}$ is weaker than initial topology of $H\backslash N$, that is, φ and so $\hat{\varphi}$ are continuous. So it is sufficient to show the followings.

Lemma 3.14. *$\hat{\varphi}^{-1}$ is continuous.*

Proof. As in the proof of lemma 2.16, it is sufficient to see that for any neighborhood W of e in N, there exists a non-zero vector u in H such that $\langle W_n u, u\rangle$ is zero for n not contained in HW.

Since the natural topology of N is induced topology by G, a neighborhood W_0 of e in G such that

$$HW_0 \cap N \subset HW \tag{3.63}$$

exists, therefore by lemma 3.8, we can select a neighborhood V of e in G such that

$$HVV^{-1}H \subset HW_0. \tag{3.64}$$

Take u, carrier of which is in $\pi(V)$, then

$$\begin{aligned} \langle W_n u, u\rangle &= \int_X u(\pi(ng))\overline{u(\pi(g))}\,d\nu(\pi(g)) \\ &= \int_G u(ng)\overline{u(g)}\,d\mu_l(g), \end{aligned} \tag{3.65}$$

is zero for n which does not belong to

$$N \cap HVV^{-1}H \subset N \cap HW_0 \subset HW. \tag{3.66}$$

This completes the proof.

§4. Duality theorem in a strong form.

1. In §2, we have proved, a duality theorem, which holds between any locally compact group G, and a set Ω of equivalence classes of (sufficiently many) representations of G. However, comparison with Pontrjagin's or Tannaka's duality theorem shows that the proved duality theorem is somewhat different from them. The main difference is that in the latter two dualities, an element in G is characterized as a vector field over the dual $\widehat{G}$ (set of all equivalence classes of *irreducible* representations) of G, instead of Ω in §2. In this point of view, we shall reformulate a duality theorem, and clarify the relation of these two types of duality.

[**Assumption**] *In what follows, we deal only separable and type I groups G. Namely, irreducible decomposability and uniqueness of the decomposition within unitary equivalence of any representation of G are provided.*

Let $\widehat{G}$ be the set of all equivalence classes of irreducible representations of G. And we attach a representative ω to each class in $\widehat{G}$, as in the case of Ω in §2. And if an operator field $\boldsymbol{T}=\{T(\omega)\}$ over $\widehat{G}$, in which each $T(\omega)$ is an operator on $\mathfrak{H}(\omega)$, is given, then for any irreducible representation ω_0, we can define unique operator as extension of $\boldsymbol{T}$ by

$$T(\omega_0)=UT(\omega)U^{-1}, \tag{4.1}$$

where $\omega(\overset{U}{\sim}\omega_0)$ is the representative of equivalence class containing ω_0. Hereafter, if it is necessary, we consider this unique extension $\widetilde{\boldsymbol{T}}$ of $\boldsymbol{T}$, under the same symbol $\{T(\omega)\}$

Definition 4.1. *An operator field $\boldsymbol{T}=\{T(\omega)\}$ over $\widehat{G}$ is called strong birepresentation when,*

(i) a) *$T(\omega)$ is an bounded operator on $\mathfrak{H}(\omega)$,*

b) *$\boldsymbol{T}$ is integrable*) on the regular representation R,*

c) *$\boldsymbol{T}$ is bounded*) on R,*

d) *$\boldsymbol{T}$ is non-zero*) on R,*

---

*) See §1.

(ii) *For the irreducible decomposition*

$$\omega_1 \otimes \omega_2 \overset{U}{\sim} \int_{\Omega(\omega_1,\omega_2)} \omega(x)\, d\nu_{1,2}(x), \tag{4.2}$$

of Kronecker product of two elements ω_1 *and* ω_2 *in* $\hat{G}$, $\boldsymbol{T}$ *is integrable on* $\omega_1 \otimes \omega_2$ *and the equation*

$$U(T(\omega_1) \otimes T(\omega_2))\, U^{-1} = \int_{\Omega(\omega_1,\omega_2)} T(\omega(x))\, d\nu_{1,2}(x), \tag{4.3}$$

is valid.

As in §2, the set of all strong birepresentations $\hat{\hat{G}}$ (bidual) becomes a group by the product operation

$$\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = \{T_1(\omega)\, T_2(\omega)\}, \quad \text{for} \quad T_j = \{T_j(\omega)\}, \ (j=1, 2), \tag{4.4}$$

and the identity $\boldsymbol{I} = \{I(\omega)\}$.

And for given g in G, $\boldsymbol{U}_g = \{U_g(\omega)\}$ gives a strong birepresentation and the map φ; $g \rightarrow \boldsymbol{U}_g$ is an algebraic *homomorphism* of G *into* $\hat{\hat{G}}$. The main proposition is as follows,

Proposition 4.1. φ *is an (algebraic)* isomorphism *of* G onto $\hat{\hat{G}}$. *That is the same, for any given strong birepresentation* $\boldsymbol{T}$, *there exists unique* g *in* G *such that*

$$\boldsymbol{U}_g = \boldsymbol{T}. \tag{4.5}$$

About the topological part, we shall consider later.

2. For any given strong birepresentation $\boldsymbol{T} = \{T(\omega)\}$, if we can define an admissible operator T over $L^2(G)$ (see §2) such that

$$U_\omega(T(\omega) \otimes T)\, U_\omega^{-1} = \sum \oplus T, \quad \text{for any } \omega \text{ in } \hat{G}, \tag{4.6}$$

where U_ω gives the equivalence relation of

$$\omega \otimes R \overset{U_\omega}{\longrightarrow} \sum \oplus R, \tag{4.7}$$

then by the reason of the results of §2, there exists an element g in G and

$$T = R_g. \tag{4.8}$$

Thus, lemma 2.4 asserts that

$$T = U_g. \tag{4.9}$$

So to prove the proposition 4.1, it is sufficient to show the existence of admissible operator T satisfying (4.6). While (4.9) results that such an admissible operator must be the form of

$$\begin{aligned} T \equiv R_g = U_R^{-1}\left(\int_{\Omega_R} U_g(\omega(x))\,d\nu_R(x)\right)U_R \\ \equiv U_R^{-1}\left(\int_{\Omega_R} T(\omega(x))\,d\nu_R(x)\right)U_R, \end{aligned} \tag{4.10}$$

for the irreducible decomposition of the regular representation

$$R \xrightarrow{U_R} \int_{\Omega_R} \omega(x)\,d\nu_R(x). \tag{4.11}$$

That is, if we show that the operator T defined by

$$T = U_R^{-1}\left(\int_{\Omega_R} T(\omega(x))\,d\nu_R(x)\right)U_R \tag{4.12}$$

on $L^2(G)$, exists and is admissible satisfying (4.6), then the proposition 4.1 is proved.

The conditions (i) a)$\sim$d) of definition 4.1 assure that (4.12) gives a non-zero bounded operator on $L^2(G)$, that is, T satisfies the condition (i) (of lemma 2.5) of admissibility.

Next, from the assumption G is type I, in the central decomposition

$$R \sim \int \tilde{\omega}(\rho)\,d\tilde{\nu}(\rho), \tag{4.13}$$

almost all components $\tilde{\omega}(\rho)$ are type I factor, namely, its operators of representation are forms of

$$U_g(\tilde{\omega}(\rho)) = I_{n(\rho)} \otimes U_g(\omega(\rho)), \tag{4.14}$$

where $\omega(\rho)$ is an irreducible representation and $I_{n(\rho)}$ is the identity operator in a $n(\rho)$-dimensional Hilbert space $\mathfrak{H}'_\rho$. (see lemma 1.7).

Because of the decomposition

$$R \sim \int \sum^{n(\rho)} \oplus\, \omega(\rho)\,d\tilde{\nu}(\rho) \tag{4.15}$$

gives the irreducible decomposition of R, almost all components of in the decomposition (4.13) are forms of

$$\widetilde{T}(\rho)\sim I_{n(\rho)}\otimes T(\omega(\rho)). \tag{4.16}$$

While for any g in G, L_g belongs to $\{R_g\}'$ then almost all components of L_g in (4.13) take forms of

$$\widetilde{L}_g(\rho)\sim L_g(\rho)\otimes I_\rho, \tag{4.17}$$

where $L_g(\rho)$ is a unitary operator in $\mathfrak{H}'_\rho$ and I_ρ is the identity operator in $\mathfrak{H}(\omega(\rho))$.

From (4.16) and (4.17), easily to see that two operators

$$T\sim\int\widetilde{T}(\rho)\,d\tilde{\nu}(\rho), \tag{4.18}$$

$$L_g\sim\int\widetilde{L}_g(\rho)\,d\tilde{\nu}(\rho), \tag{4.19}$$

mutually commute. This is the condition (ii) of admissibility.

The condition (iii) of admissibility and the equation (4.6) are proved in the same way. Let the irreducible decomposition of given representation ω_0 be

$$\omega_0\overset{U_0}{\sim}\int_{\Omega_0}\omega(x)\,d\nu_0(x). \tag{4.20}$$

Then from the corollary to lemma 1.10 the irreducible decomposition of $\omega_0\otimes R$ is given by

$$\begin{aligned}\omega_0\otimes R &\overset{U_0\otimes U_R}{\sim}\int_{\Omega_0}\omega(x)\,d\nu_0(x)\otimes\int_{\Omega_R}\omega(y)\,d\nu_R(y)\\ &\sim\int_{\Omega_0\times\Omega_R}d\nu_0(x)\,d\nu_R(y)\,\{\omega(x)\otimes\omega(y)\}\\ &\overset{\iint U_{x,y}d\nu_0(x)d\nu_R(y)}{\sim}\int_{\Omega_0\times\Omega_R}d\nu_0(x)\,d\nu_R(y)\\ &\qquad\times\left\{\int_{\Omega(x,y)}\omega(w:x,y)\,d\nu_{x,y}(w)\right\}.\end{aligned} \tag{4.21}$$

While the irreducible decomposition of $\sum_\alpha\oplus R_\alpha$ which is equivalent to $\omega_0\otimes R$ is given by

$$(4.22)\qquad \sum_{\alpha}\oplus R_\alpha \sim \sum_{\alpha}\oplus\left\{\int_{\Omega_R^\alpha}\omega(x)\,d\nu_R^\alpha(x)\right\}\sim\int_{\bigcup_\alpha\Omega_R^\alpha}\omega(x)\,d\nu_R^\alpha(x).$$

From the uniqueness of irreducible decomposition of a representation of type I group G, there exists a bimeasurable correspondence between the space $\{(w, x, y): w\in\Omega(x, y), x\in\Omega_0, y\in\Omega_R\}$ and $\bigcup_\alpha\Omega_R^\alpha$ except null set, which maps $\nu_0\times\nu_R(\times\nu_{x,y})$ to $\sum_\alpha\nu_R^\alpha$, so induces a unitary equivalence relation $\widetilde{U}$ between two representations in integral form. But obviously $\boldsymbol{T}$ is non-zero bounded, and integrable on $\sum_\alpha\oplus R_\alpha$, so that the both sides of following equation exist and define a non-zero bounded operator, and the equation is valid.

$$(4.23)\qquad \int_{\Omega_0\times\Omega_R}d\nu_0(x)\,d\nu_R(y)\left\{\int_{\Omega(x,y)}T(\omega(w: x, y))\,d\nu_{x,y}(w)\right\}$$
$$=\widetilde{U}\left\{\int_{\bigcup_\alpha\Omega_R^\alpha}T(\omega(x))\,d\nu_R^\alpha(x)\right\}\widetilde{U}^{-1}.$$

The left side of (4.23) is connected to

$$(4.24)\qquad \int_{\Omega_0}T(\omega(x))\,d\nu_0(x)\otimes\int_{\Omega_R}T(\omega(x))\,d\nu_R(x),$$

by $\left(\int U\right)$, when T is integrable on ω_0. And the right side is connected to

$$(4.25)\qquad \sum_\alpha\oplus T(R_\alpha)$$

This shows (4.6) when ω_0 is ω, and shows the condition (iii) of admissibility when ω_0 is R.

Remark. The conditions (i) b)$\sim$d) depend closely to the irreducible decomposition of R. It seems that if we don't know the exact form of decomposition of R, then we can not check whether $\boldsymbol{T}$ is a strong birepresentation or not. In this point of view, we shall consider to replace these conditions to sufficient ones which don't need the knowledge of decomposition of R.

For instance, conditions (i) c) and d) are trivially replaceable to

(i) c') $\|T(\omega)\|$ is uniformly bounded over $\widehat{G}$,

d') $\|T(\omega)\|\neq 0$, for any ω in $\widehat{G}$.

And these conditions are evidently satisfactory when

(i) c-d'') $T(\omega)$ is unitary for any ω in $\widehat{G}$.

The condition substituted for (i) b) is somewhat complicated. In [14], G. W. Mackey constructed a Borel structure on $\widehat{G}$ as follows. Let $\boldsymbol{H}_n (n=1, 2, \cdots, \infty)$ be a n-dimensional Hilbert space, and consider the set I_n of all irreducible representations of G over $\boldsymbol{H}_n$. Put the smallest Borel structure $\widetilde{\mathfrak{B}}$ in $I \equiv \bigcup I_n$ such that a) I_n is a Borel subset of I for all n, b) for each n, each u_n and v_n in $\boldsymbol{H}_n$ and each g in G, $\langle U_g(\omega) u_n, v_n \rangle$ is a Borel function on I_n. Mackey's Borel structure is the quotient Borel structure of $\widehat{G}$ as a quotient Borel space of $\{I, \mathfrak{B}\}$ with respect to equivalence relation. Under this circumstance, we consider the function $\varphi_{\{u_n, v_n\}_n}(\omega) = \{\langle T(\omega) u_n, v_n \rangle; \omega \in I_n\}_n$ on I, for an extended operator field $\widetilde{\boldsymbol{T}}$ corresponding to $\boldsymbol{T}$.

Definition 4.2. *An operator field* $\boldsymbol{T} = \{T(\omega)\}$ *over* $\widehat{G}$ *is called* integrable *on* $\widehat{G}$ *when* $\varphi_{\{u_n, v_n\}_n}$ *is* $\widetilde{\mathfrak{B}}$*-measurable (for any n, and any* u_n, v_n *in* $\mathbf{H}_n$.)

It is easy to see, when $\boldsymbol{T}$ is integrable on $\widehat{G}$ then integrable on any representation ω, especially on R. This leads us to a sufficient condition for (i) b),

(i) b') $\boldsymbol{T} = \{T(\omega)\}$ is integrable on $\widehat{G}$.

We can define a topology τ_u of I which is generated by uniform convergence on compact sets, of matrix elements $\{\langle U_g(\omega) u_n, v_n \rangle\}_n$. That is, a fundamental system of neighborhood of ω_0 is given by

$$U(\omega_0, \varepsilon, C, u, v) = \{\omega : \dim \boldsymbol{H}(\omega) = \dim \boldsymbol{H}(\omega_0) \quad \text{and} \quad |\langle (U_g(\omega) - U_g(\omega_0)) u, v \rangle| \leq \varepsilon, \text{ for any } g \text{ in } C\}, \tag{4.26}$$

for any $\varepsilon > 0$, and compact subset C in G, and vectors u, v in $\boldsymbol{H}_n (n = \dim \boldsymbol{H}(\omega_0))$. Since this set is representable as

$$U(\omega_0, \varepsilon, C, u, v) = \bigcap_{j}^{\infty} \{\omega : |\langle (U_{g_j}(\omega) - U_{g_j}(\omega_0)) u, v \rangle| \leq \varepsilon\}, \tag{4.27}$$

for countable dense set $\{g_j\}$ in C, the Borel structure generated by τ_u is equal to $\widetilde{\mathfrak{B}}$. Therefore, (i) b) is replaceable by a sufficient condition

(i) b″) $\langle T(\omega)u, v\rangle$ is τ_u-continuous for any u and v.

Combining proposition 4. 1 with (i) c-d″) and (i) b″), we get the algebraic part of Pontrjagin duality theorem*).

3. The adequate topology of $\hat{\hat{G}}$ is not so simple. From the results of §2, it is sufficient to give the weakest topology which makes any matrix element in R,

$$\int_{\Omega_R}\langle U_g(\omega)f_1(\omega), f_2(\omega)\rangle d\nu_R(\omega)\equiv\langle R_g f_1, f_2\rangle \tag{4. 28}$$

continuous, where f_1, f_2 are any elements (or it is sufficient, to run over a dense set) of $L^2(G)$ which correspond to vectors $\{f_1(\omega)\}$, $\{f_2(\omega)\}$ in the decomposition (4. 11) of R, respectively.

When G is an abelian group, for any f_1, f_2 in $L^2(G)$ and $\varepsilon>0$, there exists a compact set $C(\varepsilon)$ in $\hat{G}=\Omega_R$ such that

$$\int_{\hat{G}-C(\varepsilon)}\|f_j(\omega)\|^2 d\nu_R(\omega)<\varepsilon, \quad (j=1, 2).$$

Therefore uniform convergence of $\{U_g(\omega)f_j\}$ over compact subset of $\hat{G}$ induces the adequate topology, this results the topological part of Pontrjagin duality.

While for a compact group G, $\hat{G}$ has discrete Borel structure, so it is easy to see the adequate topology coincides to the weak topology.

4. In this section, we shall introduce some well properties which are not true in general, but are satisfied, for instance, for some semi-simple Lie group, or some semi-direct products of groups.

Definition 4. 3. *For any finite set $F=\{\omega_j: 1\leq j\leq n\}$ in $\hat{G}$, and positive integer N, let the subset $\Omega(F, N)$ of Ω be the set of all equivalence classes of representations which are, (1) direct sum of $\{\omega_{j(1)}\otimes\cdots\otimes\omega_{j(m)}, (m\leq N)\}$ and (2) its subrepresentations.*

An element of $\Omega(F, N)$ is called to be finitely generated.

Definition 4. 4. *For any countable family $S=\{\omega_j\}$ in $\hat{G}$, let the subset $\Omega(S)$ of Ω be the set of all equivalence classes of represen-*

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*) The idea to use the results of §2, for proving the Pontrjagin duality is suggested by Prof. J. Ernest to the author.

tations which are, (1) direct sum of $\{\omega_{j(1)}\otimes\cdots\otimes\omega_{j(m)}, (m=1,2,3,\cdots)\}$ *and (2) its subrepresentations.*

An element of $\Omega(S)$ is called to be countably generated

Lemma 4.1. *For representations ω_1, ω_2 in Ω with the irreducible decompositions*

$$\omega_j \overset{U_j}{\sim} \int_{\Omega_j} \omega(x) d\nu_j(x), \quad (j=1,2), \tag{4.29}$$

respectively, let an operator field $\boldsymbol{T}=\{T(\omega)\}$ over $\widehat{G}$ be integrable on ω_1, ω_2 at the same time and moreover satisfy the condition (i)—*a*) *and* (ii) *of definition 4.1, then $\boldsymbol{T}$ is integrable on $\omega_1\otimes\omega_2$.*

Proof. As in the last part of the proof of proposition 4.1,

$$\omega_1\otimes\omega_2 \overset{U}{\sim} \iint_{\Omega_1\times\Omega_2} d\nu_1(x) d\nu_2(y) \left\{\int_{\Omega(x,y)} \omega(w;\, x, y) d\nu_{x,y}(\omega)\right\}, \tag{4.30}$$

gives an irreducible decomposition of $\omega_1\otimes\omega_2$ and by the equivalence relation U which gives (4.30), $T(\omega_1)u\otimes T(\omega_2)v$ $(u\in\mathfrak{H}(\omega_1), v\in\mathfrak{H}(\omega_2))$ corresponds to

$$\left(\iint_{\Omega_1\times\Omega_2} d\nu_1(x) d\nu_2(y) \left\{\int_{\Omega(x,y)} T(\omega(w;\, x, y)) d\nu_{x,y}(w)\right\}\right) U(u\otimes v). \tag{4.31}$$

This shows that

$$\begin{aligned} &\Big\langle \Big[\Big(\iint_{\Omega_1\times\Omega_2} d\nu_1(x) d\nu_2(y) \left\{\int_{\Omega(x,y)} T(\omega(w;\, x, y)) d\nu_{x,y}(w)\right\}\Big) \\ &\qquad \times U(u\otimes v)\Big](w;\, x, y),\ U(u'\otimes v')(w;\, x, y)\Big\rangle_{\mathfrak{H}(\omega(w;\, x, y))} \\ &\quad = \langle T(\omega(w;\, x, y))\,[U(u\otimes v)(w;\, x, y)], \\ &\qquad\qquad U(u'\otimes v')(w;\, x, y)\rangle_{\mathfrak{H}(\omega(w;\, x, y))} \end{aligned} \tag{4.32}$$

is measurable on $\Omega_1\times\Omega_2(\times\Omega(x,y))$. But the vectors $\{U(u\otimes v)\}$ span the space of representation of right hand side of (4.30), thus integrability of $\boldsymbol{T}$ on $\omega_1\otimes\omega_2$ is deduced.

Lemma 4.2. *If ω_0 is countably generated, then any operator fileld $\boldsymbol{T}=\{T(\omega)\}$ over $\widehat{G}$ which satisfies the conditions* (i)—*a*) *and* (ii) *of definition 4.1 is integrable on ω_0.*

Proof. Repeated applications of lemma 4.1 result that $\boldsymbol{T}$ is integrable on $\tilde{\omega}\equiv\omega_{j(1)}\otimes\cdots\otimes\omega_{j(m)}$, $(\omega_j\in\widehat{G})$.

Now let ω_0 be a direct sum of $\tilde{\omega}^{\alpha}$'s of above form. Then for any vectors u, v in $\mathfrak{H}(\omega_0)$, there exist countable members $\{\tilde{\omega}^{\alpha_j}\}$ such that

$$u, v\in\sum_j^{\infty}\oplus\mathfrak{H}(\tilde{\omega}^{\alpha_j}). \tag{4.33}$$

This means that the corresponding vector valued functions $\{u(\omega)\}$, $\{v(\omega)\}$ have carriers in $\bigcup_j^{\infty}\Omega(\tilde{\omega}^{\alpha_j})$. So

$$\langle T(\omega)u(\omega), v(\omega)\rangle_{\mathfrak{H}(\omega)}=\sum_j\langle T(\omega)u(\omega), v(\omega)\rangle_{\mathfrak{H}(\omega)}\chi_{\Omega(\tilde{\omega}^{\alpha_j})} \tag{4.34}$$

is measurable with respect to the measure of irreducible decomposition of ω_0, as a countable sum of measurable functions.

Lastly if ω_0 is general element in $\Omega(S)$ that is, a subrepresentation of $\sum_{\alpha}\oplus\tilde{\omega}^{\alpha}$, from lemma 1.13, $\boldsymbol{T}$ is integrable on ω_0.

Lemma 4.3. *If ω_0 is finitely generated then any operator field $\boldsymbol{T}=\{T(\omega)\}$ over $\widehat{G}$ which satisfies the conditions* (i)—*a*) *and* (ii) *of definition 4.1 is bounded on ω_0.*

Proof. From the relation (lemma 1.3)

$$\| T(\omega_{j(1)})\otimes\cdots\otimes T(\omega_{j(m)})\|=\prod_k^{m}\| T(\omega_{j(k)})\| \tag{4.35}$$

gives that

$$\nu_{\omega_0}\text{-ess-sup}\| T(\omega)\|_{\mathfrak{H}(\omega)}\leqq(\max_j(1, \| T(\omega_j)\|))^N. \tag{4.36}$$

Lemma 4.4. *If ω_0 is countably generated and $\boldsymbol{T}$ is an operator field over $\widehat{G}$ which satisfies the conditions* (i)—*a*) *and* (ii) *of definition 4.1. Moreover if $\{T(\omega)\}$ satisfies one of the following conditions for generator $\{\omega_j\}$ of ω_0,*

a) $(T(\omega_j))^{-1}(0)=\{0\}$,

b) *$T(\omega_j)\mathfrak{H}(\omega_j)$ is dense in $\mathfrak{H}(\omega_j)$,*

then $\boldsymbol{T}$ is non-zero on ω_0.

Proof. By the definition, ω_0 is a subrepresentation of $\sum_\alpha \oplus \tilde{\omega}^\alpha$, therefore if T is zero on ω_0, there exists an α and non-zero subrepresentation $\tilde{\omega}_1$ of $\tilde{\omega}^\alpha$ such that T is zero on $\tilde{\omega}_1$.

So it is enough to show that if $T(\omega_j)$ satisfies one of the conditions a), b), then $T(\omega_{j(1)})\otimes\cdots\otimes T(\omega_{j(m)})$ has no non-trivial G-invariant zero-space, that is the same, since this operator reduces G-invariant space, the minimal G-invariant closed subspace which contains the range of this operator is the whole space. We shall show this property by induction with respect to m.

At first, for $m=1$, since ω_j is irreducible so non-trivial G-invariant space must be whole $\mathfrak{H}(\omega_j)$, but the both of conditions a) and b) assure $T(\omega_j)\neq 0$, this shows, the above is true in this case.

Next, we assume that $B\equiv T(\omega_{j(1)})\otimes\cdots\otimes T(\omega_{j(m-1)})$ has the above property on $\mathfrak{H}\equiv\mathfrak{H}(\omega_{j(1)})\otimes\cdots\otimes\mathfrak{H}(\omega_{j(m-1)})$, and consider $B\otimes T(\omega_{j(m)})$ on $\mathfrak{H}\otimes\mathfrak{H}(\omega_{j(m)})$. Take a complete orthonormal system $\{v_l\}$ in $\mathfrak{H}$, and $\{w_l\}$ in $\mathfrak{H}(\omega_{j(m)})$, then any vector in $(B\otimes T(\omega_{j(m)}))^{-1}(0)$ is written as $\sum_l a_l u_l \otimes w_l$ uniquely, in which u_l are vectors in $\mathfrak{H}$ and $\{a_l\}$ satisfies

$$(4.37)\qquad \sum_l |a_l|^2\cdot\|u_l\|_{\mathfrak{H}}^2<+\infty.$$

While

$$(4.38)\qquad 0=(B\otimes T(\omega_{j(m)}))(\sum_l a_l u_l\otimes w_l)=\sum_l a_l(Bu_l)\otimes(T(\omega_{j(m)})w_l)$$
$$=\sum_{l,k} a_l\langle Bu_l, v_k\rangle v_k\otimes(T(\omega_{j(m)})w_l).$$

(4.37) shows the convergence of vectors $\sum_l a_l\langle Bu_l, v_k\rangle w_l$ for any k, so that from the boundedness of $T(\omega_{j(m)})$,

$$(4.39)\qquad w_k=T(\omega_{j(m)})[\sum_l a_l\langle Bu_l, v_k\rangle w_l]=\sum_l a_l\langle Bu_l, v_k\rangle T(\omega_{j(m)})(w_l),$$

exists for any k. So from (4.38)

$$(4.40)\qquad \widetilde{w}_k=0, \qquad \text{for any } k.$$

Now, if $T(\omega_{j(m)})$ satisfies the condition a) then

$$(4.41)\qquad \sum_l a_l\langle Bu_l, v_k\rangle w_l=0, \qquad \text{for any } k,$$

(4. 42) $a_l\langle Bu_l, v_k\rangle=0$, for any l and k.

This shows that, if $a_l\neq 0$ then $Bu_l=0$.

To belong the vector $\sum_l a_l u_l\otimes w_l$ in G-invariant zero-space of $B\otimes T(\omega_{j(m)})$, $(U_g(\omega_{j(1)})\otimes\cdots\otimes U_g(\omega_{j(m)}))(\sum_l a_l u_l\otimes w_l)$ $=(\sum_l a_l((U_g(\omega_{j(1)})\otimes\cdots\otimes U_g(\omega_{j(m-1)}))u_l\otimes U_g(\omega_{j(m)})w_l)$ must be in $(B\otimes T(\omega_{j(m)}))^{-1}(0)$. Thus from the results of above arguments, and arbitrariness of complete orthonormal system $\{w_l\}$, u_l must be in G-invariant zero-space of B. Therefore $a_l u_l=0$ for all l, and the assertion is proved in this case.

On the other hand, let $T(\omega_{j(m)})\mathfrak{H}(\omega_{j(m)})$ be dense in $\mathfrak{H}(\omega_{j(m)})$, of course, $U_g(\omega_{j(m)})T(\omega_{j(m)})\mathfrak{H}(\omega_{j(m)})$ is dense for any g in G. From the assumption, for any u in $\mathfrak{H}$ and $\varepsilon>0$, there exist a $\{g_k\}$ in G and $\{v_k\}$ in $\mathfrak{H}$ such that $\|u-\sum_k(U_{g_k}(\omega_{j(1)})\otimes\cdots\otimes U_{g_k}(\omega_{j(m-1)}))Bv_k\|<\varepsilon$. And for any w in $\mathfrak{H}(\omega_{j(m)})$ and g_k as above we can select t_k in $\mathfrak{H}(\omega_{j(m)})$ such that $\|w-U_{g_k}(\omega_{j(m)})T(\omega_{j(m)})t_k\|<\varepsilon$. So

$$(4.43)\quad \|u\otimes w-\sum_k(U_{g_k}(\omega_{j(1)})\otimes\cdots\otimes U_{g_k}(\omega_{j(m)}))(B\otimes T(\omega_{j(m)}))(v_k\otimes t_k)\| \leq\|u\|\varepsilon+\|w\|\varepsilon+\varepsilon^2.$$

This shows that $B\otimes T(\omega_{j(m)})$ has the above property.

Lemma 4. 5. *If ω_0 is finitely generated, then the weakest topology of G which makes $\langle U_g(\omega_0)v, u\rangle$ continuous for any u, v in $\mathfrak{H}(\omega_0)$ is weaker than the weakest topology which makes $\{\langle U_g(\omega_j)v_j, u_j\rangle\}$ continuous for any ω_j and any u_j, v_j in $\mathfrak{H}(w_j)$.*

Proof. All the operator $U_g(\omega_0)$, $U_g(\omega_j)$ are unitary. So these topology are equivalent to the topology which make $U_g(\omega_0)v$, $\{U_g(\omega_j)v_j\}_j$ continuous in strong topology of vectors, respectively. But ω_0 is a subrepresentation of $\tilde{\omega}=\sum\oplus(\omega_{j(1)}\otimes\cdots\otimes\omega_{j(m)})$, so, it is sufficient to show that $U_g(\tilde{\omega})v$ is continuous in the topology induced by $\{U_g(\omega_j)v_j\}_j$. For the vector of type $v\equiv v_{j(1)}\otimes\cdots\otimes v_{j(m)}$, the inequality

$$(4.44)\quad \|v_{j(1)}\otimes\cdots\otimes v_{j(m)}-(U_g(\omega_{j(1)})\otimes\cdots\otimes U_g(\omega_{j(m)})v_{j(1)}\otimes\cdots\otimes v_{j(m)}\|$$
$$\leqq\sum_k^m\|v_{j(1)}\otimes\cdots\otimes v_{j(k-1)}\otimes(I-U_g(\omega_{j(k)}))v_{j(k)}\otimes U_g(\omega_{j(k+1)})v_{j(k+1)}\otimes\cdots\otimes U_g(\omega_{j(m)})v_{j(m)}\|$$
$$\leqq\sum_k^m\|v_{j(k)}-U_g(\omega_{j(k)})v_{j(k)}\|\prod_{l\neq k}\|v_{j(l)}\|.$$

shows that $U_g(\tilde{\omega})v$ is continuous. And for general vector $v\equiv\sum_s^\infty v^s$, we can take N such that,

$$(4.45)\qquad \|v-\sum_s^N v^s\|<\varepsilon.$$

Take g sufficiently near to e, then

$$(4.46)\qquad \|v-\sum_s\oplus(U_g(\omega^s_{j(1)})\otimes\cdots\otimes U_g(\omega^s_{j(m_s)}))v^s\|$$
$$\leqq\|v-\sum_s^N v^s\|+\sum_s^N\|v^s_{j(1)}\otimes\cdots\otimes v^s_{j(m_s)}-(U_g(\omega^s_{j(1)})\otimes\cdots\otimes U_g(\omega^s_{j(m_s)}))(v^s_{j(1)}\otimes\cdots\otimes v^s_{j(m_s)})\|$$
$$+\|(\sum_s\oplus U_g(\omega^s_{j(1)})\otimes\cdots\otimes U_g(\omega^s_{j(m_s)}))(v-\sum_s^N v^s)\|$$
$$\leqq 2\varepsilon+\sum_s^N\sum_k^{m_s}\|v^s_{j(k)}-U_g(\omega^s_{j(k)})v^s_{j(k)})\|\prod_{l\neq k}\|v^s_{j(l)}\|$$

becomes very small.

Combining these results, we can loosen the conditions of definition 4.1 in special cases.

Proposition 4.2. *If R is countably generated, then for opetator fields $\boldsymbol{T}=\{T(\omega)\}$ over $\hat{G}$, which satisfies only the conditions* (i) *a*), *c*) *and* (ii) *of definition 4.1, and the condition of lemma 4·4 for the generators $\{\omega_j\}$ of R, the same result as proposition 4.1 is valid.*

Proposition 4.3. *If R is finitely generated, then for operator fields $\boldsymbol{T}=\{T(\omega)\}$ over $\hat{G}$, which satisfies only the conditions* (i) *a*), *and* (ii) *of definition 4.1, and the condition of lemma 4.4 for the generators $\{\omega_j\}$ of R, the same result as proposition 4.1 is valid.*

Moreover, in this case, the topology of G coincides the weakest topology of G which makes the matrix elements $\{\langle U_g(\omega_j)v, u\rangle : u, v\in\mathfrak{H}(\omega_j)\}_j$ continuous for the generator $\{\omega_j\}$ of R.

We shall show later, the situation of proposition 4. 2 is the case of some semi-direct product of groups (see §7), and that of proposition 4. 3 is the case of connected non-compact simple Lie group with finite centre, but in the latter case, the condition of lemma 4. 4 can be loosen again.

We shall call the duality theorem under the special situation as above by,

Definition 4. 5. *If for any operator field* $\boldsymbol{T}$ *over* $\widehat{G}$, *the condition* (i)—*b*) *of definition 4. 1 follows from the other conditions, then we call that G satisfies the* duality theorem of the first kind.

Definition 4. 6. *Analogously, if the conditions* (i)—*b*) *and c*) *follows from the others, then we call that G satisfies the* duality theorem of the second kind.

5. Example. (The group of linear transformations on the straight line.)

As shown by I. M. Gel'fand and M. A. Naimark [6] this group is representable as a matrix group

$$G=\left\{\begin{pmatrix} a, & b \\ 0, & 1 \end{pmatrix}:\ a>0,\ -\infty<b<\infty\right\}, \tag{4. 47}$$

which is the semi-direct product of normal subgroup

$$N=\left\{\begin{pmatrix} 1, & b \\ 0, & 1 \end{pmatrix}\right\}, \tag{4. 48}$$

and closed subgroup

$$K=\left\{\begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}\right\}. \tag{4. 49}$$

And irreducible representations of G are given by,

$$\begin{cases} \text{i)} & \omega_\chi\equiv\left\{\chi(g)\equiv\chi(a)\ \left(g=\begin{pmatrix} a, & b \\ 0, & 1 \end{pmatrix}\right),\ \boldsymbol{C}\right\}, \\ & \qquad\qquad\qquad \text{for a unitary character } \chi \text{ of } K, \\ \text{ii)} & \omega_+\equiv \underset{N\to G}{Ind}\, e^{ib}, \\ \text{iii)} & \omega_-\equiv \underset{N\to G}{Ind}\, e^{-ib}, \end{cases} \tag{4. 50}$$

which correspond respectively to the G-orbits in $\widehat{N}(\sim\boldsymbol{R})$, such that

$$(4.51)\qquad \begin{cases} \text{i)} & l_0\equiv\{0\}, \\ \text{ii)} & l_+\equiv\{x;\ x>0\}, \\ \text{iii)} & l_-\equiv\{x;\ x<0\}. \end{cases}$$

For a function $f(g)\equiv f\left(\begin{pmatrix}1, & b\\ 0, & 1\end{pmatrix}\begin{pmatrix}a, & 0\\ 0, & 1\end{pmatrix}\right)$ in $L^2(G)$, the Fourier transform with respect to b is given by

$$(4.52)\qquad \tilde{f}(a, x)\equiv c_0\int_{-\infty}^{\infty} f\left(\begin{pmatrix}1, & b\\ 0, & 1\end{pmatrix}\begin{pmatrix}a, & 0\\ 0, & 1\end{pmatrix}\right)e^{-ibx}db,$$

(c_0; normalizing constant).

And it gives a decomposition of the regular representation R of G,

$$(4.53)\qquad R\sim\int_{-\infty}^{\infty} \underset{N\to G}{Ind}\, e^{ibx}dx,$$

where the component $\underset{N\to G}{Ind}\, e^{ibx}$ is equivalent to ω_+ when $x>0$, and is equivalent to ω_- when $x<0$.

Consequently the regular representation R of G is decomposed to a direct sum of multiples of ω_+ and ω_-,

$$(4.54)\qquad R\sim\sum_{j=1}^{\infty}\oplus\omega_+^j\oplus\sum_{j=1}^{\infty}\oplus\omega_-^j,\ (\omega_+^j\sim\omega_+,\ \omega_-^j\sim\omega_-).$$

That is, R is finitely generated with the generators $\{\omega_+, \omega_-\}$. And G satisfies the duality theorem of second kind.

§5. Connected semi-simple Lie groups.

1. Let G be a connected semi-simple Lie group with finite centre. This § is devoted to show the followings.

Lemma 5.1. *For such a* G, *the regular representation* R *is countably generated.*

Moreover, if G *has no compact factors**) *then the regular representation is finitely generated.*

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*) Let the decomposition of G as a direct product of simple Lie groups G_j be $G=G_1\times\cdots\times G_n$. We call each G_j as a factor of G, and G has no compact factors, when all G_j's are non-compact.

Lemma 5. 2. *For such a G, the condition* (i) d) *of definition* 4. 1 (*non-zero property of* $\boldsymbol{T}$ *on* R) *follows from*

(i) d''') $T(\omega)\neq 0$, *for some irreducible representation ω which is induced by a representation of the subgroup Γ (see (5. 3)), as in the paper of F. Bruhat* [1],

and the other conditions of definition 4. 1.

And from the results of §4, we get,

Proposition 5. 1. *A connected semi-simple Lie group with finite centre satisfies the duality theorem of the first kind.*

Proposition 5. 2. *A connected semi-simple Lie group with finite centre without compact factors satisfies the duality theorem of the second kind.*

And in this case, the weak topology of $\hat{\hat{G}}$ coincides with the initial topology of G.

2. At first we shall quote some results about connected semi-simple Lie groups without proofs or with simple proofs.

Lemma 5. 3. (*Harish-Chandra* [*9*]). *A connected semi-simple Lie group is a type I group.*

Corollary. *To prove the lemmata 5. 1 and 5. 2 for G, it is sufficient to show the same assertions for each factor G_j of G.*

Proof. It is easy to see that the regular representation R of G is equivalent to the outer Kronecker product of regular representations R_j $(1\leq j\leq n)$ of each factor G_j of G.

While from lemma 1. 2, for any irreducible representation ω_j of each G_j, $\omega_1\hat{\otimes}\omega_2\hat{\otimes}\cdots\hat{\otimes}\omega_n$ is irreducible. So if each R_j are countably (resp. finitely) generated and its generators are $\{\omega_j^k\}_k$, then $\{\omega_1^{k_1}\hat{\otimes}\omega_2^{k_2}\hat{\otimes}\cdots\hat{\otimes}\omega_n^{k_n}\}_{(k_1,\cdots,k_n)}$ are all irreducible and construct a countable (resp. finite) family of generators of R.

Moreover, since $T_1(R_1)\hat{\otimes}T_2(R_2)\hat{\otimes}\cdots\hat{\otimes}T_n(R_n)=0$, if and only if there exists j such that $T_j(R_j)=0$, so the non-zero property of $T_j(R_j)$

results that of $T(R)$.

Lemma 5.4. *For any compact Lie group, the regular representation R is countably generated.*

Proof. Since a compact Lie group is separable, so $L^2(G)$ is separable, that is, R is decomposed to a discrete direct sum of countable irreducible representations.

For any connected simple Lie group G with finite centre $Z(G)$, and with Lie algebra $\mathfrak{g}$, let its Iwasawa decomposition be

(5.1) $$G=KHN,$$

where H is a closed simply connected abelian subgroup, N is a closed nilpotent subgroup and K is a compact subgroup, and any element g of G is represented uniquely by elements k in K, h in H and n in N as

(5.2) $$g=khn.$$

Put M the centralizer of H in K, then M contains the centre $Z(G)$ of G, and

(5.3) $$\Gamma\equiv MHN$$

becomes a closed subgroup of G, containing $Z(G)$.

Lemma 5.5. *If G is non-compact, then*

(5.4) $$\Gamma\neq G.$$

Proof. For non-compact G, the subgroup H, therefore, HN is a non-trivial closed subgroup of G. But it is easy to see that M is contained in the normalizer of HN (see, F. Bruhat [1] p. 186). So if $\Gamma=G$, that is, if $K=M$, HN must be a normal subgroup of G. This contradicts to the simplicity of G.

Take a unitary character φ of abelian group H, such that, for any Weyl transformation s on H,

(5.5) $$\varphi^s\neq\varphi,$$

where φ^s is the unitary character defined by

$$\varphi^s(h) = \varphi(s(h)), \qquad \text{for any } h \text{ in } H. \tag{5.6}$$

Now take any irreducible unitary representation $\sigma \equiv \{V_m(\sigma), \boldsymbol{H}(\sigma)\}$ of compact group M, and consider the unitary representation of Γ on $\boldsymbol{H}(\sigma)$ defined by

$$\begin{cases} \tau \equiv \{W_\gamma, \boldsymbol{H}(\sigma)\}, \\ W_\gamma = W_{mhn} \equiv \varphi(h) V_m(\sigma), \qquad \text{for } \gamma \equiv mhn \text{ in } \Gamma. \end{cases} \tag{5.7}$$

Obviously τ give an irreducible representation of Γ.

Lemma 5.6. (*F. Bruhat* [*1*]). *The representation*

$$\omega \equiv \underset{\Gamma \to G}{Ind}\, \tau \tag{5.8}$$

of G is irreducible.

Remark. F. Bruhat's original form of this lemma is based on the property

$$\tau^s \neq \tau, \quad \text{for any Weyl transformation } s, \tag{5.9}$$

instead of (5.5). That is, (5.5) gives only a sufficient condition of (5.9). But for our aim, it is enough to consider only representations of such a kind.

Lemma 5.7. *We can select finite representations* $\{\tau_j\}$ $(1 \leq j \leq n)$ *of the type* (*5.7*), *for which the restriction of*

$$\tau_0 \equiv \sum_j^n \oplus \tau_j \tag{5.10}$$

to $Z(G)$ contains a subrepresentation which is equivalent to the regular representation of $Z(G)$.

Proof. From the assumption, $Z(G)$ is a finite group. Then its regular representation is finite dimensional, and is a discrete direct sum of finite irreducible components. The Frobenius reciprocity theorem on induced representations, being applied to M and $Z(G)$, assures that for each irreducible representation ρ_j of $Z(G)$, there exists an irreducible representation σ_j of M such that the restriction of σ_j to $Z(G)$ contains a component which is equivalent to ρ_j. Thus construct τ_j from some fixed φ as (5.5) and σ_j by (5.7), then the assertion of this lemma is immediately.

Let the projection onto the subrepresentation, which is equivalent to the regular representation of $Z(G)$, in $\boldsymbol{H}(\tau_0)(\equiv\sum_j^{n}\oplus\boldsymbol{H}(\sigma_j))$ be P_0. And for $\tau_0\equiv\sum_j^{n}\oplus\tau_j$, put

$$\omega_j\equiv\underset{\Gamma\to G}{Ind}\,\tau_j\,. \tag{5.11}$$

Then from lemma 1.16,

$$\omega_0\equiv\sum_j^{n}\oplus\omega_j\sim\underset{\Gamma\to G}{Ind}(\sum_j^{n}\oplus\tau_j)=\underset{\Gamma\to G}{Ind}\,\tau_0\,. \tag{5.12}$$

Lemma 5.8. *There exists a function $P_1(g)$ on G, values of which are projections on $\boldsymbol{H}(\tau_0)$, and the restriction of τ_0 to $Z(G)$ operates as the regular representation on the range of this projections. Moreover,*

$$(\widetilde{P}_1 f)(g)\equiv P_1(g)f(g) \tag{5.13}$$

gives a projection on $\mathfrak{H}(\omega_0)$.

Proof. Take a complete orthonormal system $\{v_j\}$ in $P_0\boldsymbol{H}(\tau_0)$, then from strong continuity of $W_\gamma(\tau_0)$ and finiteness of the dimension of $P_0\boldsymbol{H}(\tau_0)$, there exists a neighborhood V of e in Γ such that

$$\|W_\gamma(\tau_0)v_j-v_j\|\leq(1/2)(<(1/\sqrt{2})), \tag{5.14}$$
$$\text{for any } j \text{ and any } \gamma \text{ in } V.$$

One can select an open relative compact neighborhood U of e in G such that

$$UU^{-1}\cap\Gamma\subset V. \tag{5.15}$$

Let C be a compact neighborhood contained in U, then there exists a continuous function f satisfying

$$\begin{cases}\text{i)} & 0\leq f(g)\leq 1, \quad \text{for any } g \text{ in } G,\\ \text{ii)} & f(g)=1, \quad \text{on } C;\ \text{and}=0, \text{ on the outside of } U.\end{cases} \tag{5.16}$$

Define vector valued functions

$$(5.17)\quad f_j(g)\begin{cases}=\left(\int_\Gamma f(\gamma g)\,d\mu_\Gamma(\gamma)\right)^{-1}\int_\Gamma f(\gamma g)\,W_{\gamma^{-1}}(\tau_0)v_j\,d\mu_\Gamma(\gamma), & \text{on } \Gamma C,\\ =0, & \text{on the outside of } \Gamma C,\end{cases}$$

then obviously $\langle f_j(g), v\rangle$ are measurable on G for any vector v in $\boldsymbol{H}(\tau_0)$ and j. Moreover,

$$(5.18)\qquad f_j(\gamma g)=W_\gamma(\tau_0)f_j(g),$$
for any γ in Γ and any g in G,

and if g is in C then

$$(5.19)\quad \|f_j(g)-v_j\|=\left(\int_\Gamma f(\gamma g)\,d\mu_\Gamma(\gamma)\right)^{-1}\left\|\int_\Gamma f(\gamma g)\,W_{\gamma^{-1}}(\tau_0)v_j\,d\mu_\Gamma(\gamma)-\int_\Gamma f(\gamma g)v_j\,d\mu_\Gamma(\gamma)\right\|$$
$$\leq\left(\int_\Gamma f(\gamma g)\,d\mu_\Gamma(\gamma)\right)^{-1}\int_\Gamma f(\gamma g)\,\|W_{\gamma^{-1}}(\tau_0)v_j-v_j\|\,d\mu_\Gamma(\gamma)\leq(1/2).$$

The last inequality follows from that for any g in C the integral domain is

$$(5.20)\qquad \{\gamma:\gamma g\in U,\ \gamma\in\Gamma\}\subset UC^{-1}\cap\Gamma\subset UU^{-1}\cap\Gamma\subset V.$$

Let $\{g_k\}$ be a countable set such that both of $\{\pi(C\cdot g_k)\}$ and $\{\pi(U\cdot g_k)\}$ give locally finite coverings of G/Γ, where π is the canonical map from G onto G/Γ. Such a set is given, for instance, by considering finite coverings by $\{\pi(C\cdot g_j)\}$ of the compact sets

$$(5.21)\qquad \overline{\pi(U^n)}-\pi(U^{n-1}).$$

Next, denote the functions

$$(5.22)\qquad f_{j,k}(g)\equiv f_j(gg_k^{-1}),$$

and lastly construct a family of functions on G by

$$(5.23)\qquad \tilde{f}_j(g)\equiv f_{j,k}(g),\qquad \text{for } g \text{ in } \left(\Gamma Cg_k-\bigcup_{l=1}^{k-1}\Gamma C\cdot g_l\right),$$

then $\tilde{f}_j$'s are measurable functions on G, and from (5.18), (5.22) and (5.23), satisfy the relation

$$(5.24)\qquad \tilde{f}_j(\gamma g)=W_\gamma(\tau_0)\tilde{f}_j(g),\qquad \text{for any } \gamma \text{ in } \Gamma \text{ and } g \text{ in } G.$$

We shall show $\{\tilde{f}_j(g)\}_j$ are mutually linearly independent for any g in G. In fact, if g is in $(\Gamma\cdot Cg_k - \bigcup_{l=1}^{k-1}\Gamma\cdot Cg_l)$. g is represented as

$$g=\gamma\cdot c\cdot g_k, \quad \text{for } \gamma \text{ in } \Gamma,\ c \text{ in } C. \tag{5.25}$$

And for any j,

$$\tilde{f}_j(g)=f_{j,k}(\gamma c g_k)=W_\gamma(\tau_0)f_{j,k}(cg_k)=W_\gamma(\tau_0)f_j(c). \tag{5.26}$$

But using (5·19),

$$\|W_\gamma(\tau_0)f_j(c)-W_\gamma(\tau_0)v_j\|=\|f_j(c)-v_j\|\leq(1/2), \quad \text{for any } j. \tag{5.27}$$

Since $\{v_j\}$ is a complete orthonormal system in $P_0\boldsymbol{H}(\tau_0)$, so $\{f_j(c)\}_j$, therefore $\{\tilde{f}_j(g)\}_j$ are mutually linearly independent.

This results that $\{\tilde{f}_j(g)\}_j$ spans a vector subspace of $\boldsymbol{H}(\tau_0)$ which has same dimension with $P_0\boldsymbol{H}(\tau_0)$. While since $Z(G)$ is the centre of G, the $W_z(\tau_0)$ $(z\in Z(G))$ operates on $\{\tilde{f}_j(g)\}_j$ as follows,

$$\begin{aligned} W_z(\tau_0)\tilde{f}_j(g) &= W_z(\tau_0)\, W_\gamma(\tau_0)f_j(c) \\ &= (\text{const.})\times W_z(\tau_0)\, W_\gamma(\tau_0)\int_\Gamma f(\gamma c)\, W_{\gamma^{-1}}(\tau_0)v_j\, d\mu_\Gamma(\gamma) \\ &= (\text{cost.})\times W_\gamma(\tau_0)\int_\Gamma f(\gamma c)\, W_{\gamma^{-1}}(\tau_0)\,(W_z(\tau_0)v_j)\, d\mu_\Gamma(\gamma). \end{aligned} \tag{5.28}$$

This shows that $\{\tilde{f}_j(g)\}_j$ transfer under the operation of $W_z(\tau_0)$ as same manner as $\{v_j\}_j$, from the definition of $\{v_j\}_j$, $\{W_z(\tau_0)\}_z$ is equivalent to the regular representation of $Z(G)$ on $P_0\boldsymbol{H}(\tau_0)$, that is, on the space $\boldsymbol{H}(g)$ spanned by $\{\tilde{f}_j(g)\}_j$. Put the projection from $\boldsymbol{H}(\tau_0)$ onto $\boldsymbol{H}(g)$ as $P_1(g)$. Then $P_1(g)$ is given as

$$P_1(g)f(g)=\sum c_j(g)\langle f(g),\tilde{f}_j(g)\rangle\tilde{f}_j(g), \tag{5.29}$$

where $c_j(g)$ are measurable and not all zero at same time, satisfying,

$$c_j(\gamma g)=c_j(g), \quad \text{for } \gamma \text{ in } \Gamma,\ g \text{ in } G. \tag{5.30}$$

Thus $P_1(g)f(g)$ belongs in $\mathfrak{H}(\omega_0)$, and it is easy to see that $\tilde{P}_1$ given in (5.13) is a projection over $\mathfrak{H}(\omega_0)$.

Lemma 5.9. *The restriction of $\tau_0\otimes\cdots\otimes\tau_0$ to $Z(G)$ contains a subreprentation which is equivalent to a multiple of the regular representation of $Z(G)$. And $P_1(g_1)\otimes P_1(g_2)\otimes\cdots\otimes P_1(g_n)$ give a projection of $\boldsymbol{H}(\tau_0)\otimes\cdots\otimes\boldsymbol{H}(\tau_0)$ onto the subrepresentation for any $(g_1, \cdots, g_n)$, where $P_1(g)$ is given in lemma 5.8.*

Proof. From lemma 5.7, $\tau_0|_{Z(G)}$ contains a component being equivalent to the regular representation, and lemma 5.8 claims that $P_1(g)$ gives a projection to this component for any g in G. By lemma 1.25, the Kronecker product of this components is equivalent to a multiple of the regular representation. Therefore,

$$\tau_0|_{Z(G)}\otimes\cdots\otimes\tau_0|_{Z(G)}\sim(\tau_0\otimes\cdots\otimes\tau_0)|_{Z(G)} \tag{5.31}$$

contains a component which is equivalent to a multiple of the regular representations, and $P_1(g_1)\otimes P_1(g_2)\otimes\cdots\otimes P_1(g_n)$ gives the projection onto this component.

Let M be a locally compact space, and G be a Lie group which operates as a transformation group over M, such that $(m,g)\rightarrow mg$ gives a continuous map of $M\times G$ to M, and $me=m$ for any m in M and the unit element e in G.

Now we call the closed subgroup

$$\Gamma(m)=\{g: mg=m\} \tag{5.32}$$

of G, defined for any given m in M, the isotropy subgroup on m.

Lemma 5.10. (*J. Glimm* [*8*]). *The dimension of the closed subgroup $\Gamma(m)$ is a upper semi-continuous function on M.*

3. Now we shall entre to the proof of lemmata 5.1 and 5.2.

From corollary to lemma 5.3, for proving lemmata 5.1 and 5.2, it is sufficient to show that these lemmata are true for any simple factor of G.

If G is compact then lemma 5.1 is same to lemma 5.4. And non-zero property of $\boldsymbol{T}=\{T(\omega)\}$ on R follows from existence of ω in $\widehat{G}$ for which $T(\omega)\neq 0$, because any irreducible representation of compact group is contained in R as a discrete component. This is namely lemma 5. 2.

Thus in what follows, we restrict us to the case that G is a non-compact connected simple group with finite centre.

For such a G, from lemma 5. 5, $\Gamma \neq G$. Put

(5. 33) $$\dim \Gamma \equiv d.$$

Lemma 5. 11. *For* $t \geqq 2d+2$, $\overbrace{\omega_0 \otimes \cdots \otimes \omega_0}^{t}$ (ω_0 *is the representation given in (5. 12)) contains a subrepresentation which is equivalent to* R.

That is, for such a G *Lemma 5. 1 is true.*

Proof. In §6, it is shown that two closed subgroups Γ^t $(\equiv \overbrace{\Gamma \times \cdots \times \Gamma}^{t})$ and $\widetilde{G}_t (\equiv \{(g, \cdots, g) : g \in G\})$ in $G^t (\equiv \overbrace{G \times \cdots \times G}^{t})$ are regularly related. So we can apply G. W. Mackey's results (lemma 1. 23) to this case.

(5. 34) $$\omega_0 \otimes \cdots \otimes \omega_0 = (\underset{\Gamma \to G}{Ind}\, \tau_0) \otimes \cdots \otimes (\underset{\Gamma \to G}{Ind}\, \tau_0)$$
$$\sim \int_{\Gamma^t \backslash G^t / \widetilde{G}_t} \omega(g_1(\tau_0) \otimes \cdots \otimes g_t(\tau_0) : \Gamma^t(\hat{g}))\, d\nu(\hat{g}),$$

where $\hat{g} = (g_1, \cdots, g_t)$ runs over the set of representatives of $(\Gamma^t, \widetilde{G}_t)$-double cosets in G^t, and ν is a measure over $\Gamma^t \backslash G^t / \widetilde{G}_t$ such that a double cosetwise set E in G^t is a null set with respect to a Haar measure $\mu^t (\equiv \overbrace{\mu \times \cdots \times \mu}^{t})$ over G^t, if and only if its canonical image $\widetilde{E}$ in $\Gamma^t \backslash G^t / \widetilde{G}_t$ is a ν-null set. $\Gamma^t(\hat{g})$ is a closed subgroup of G such that

(5. 35) $$\Gamma^t(\hat{g}) \equiv g_1^{-1} \Gamma g_1 \cap \cdots \cap g_t^{-1} \Gamma g_t\,,$$

then $\Gamma^t(\hat{g})$ contains $Z(G)$ always.

Write by $g_j(\tau_0)$ the representation of $g_j^{-1} \Gamma g_j$, space of representation of which is the space $\boldsymbol{H}(\tau_0)$ as same as τ_0, and operators of which is given by

(5. 36) $$W_{g_j \gamma' g_j^{-1}}(\tau_0), \qquad \text{for } \gamma' \text{ in } g_j^{-1} \Gamma g_j\,.$$

Then the outer Kronecker product $g_1(\tau_0)\hat{\otimes}\cdots\hat{\otimes}g_t(\tau_0)$ is defined as a representation of the subgroup $g_1^{-1}\Gamma g_1\times g_2^{-1}\Gamma g_2\times\cdots\times g_t^{-1}\Gamma g_t$ in G^t. So we can consider its restriction to the subgroup $\widetilde{\Gamma}^t(\hat{g})\equiv\{(\gamma,\cdots,\gamma):\gamma\in\Gamma^t(\hat{g})\}$ which is isomorphic to $\Gamma^t(\hat{g})$. Regarding this restriction as a representation

$$(5.37)\qquad g_1(\tau_0)\,|_{\Gamma^t(\hat{g})}\otimes\cdots\otimes g_t(\tau_0)\,|_{\Gamma^t(\hat{g})}(\equiv\rho(\hat{g},\Gamma^t(\hat{g})))$$

of $\Gamma^t(\hat{g})$, put

$$(5.38)\qquad \omega(g_1(\tau_0)\otimes\cdots\otimes g_t(\tau_0):\Gamma^t(\hat{g}))\equiv\underset{\Gamma^t(\hat{g})\to G}{Ind}\,\rho(\hat{g},\Gamma^t(\hat{g})),$$

a representation of G induced by the representation $\rho(\hat{g},\Gamma^t(\hat{g}))$ of $\Gamma^t(\hat{g})$.

In the other hand, because of invariancy of elements in $Z(G)$ by inner automorphisms,

$$(5.39)\qquad g_j(\tau_0)\,|_{Z(G)}\sim\tau_0\,|_{Z(G)}\,.$$

So

$$(5.40)\qquad g_1(\tau_0)\,|_{Z(G)}\otimes\cdots\otimes g_t(\tau_0)\,|_{Z(G)}\sim\tau_0\,|_{Z(G)}\otimes\cdots\otimes\tau_0\,|_{Z(G)}$$
$$\sim(\tau_0\otimes\cdots\otimes\tau_0)\,|_{Z(G)}\,.$$

From lemma 5.9, the last representation contains a subrepresentation which is equivalent to a multiple of the regular representation of $Z(G)$, so by the step theorem of induced representations (corollary to lemma 1.18)

$$(5.41)\qquad \underset{Z(G)\to G}{Ind}(g_1(\tau_0)\,|_{Z(G)}\otimes\cdots\otimes g_t(\tau_0)\,|_{Z(G)})$$

contains a subrepresentation which is equivalent to a multiple of R.

Lemma 5.12. *For $t\geqq 2d+2$, the set*

$$(5.42)\qquad E\equiv\{\hat{g}:\Gamma^t(\hat{g})\neq Z(G)\},$$

is μ^t-measure zero in G^t. That is, the set of double cosets, for any element of which the isotropic subgroup in $\widetilde{G}_t$ differs from $Z(G)$, is ν- measure zero in $\Gamma^t\backslash G^t/\widetilde{G}_t$.

The proof of this lemma is given in the next section.

When we assume the above lemma is true, then for a double cosets in $C(E)$ (complement of E), which contains the representative $\hat{g}$,

$$(5.43)\quad \omega(g_1(\tau_0)\otimes\cdots\otimes g_t(\tau_0):\Gamma'(\hat{g}))=\omega(g_1(\tau_0)\otimes\cdots\otimes g_t(\tau_0):Z(G))$$
$$\sim \underset{Z(G)\to G}{Ind}\, g_1(\tau_0)\,|_{Z(G)}\otimes\cdots\otimes g_t(\tau_0)\,|_{Z(G)}$$
$$\sim \underset{Z(G)\to G}{Ind}\,(\tau_0\otimes\cdots\otimes\tau_0)\,|_{Z(G)}$$

contains a subrepresentation which is equivalent to a multiple of R.

By lemma 1.23, the correspondence of vectors in the decomposition (5.34) is generated by

$$(5.44)\qquad (f_1\otimes\cdots\otimes f_t)\to f_1(g_1g)\otimes\cdots\otimes f_t(g_tg),$$

as a function of g.

So it is easy to see, using the notation $\widetilde{P}_1$ given in (5.13), the projection $\widehat{P}_1$ generated by

$$(5.45)\qquad \widehat{P}_1(f_1(g_1\cdot)\otimes\cdots\otimes f_t(g_t\cdot))(g)$$
$$=(\widetilde{P}_1f_1)(g_1g)\otimes(\widetilde{P}_1f_2)(g_2g)\otimes\cdots\otimes(\widetilde{P}_1f_t)(g_tg),$$

on the space of the right hand side of (5.34) has the range, on which the restriction of $\omega(g_1(\tau_0)\otimes\cdots\otimes g_t(\tau_0):\Gamma'(\hat{g}))$ is equivalent to a multiple of R. This shows, the right hand side of (5.34) contains a direct integral, almost all components of which are equivalent to a multiple of R. But from the construction of $\widetilde{P}_1$, this direct integral is equivalent to a direct integral of a multiple of R with same multiplicity over $C(E)$, that is, the result of lemma 1.8 asserts that this direct integral is equivalent to a multiple of R.

Thus it is remained to prove lemma 5.12.

4. Proof of lemma 5.12. By lemma 5.5, $G\neq\Gamma$, so we consider G as a transformation group over the homogeneous space $M=\Gamma'\backslash G'$ which maps a coset m containing $(\hat{g}')\equiv(g_1,\cdots,g_t)$ to the cosets mg containing $(\hat{g}'\cdot g)\equiv(g_1g,\cdots,g_tg)$. It is easy to show that the isotropy group of this coset m is $\Gamma'(\hat{g}')$. By lemma 5. 10, $\dim\Gamma'(\hat{g}')$ is a upper semi-continuous function on G'. So the set

$$E'_m \equiv \{\hat{g}' : \dim\Gamma'(\hat{g}') \geqq m+1\} = \{\hat{g}' : \dim\Gamma'(\hat{g}') \gneqq m\} \tag{5. 46}$$

is a closed set in G'.

Lemma 5. 13. *For* $m \geqq (\dim\Gamma) - l + 1$,

$$\mu'(E'_m) = 0. \tag{5. 47}$$

Before stating the proof of lemma 5. 13, we have to set some lemmata about simple Lie algebra $\mathfrak{g}$ of G.

Lemma 5. 14. *For any element X in $\mathfrak{g}$ and any proper subspace V in $\mathfrak{g}$,*

$$\mu(\{g: (adg)X \in V\}) = 0. \tag{5. 48}$$

Proof. Since $\{g: (adg)X \in V\} \subseteq \{g: (adg)X \in V_0\}$ for any V_0 containing V, without loss of generality, we can assume V is a hyperplane in $\mathfrak{g}$. By the reason of simplicity of $\mathfrak{g}$, its Killing form $B(X, Y)$ is non-degenerated, so there exists an element X_0 in $\mathfrak{g}$ such that X_1 is in V if and only if

$$B(X_0, X_1) = 0. \tag{5. 49}$$

For given X, put $X_1 = (adg)X$, then $B(X_0, (adg)X)$ is a non-zero analytic function of g. Therefore, this asserts that

$$\mu(\{g: (adg)X_1 \subset V\}) = \mu(\{g: B(X_0, (adg)X) = 0\}) = 0. \tag{5. 50}$$

Lemma 5. 15. *For any proper subspace V_1 and V_2 in $\mathfrak{g}$,*

$$\mu(\{g: (adg)\, V_1 \subset V_2\}) = 0. \tag{5. 51}$$

Proof. Let a basis in V_1 be $\{X_j\}_j$, then

$$\{g: (adg)\, V_1 \subset V_2\} = \bigcap_j \{g: (adg)X_j \subset V_2\}. \tag{5. 52}$$

So (5. 48) results (5. 51).

Proof of Lemma 5. 13. For any given two proper closed subgroups K_1, K_2 in G, with Lie algebras $\mathfrak{k}_1$, $\mathfrak{k}_2$ respectively. Consider the set

$$N(K_1, K_2) = \{g: \dim K_1 = \dim(K_1 \cap g^{-1}K_2 g)\}. \tag{5.53}$$

Since K_1, K_2 and $K_1 \cap g^{-1}K_2 g$ are Lie groups too, so the equation

$$\dim K_1 = \dim(K_1 \cap g^{-1}K_2 g) \tag{5.54}$$

means that the connected components K_1^0 of e in K_1 and $(K_1 \cap g^{-1}K_2 g)^0$ of e in $K_1 \cap g^{-1}K_2 g$ coincide, that is, (5.54) is equivalent to

$$K_1^0 \subset g^{-1}K_2^0 g. \tag{5.55}$$

But the relation (5.55) is transfered to the relation between Lie abgebras, as

$$\mathfrak{k}_1 \subset (adg)\mathfrak{k}_2. \tag{5.56}$$

Therefore,

$$N(K_1, K_2) = \{g: \mathfrak{k}_1 \subset (adg)\mathfrak{k}_2\}. \tag{5.57}$$

So lemma 5.15 results

$$\mu(N(K_1, K_2)) = 0. \tag{5.58}$$

We apply (5.58) to the case of that

$$\begin{aligned} &K_1^{(1)} = K_2^{(1)} = \Gamma;\ K_1^{(2)} = \Gamma \cap g_1^{-1}\Gamma g_1,\ K_2^{(2)} = \Gamma;\ \cdots \\ &\cdots;\ K_1^{(j)} = \Gamma \cap g_1^{-1}\Gamma g_1 \cap \cdots \cap g_{j-1}^{-1}\Gamma g_{j-1},\ K_2^{(j)} = \Gamma;\ \cdots, \end{aligned} \tag{5.59}$$

inductively, and we get

$$\begin{aligned} \mu^j(\{(g_1, \cdots, g_j):\ &\dim(\Gamma \cap g_1^{-1}\Gamma g_1 \cap \cdots \\ &\cdots \cap g_j^{-1}\Gamma g_j) \gneqq (\dim \Gamma) - j\}) = 0. \end{aligned} \tag{5.60}$$

This implies immediately

$$\begin{aligned} \mu^{j+1}(\{(g_1, \cdots, g_{j+1}):\ &\dim(g_1^{-1}\Gamma g_1 \cap \cdots \\ &\cdots \cap g_{j+1}^{-1}\Gamma g_{j+1}) \gneqq (\dim \Gamma) - j\}) = 0. \end{aligned} \tag{5.61}$$

Put $j+1 = l$, we get (5.47).

Lemma 5.16. *For two discrete subgroups D_1, D_2 in G,*

$$\mu(\{g: D_1 \cap g^{-1}D_2 g \not\subset Z(G)\}) = 0. \tag{5.62}$$

Proof. Because of σ-compactness of G, D_1 and D_2 are both countable sets. And the set

$$\{g\colon D_1\cap g^{-1}D_2g\not\subset Z(G)\}=\bigcup_{d\in D_2\cap C(Z(G))}\{g\colon g^{-1}dg\in D_1\}$$

is F_σ, therefore measurable.

Now consider for any elements d_1 in D_1, d_2 in D_2,

$$N(d_1, d_2)\equiv\{g\colon d_1=g^{-1}d_2g\}. \tag{5.63}$$

Obviously this set is shown as

$$N(d_1, d_2)=\mathfrak{c}(d_1)n_0=n_0\mathfrak{c}(d_2), \tag{5.64}$$

where $\mathfrak{c}(d_1)$, $\mathfrak{c}(d_2)$ are the centralizer of d_1, d_2, respectively, and n_0 is any element of $N(d_1, d_2)$. Of course $\mathfrak{c}(d_1)$, $\mathfrak{c}(d_2)$ are closed subgroups of G. So from connectedness and simplicity of G, if d_1 or d_2 is not in $Z(G)$, $N(d_1, d_2)$ is lower dimensional than G, and

$$\mu(N(d_1, d_2))=0. \tag{5.65}$$

If $D_1\cap g^{-1}D_2g\not\subset Z(G)$, then there exists a couple of d_1 in D_1 and d_2 in D_2, such that both of them are not in $Z(G)$ and

$$d_1=g^{-1}d_2g. \tag{5.66}$$

So such a g is contained in $N(d_1, d_2)$. That is

$$\{g\colon D_1\cap g^{-1}D_2g\not\subset Z(G)\}\subset\bigcup_{(d_1, d_2)\not\in(Z(G), Z(G))}N(d_1, d_2). \tag{5.67}$$

Since D_1, D_2 are countable sets,

$$\mu\Big(\bigcup_{(d_1, d_2)\not\in(Z(G), Z(G))}N(d_1, d_2)\Big)=0. \tag{5.68}$$

Therefore (5.67) results (5.62).

Lemma 5.17. *$\Gamma^l(\hat{g}^l)$ contains $Z(G)$ for any l, and*

$$\{\hat{g}^l\colon \Gamma^l(\hat{g}^l)\neq Z(G)\}=\{\hat{g}^l\colon \Gamma^l(\hat{g}^l)\not\subset Z(G)\} \tag{5.69}$$

is a $(\Gamma^l, \widetilde{G}_l)$-double cosetwise measurable set in G^l.

Proof. Since Γ contains $Z(G)$, and since $Z(G)$ is invariant with respect to any automorphism of G, so any $g^{-1}\Gamma g$ and $\Gamma^l(\hat{g}^l)$ contain $Z(G)$.

Obviously, from the definition of $\Gamma^l(\hat{g}^l)$

(5. 70) $\qquad \Gamma'(\hat{\gamma}'\hat{g}')=\Gamma'(\hat{g}'), \quad$ for $\hat{\gamma}'=(\gamma_1, \cdots, \gamma_l)$ in Γ',

and

(5. 71) $\qquad \Gamma'(\hat{g}'\tilde{g})=g^{-1}\Gamma'(\hat{g}')g, \quad$ for $\tilde{g}=(g, \cdots, g)$ in $\widetilde{G}_l$.

This shows that the set given in (5. 69) is $(\Gamma', \widetilde{G}_l)$-double cosetwise set in G'.

Lastly we consider $\widetilde{G}_l \sim G$ as a transformation group over $M=\Gamma' \diagdown G'$. For proving the set given in (5. 69) is measurable, it is equivalent to show the set in M,

(5. 72) $\qquad M_0=\{m: m \neq mg, \text{ for any } g \notin Z(G)\}$

is measurable. Because for any $\hat{g}'$ belonging to a Γ'-coset m in M_0, the isotropy group $\Gamma'(\hat{g}')$ of m is equal to $Z(G)$.

The set given in (5. 69) is the inverse image by π, of the complement of M_0.

Now, for any m and g such that $mg \neq m$, there exist a neighborhood V of m and a neighborhood U of g such that

(5. 73) $\qquad VU \cap V = \phi.$

In fact, since $mg \neq m$, there exist a neighborhood V_1 of m such that

(5. 74) $\qquad V_1 \cap V_1 g = \phi.$

But the map $(m, g) \rightarrow mg$ is continuous, so we can take a neighborhood V of m and a neighborhood U_0 of e and

(5. 75) $\qquad VU_0 \subset V_1,$

then, the pair of neighborhoods V of m and $U=U_0 g$ of g satisfies (5. 73).

Next we shall show the set

(5. 76) $\qquad M(K)=\{m: mg \neq m, \text{ for any } g \text{ in } K\}$

is open, for any compact set K in G.

In fact, for any m and g such that $mg \neq m$, there exist neighborhoods $V(m)$, $U(g)$ such that

$$V(m)U(g) \cap V(m) = \phi. \tag{5.77}$$

For any fixed m in $M(K)$, and for any g in K we take $V(g, m)$, $U(g)$ as (5.77). Since K is compact so there exists a finite covering $K \subset \bigcup_j U(g_j)$. Let $V = \bigcap_j V(g_j, m)$, then for any m_1 in V and any g in K, there exists a $U(g_j)$ which contains g, so

$$V \cap VU(g_j) \subseteq V(g_j\colon m) \cap V(g_j\colon m)U(g_j) = \phi, \tag{5.78}$$

that is,

$$V \cap VK = \phi. \tag{5.79}$$

This shows $M(K) \supset V$, that is, $M(K)$ is open.

Finally $G - Z(G)$ is σ-compact, so we can take a countable compact cover $G - Z(G) = \bigcup_j^{\infty} K_j$. Under these notations, M_0 is represented as

$$M_0 = \bigcap_j^{\infty} M(K_j), \tag{5.80}$$

therefore M_0 is a G_δ-set and measurable.

Now we are on the step to prove lemma 5.12. As shown in lemma 5.17, the set $E \equiv \{\hat{g}^l\colon \Gamma^l(\hat{g}^l) \neq Z(G)\}$ is measurable.

While from (5.60) and (5.61), putting $l = d+1$, we get,

$$\mu^l(\{(g_1, \cdots, g_l)\colon \dim(g_1^{-1}\Gamma g_1 \cap \cdots \cap g_l^{-1}\Gamma g_l) \gneqq 0\}) = 0, \tag{5.81}$$

$$\begin{aligned} \mu^{l-1}(\{(g'_1, \cdots, g'_{l-1})\colon &\dim(\Gamma \cap g_1'^{-1}\Gamma g'_1 \cap \cdots \\ &\cdots \cap g_{l-1}'^{-1}\Gamma g'_{l-1}) \gneqq 0\}) = 0. \end{aligned} \tag{5.82}$$

That is, $\Gamma^l(\hat{g}^l) = g_1^{-1}\Gamma g_1 \cap \cdots \cap g_l^{-1}\Gamma g_l$ and $\Gamma \cap \Gamma^{l-1}(\hat{g}'^{l-1}) = \Gamma \cap g_1'^{-1}\Gamma g'_1 \cap \cdots \cap g_{l-1}'^{-1}\Gamma g'_{l-1}$ are discrete subgroups for almost all $\hat{g}^l$ and $\hat{g}'^{l-1}$.

But from lemma 5.16, for such $\hat{g}^l$ and $\hat{g}'^{l-1}$,

$$\begin{aligned} \mu(\{g\colon g_1^{-1}\Gamma g_1 \cap \cdots \cap g_l^{-1}\Gamma g_l \cap g^{-1}(g_1'^{-1}\Gamma g'_1 \cap \cdots \\ \cdots \cap g_{l-1}'^{-1}\Gamma g'_{l-1} \cap \Gamma)g \neq Z(G)\}) = 0. \end{aligned} \tag{5.83}$$

From the measurability of mapping

$$\begin{aligned} \hat{g}^{2l+1} \equiv (g_1, \cdots, g_l, g_{l+1}, \cdots, g_{2l-1}, g_{2l}) \rightarrow \\ \rightarrow (g_1, \cdots, g_l, g_{l+1}g_{2l}, \cdots, g_{2l-1}g_{2l}, g_{2l}), \end{aligned} \tag{5.84}$$

and the measurability of E, the measurability of the set

$$E' \equiv \{\hat{g}^{2l+1}\colon g_1^{-1}\Gamma g_1 \cap \cdots \cap g_l^{-1}\Gamma g_l \cap g_{2l}^{-1}(g_{l+1}^{-1}\Gamma g_{l+1} \cap \cdots \cdots \cap g_{2l-1}^{-1}\Gamma g_{2l-1} \cap \Gamma) g_{2l} \neq Z(G)\} \tag{5.85}$$

follows. So the above arguments result.

$$\mu^{2l}(E') \equiv \mu^{2d+2}(E') = 0. \tag{5.86}$$

By the mapping (5.84), (5.86) is equivalent to

$$\mu^{2d+2}(E) = 0. \tag{5.87}$$

For $t \gtreqqless 2d+2$, (5.87) assures that for almost all $\hat{g}^t$

$$\Gamma^t(\hat{g}^t) = \Gamma^{2l}(\hat{g}^{2l}) \cap \Gamma^{t-2l}(\hat{g}^{t-2l}) = Z(G) \cap \Gamma^{t-2l}(\hat{g}^{t-2l}) = Z(G), \tag{5.88}$$

where $\hat{g}^t = (\hat{g}^{2l}, \hat{g}^{t-2l})$. This is the required result.

5. Now we shall prove lemma 5.2 for non-compact simple Lie group G with finite centre.

At first, let σ_r be the regular representation of M and τ_r be the representation of Γ defined as in (5.7) from any fixed φ and σ_r instead of σ.

Lemma 5.18. *The restriction of τ_r to $Z(G)$ is equivalent to a multiple of the regular representation of $Z(G)$.*

proof. For characters $\{\chi_j\}$ of $Z(G)$, by joining the functions

$$f_j(m) \equiv \int_{Z(G)} \chi_j(z) f(z \cdot m)\, dz, \quad (f \in C_0(M)), \tag{5.89}$$

one can get a family of functions $\{f_j\}$ such that

$$\begin{cases} |f_j(m)| = 1, \\ f_j(zm) = \chi_j(z) f_j(m), \end{cases} \qquad \text{for any } m \text{ in } M. \tag{5.90}$$

Take a complete orthonormal system $\{h_k(\widetilde{m})\}$ of $L^2(M/Z(G))$, then the system $\{h_k(\widetilde{m}) f_j(m)\}$ constructs a complete orthonormal system of $L^2(M)$. Moreover the closed space $\boldsymbol{H}_k$ spanned by $\{h_k(m) f_j(m)\}_j$ for fixed k, is a subspace on which the operators $\{W_z^{\tau_r}\}$ operate as the regular representation of $Z(G)$, and evidently,

(5. 91) $$\boldsymbol{H}(\tau_r)=\sum\oplus\boldsymbol{H}_k.$$

This shows the assertion.

Completely analogous arguments as in the proof of lemma 5. 11, shows,

Lemma 5.19. *Put*

(5. 92) $$\omega_r=\underset{\Gamma\to G}{Ind}\,\tau_r$$

then, for $t\geqq 2d+2$ $(d=dim\ \Gamma)$,

(5. 93) $$(\omega_r)^t\equiv\overbrace{\omega_r\otimes\cdots\otimes\omega_r}^{t}$$

is equivalent to a multiple of the regular representation R of G.

Proof. As in lemma 5. 11, one gets an analogous formula to (5. 34),

(5. 94) $$(\omega_r)^t\sim\int_{\Gamma^t\backslash G^t/\tilde{G}_t}\omega(g_1(\tau_r)\otimes\cdots\otimes g_t(\tau_r):\Gamma^t(\hat{g}))d\nu(\hat{g}).$$

where almost all $\omega(g_1(\tau_r)\otimes\cdots\otimes g_t(\tau_r):\Gamma^t(\hat{g}))$ are the representation of G induced by $\overbrace{\tau_r\otimes\cdots\otimes\tau_r}^{t}|_{Z(G)}$ of $Z(G)$. But the last representations of $Z(G)$ are multiples of the regular representation of $Z(G)$ with same multiplicities for any such a $\hat{g}$. So $\omega(g_1(\tau_r)\otimes\cdots\otimes g_t(\tau_r):\Gamma^t(\hat{g}))$ are multiples of R with same multiplicities for almost all $\hat{g}$. The results is easily deduced by applying of lemma 1. 8.

Lemma 5.20. *Any irreducible representation ω induced by a irreducible representation τ of Γ as (5. 7) is a subrepresentaiton of ω_r.*

Proof. Because of compactness of M, σ_r contains any irreducible representation σ. Therefore, τ_r contains $\tau\equiv\varphi\cdot\sigma$. This asserts that $\omega_r\equiv\underset{\Gamma\to G}{Ind}\,\tau_r$ contains $\omega\equiv\underset{\Gamma\to G}{Ind}\,\tau$.

Proof of Lemma 5.2. Let $\boldsymbol{T}=\{T(\omega)\}$ be zero on R, then from the equivalence relation given in lemma 5. 18, $\boldsymbol{T}$ is zero on $(\omega_r)^t$, but this is true only when $\boldsymbol{T}$ is zero on ω_r. Because of lemma 5. 20,

ω_r contains the irreducible representation ω as a component, so $T(\omega)$ must be zero. This completes the proof.

§6. Orbits space on semi-simple Lie groups.

1. For proving lemma 5.11 in §5, we have used the property that in any connected simple Lie group G with finite centre, two closed subgroups Γ^t and $\widetilde{G}_t$ in G^t are regularly related. In this §, we shall give the proof of this.

Lemma 6.1. *Two closed subgroups Γ^t and $\widetilde{G}_t$ in G^t are regularly related in the sense of G. W. Mackey. That is, there exists a countable family $\{E_j\}$ of $\Gamma^t-\widetilde{G}_t$ double cosetwise measurable sets where E_0 is μ-null set and other $\Gamma^t-\widetilde{G}_t$ double coset is represented as an intersection of E_j's. (countably separated except* E_0).

2. Before to prove this lemma, we quote useful results.

Lemma 6.2. (*J. Glimm* [8] *Th. 1*). *Let G be a separable locally compact*) topological transformation group acting on a separable locally compact*) space M as given in §5.4. Then the followings are equivalent*:

(1) *The space M/G of G-orbits on M is a T_0-space by induced topology from M,*

(2) *M/G is countably separated,*

(3) *for each m in M, the map $\Gamma(m)g\rightarrow mg$ from $\Gamma(m)\backslash G$ onto mG is a homeomorphism, where $\Gamma(m)$ is the isotropy group of m and mG has the relative topology as a subset of M.*

Lemma 6.3. (*F. Bruhat* [1] *Th. 7.1*). *When G and Γ are as in §5.2, then the space of $\Gamma-\Gamma$ double cosets in G, is a finite set.*

Lemma 6.4. (*F. Bruhat* [1], *Chap. III §7.3*). *For any g in G, there exists a s(g) in $\Gamma g\Gamma$ such that*

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*) We assume T_2-property in the word "locally compact".

$$(6.1)\qquad \begin{cases} \Gamma\cap s(g)^{-1}\Gamma s(g)=MHN_{s(g)}, \\ N_{s(g)}=N\cap s(g)^{-1}Ns(g). \end{cases}$$

Corollary. *If*

$$(6.2)\qquad \dim(\Gamma g\Gamma)=\dim G,$$

then there exists a $s(g)$ in $\Gamma g\Gamma$, such that

$$(6.3)\qquad \Gamma\cap s(g)^{-1}\Gamma s(g)=MH.$$

Proof. It is easy to see, for Lie algebras $\mathfrak{g}, \mathfrak{c}, \mathfrak{m}, \mathfrak{h}, \mathfrak{n}$ of G, Γ, M, H, N respectively,

$$(6.4)\qquad \dim G=\dim\mathfrak{g}=\dim\mathfrak{n}+\dim\mathfrak{c}= =2\dim\mathfrak{n}+\dim\mathfrak{h}+\dim\mathfrak{m}.$$

(see F. Bruhat [1] proof of lemma 7.1, p 188). From the assumption, using $s(g)$ in the result of lemma 6.4,

$$\begin{aligned}(6.5)\qquad \dim G&=\dim\Gamma g\Gamma=2\dim\Gamma-\dim(\Gamma\cap g^{-1}\Gamma g)\\ &=2(\dim\mathfrak{n}+\dim\mathfrak{h}+\dim\mathfrak{m})-\dim(\Gamma\cap s(g)^{-1}\Gamma s(g))\\ &=2\dim\mathfrak{n}+2\dim\mathfrak{h}+2\dim\mathfrak{m}-(\dim M+\dim H+\dim N_{s(g)})\\ &=2\dim\mathfrak{n}+\dim\mathfrak{h}+\dim\mathfrak{m}-\dim N_{s(g)}.\end{aligned}$$

Equate the right hand sides of (6–4) and (6–5), we get

$$(6.6)\qquad \dim N_{s(g)}=0.$$

But as shown in F. Bruhat's paper [1] p 189, $N_{s(g)}$ is connected, so that $N_{s(g)}$ must be $\{e\}$ and we get (6–3) from (6–1) immediately.

3. Proof of lemma 6.1. Consider a map on G^t to G^{t-1} defined by

$$(6.7)\qquad \varphi:(g_1,\cdots,g_t)\to(g_1g_t^{-1},\cdots,g_{t-1}g_t^{-1}).$$

It is easy to see that φ maps any $\Gamma^t-\widetilde{G}_t$ double coset in G^t to a $\Gamma^{t-1}-\widetilde{\Gamma}_{t-1}$ double coset in G^{t-1}, one-to-one way. Therefore, an one-to-one correspondence $\hat{\varphi}$ between $\Gamma^t\backslash G^t/\widetilde{G}_t$ and $\Gamma^{t-1}\backslash G^{t-1}/\widetilde{\Gamma}_{t-1}$ is established by φ. Since φ is continuous and open, $\hat{\varphi}$ gives a homeomorphism of these spaces with canonically induced topologies from the topologies of G^t, G^{t-1} respectively. And the inverse image of μ^{t-1}null set in G^{t-1} by φ is μ^t-null set in G^t. That is, if we show $\Gamma^{t-1}\backslash G^{t-1}/\widetilde{\Gamma}_{t-1}$ is countably separated except null set, then lemma 6.1 is proved.

While from lemmata 6.2 and 6.3, $\Gamma\backslash G/\Gamma$ is a T_0-space and the union of all lower dimensional $\Gamma-\Gamma$ double cosets in G becomes a μ-null set F in G. Put

$$G'=G-F, \tag{6.8}$$

then G' is open as a union of open cosets which has same dimension with G. And $\Gamma^{t-1}\backslash(G')^{t-1}\sim(\Gamma\backslash G')^{t-1}$ is Γ^{t-1}-orbitwise open set in $\Gamma^{t-1}\backslash G^{t-1}$, especially $\widetilde{\Gamma}_{t-1}$-orbitwise open set in $\Gamma^{t-1}\backslash G^{t-1}$, and locally compact. Therefore, to prove that $\Gamma^{t-1}\backslash(G')^{t-1}/\widetilde{\Gamma}_{t-1}$ is countably separated, it is sufficient to show this space is a T_0-space, by lemma 6.2, this results lemma 6.1 soon.

For showing this, we shall prove the following.

Lemma 6.5. *For fixed l and closed subgroups $A\supset B$ in Γ^l, if $\Gamma^l\backslash(G')^l/A$ and $\hat{g}^{-1}\Gamma^l\hat{g}\cap A\backslash A/B$ are a T_0-space for any $\hat{g}\equiv(g_1, \cdots, g_l)$ in $(G')^l$, then $\Gamma^l\backslash(G')^l/B$ is a T_0-space.*

Proof. Let $\hat{g}$, $\hat{g}'$ be representatives of two different Γ^l-B double cosets in $(G')^l$.

If $\hat{g}$, $\hat{g}'$ belong to mutually different Γ^l-A double cosets, then from T_0-property of $\Gamma^l\backslash(G')^l/A$ there exists a Γ^l-A double cosetwise open set which contains one of $\hat{g}$ and $\hat{g}'$ and does not contain the other. Since $A\supset B$, a Γ^l-A double cosetwise set is Γ^l-B double cosetwise set too. This shows the separating property of $\Gamma^l\backslash(G')^l/B$ about $(\hat{g}, \hat{g}')$.

Next if $\hat{g}, \hat{g}'$ belong to a same Γ^l-A double coset $\Gamma^l\hat{g}A=\Gamma^l\hat{g}'A$. From lemma 6.2 and T_0-property of $\Gamma^l\backslash(G')^l/A$, this double coset is homeomorphic to the homogeneous space $\hat{g}^{-1}\Gamma^l\hat{g}\cap A\backslash A$, especially locally compact. The group B operates on this homogeneous space $\hat{g}^{-1}\Gamma^l\hat{g}\cap A\backslash A$ which is homeomorphic to $\Gamma^l\hat{g}A$, as a locally compact topological transformation group. By the homeomorphism between $\Gamma^l\hat{g}A$ and $\hat{g}^{-1}\Gamma^l\hat{g}\cap A\backslash A$, a Γ^l-B double coset is mapped to some $(\hat{g}^{-1}\Gamma^l\hat{g}\cap A)-B$ double coset in A. So the existence of Γ^l-B double cosetwise open set separating the cosets which contain $\hat{g}, \hat{g}'$ respectively, follows from T_0-property of $\hat{g}^{-1}\Gamma^l\hat{g}\cap A\backslash A/B$.

We put A and B in lemma 6.5 as

$$\text{(6.9)} \qquad \begin{cases} A=\widetilde{\Gamma}_{l-1}\times\Gamma=\{(\overbrace{\gamma,\cdots,\gamma}^{l-1},\gamma')\in\Gamma^l\}, \\ B=\widetilde{\Gamma}_l. \end{cases}$$

Then from

$$\text{(6.10)} \qquad \begin{aligned} \Gamma^l\backslash(G')^l/A &= \Gamma^l\backslash(G')^l/(\widetilde{\Gamma}_{l-1}\times\Gamma) \\ &= (\Gamma^{l-1}\backslash(G')^{l-1}/\widetilde{\Gamma}_{l-1})\times(\Gamma\backslash G'/\Gamma), \end{aligned}$$

and

$$\text{(6.11)} \qquad \begin{aligned} \hat{g}^{-1}\Gamma^l\hat{g}\cap A &= \{(\overbrace{\gamma,\cdots,\gamma}^{l-1},\gamma'): \\ &:\gamma\in\Gamma\cap g_1^{-1}\Gamma g_1\cap\cdots\cap g_{l-1}^{-1}\Gamma g_{l-1},\ \gamma'\in\Gamma\cap g_l^{-1}\Gamma g_l\} \\ &= \{(\overbrace{\gamma,\cdots,\gamma}^{l-1}):\gamma\in\Gamma\cap g_1^{-1}\Gamma g_1\cap\cdots\cap g_{l-1}^{-1}\Gamma g_{l-1}\}\times\Gamma\cap g_l^{-1}\Gamma g_l. \end{aligned}$$

We shall use the notation

$$\text{(6.12)} \qquad \hat{g}_0^{l-1}\equiv(e, g_1, \cdots, g_{l-1}),$$

then (6.11) is shown by the notation in §5, as

$$\text{(6.11}'\text{)} \qquad \hat{g}^{-1}\Gamma^l\hat{g}\cap A=\widetilde{\{\Gamma^l(\hat{g}_0^{l-1})\}}_l\times\Gamma^2(\hat{g}_0^1).$$

And

$$\text{(6.13)} \qquad \begin{aligned} (\hat{g}^{-1}\Gamma^l\hat{g}\cap A)\backslash A/B &= \widetilde{\{\Gamma^l(\hat{g}_0^{l-1})\}}_l\times\Gamma^2(\hat{g}_0^1)\backslash\widetilde{\Gamma}_{l-1}\times\Gamma/\widetilde{\Gamma}_l \\ &\sim\Gamma^l(\hat{g}_0^{l-1})\times\Gamma^2(\hat{g}_0^1)\backslash\Gamma^2/\widetilde{\Gamma}_2, \end{aligned}$$

because of $\widetilde{\Gamma}_{l-1}\sim\Gamma$ and $\widetilde{\{\Gamma^l(\hat{g}_0^{l-1})\}}_l\sim\Gamma^l(\hat{g}_0^{l-1})$.

Using analogous mapping as (6-7), the last space is homeomorphic to

$$\text{(6.14)} \qquad \Gamma^l(\hat{g}_0^{l-1})\times\Gamma^2(\hat{g}_0^1)\backslash\Gamma^2/\widetilde{\Gamma}_2\sim\Gamma^l(\hat{g}_0^{l-1})\backslash\Gamma/\Gamma^2(\hat{g}_0^1).$$

By lemmata 6.2 and 6.3, $\Gamma\backslash G'/\Gamma$ is T_0, therefore if the space given in (6.14) is T_0, then using lemma 6.5 for (6.10) and (6.13) we can prove the T_0-property of $\Gamma^l\backslash(G')^l/\widetilde{\Gamma_l}$ by induction with respect to l.

Consequently the problem is reduced to show that the space given in (6.14) is T_0.

By the corollary to lemma 6.4, for any g in G', $\Gamma^2(\hat{g}_0^1)$ is conjugate to MH in Γ. So the space $\Gamma/\Gamma^2(\hat{g}_0^1)\sim\Gamma/MH$ is homeomorphic to N, and from the simply connectedness of N, this space is homeomorphic to its Lie algebra $\mathfrak{n}$. Thus the space $\Gamma^l(\hat{g}_0^{l-1})\backslash\Gamma/\Gamma^2(\hat{g}_0^1)\sim\Gamma^l(\hat{g}_0^{l-1})\backslash\Gamma/MH$ is homeomorphic to the orbits space by the operations $\{\mathrm{ad}\gamma\}$ on adjoint representation restricted to $\mathfrak{n}$, for γ in $\Gamma^l(\hat{g}_0^{l-1})$ which is conjugate to a subgroup Γ' of MH in Γ.

While the general theory of Lie algebras asserts that $\mathfrak{n}$ is spanned by the root vectors E_α of ad h such that

$$(6.15)\qquad (\mathrm{ad}\, h)E_\alpha = e^{\alpha(Y)}E_\alpha, \qquad \text{for } h\equiv\exp Y \text{ in } H.$$

We denote by $\mathfrak{n}_\lambda$ the subspace of $\mathfrak{n}$ which is spanned by E_α's such that α's give a same linear form λ on H, then $\mathfrak{n}$ is represented as a direct sum of $\mathfrak{n}_\lambda$'s. For any X in $\mathfrak{n}$, let X_λ be the component of X in $\mathfrak{n}_\lambda$. And for some set J of indices, put

$$(6.16)\qquad \Omega_J\equiv\{X: X_{\lambda_j}=0,\ j\in J\},$$

then this space is a closed subspace of $\mathfrak{n}$. It is sufficient to show that each orbit $\{(\mathrm{ad}\gamma)X:\ \gamma\in\Gamma'\}$ is closed in $\Omega_J-\bigcup_{J_1\supsetneq J}\Omega_{J_1}$ which contains X.

While since any m in M commutes with all h in H, so $\{\mathrm{ad}\ m\}$ makes invariant each subspaces $\mathfrak{n}_\lambda$'s. As shown in Harish-Chandra's paper [10], for adequate inner product ad m becomes an orthogonal transformation and ad h is given as in (6.15).

For given $\hat{g}_0^{l-1}=(e, g_1, \cdots, g_{l-1})$, we take $\{s(g_j)(\equiv s_j)\}_j$ as in lemma 6.4 corresponding to each $g_j(1\leqq j\leqq l-1)$. Put

$$(6.17)\qquad g_j=\gamma_j' s_j\gamma_j, \quad (\gamma_j, \gamma_j', \in\Gamma),$$

and

(6. 18) $\gamma_{l-1}\gamma_j^{-1}=n_jh_jm_j, \quad n_j\in N, h_j\in H, m_j\in M.$

This shows that

(6. 19)
$$\begin{aligned}\Gamma'(\hat{g}_0^{l-1}) &= \Gamma\cap g_1^{-1}\Gamma g_1\cap\cdots\cap g_{l-1}^{-1}\Gamma g_{l-1}\\ &= \Gamma\cap\gamma_1^{-1}s_1^{-1}\Gamma s_1\gamma_1\cap\cdots\cap\gamma_{l-1}^{-1}s_{l-1}^{-1}\Gamma s_{l-1}\gamma_{l-1}\\ &= \gamma_1^{-1}(\Gamma\cap s_1^{-1}\Gamma s_1)\gamma_1\cap\cdots\cap\gamma_{l-1}^{-1}(\Gamma\cap s_{l-1}^{-1}\Gamma s_{l-1})\gamma_{l-1}\\ &= \gamma_1^{-1}MH\gamma_1\cap\cdots\cap\gamma_{l-1}^{-1}MH\gamma_{l-1}\\ &= \gamma_{l-1}^{-1}\{(\gamma_{l-1}\gamma_1^{-1})MH(\gamma_{l-1}\gamma_1^{-1})^{-1}\cap\cdots\\ &\qquad\cdots\cap(\gamma_{l-1}\gamma_{l-2}^{-1})\, MH(\gamma_{l-1}\gamma_{l-2}^{-1})^{-1}\cap MH\}\gamma_{l-1}\\ &= \gamma_{l-1}^{-1}\{n_1MHn_1^{-1}\cap\cdots\cap n_{l-2}MHn_{l-2}^{-1}\cap MH\}\gamma_{l-1}.\end{aligned}$$

Thus $\Gamma'(\hat{g}_0^{l-1})$ is conjugate in Γ to

(6. 20) $\Gamma'=n_1MHn_1^{-1}\cap\cdots\cap n_{l-2}MHn_{l-2}^{-1}\cap MH$

While for any γ in Γ the decomposition

(6. 21) $\gamma=\mathrm{mhn}, \quad m\in M,\ h\in H,\ n\in N,$

is unique. So an element γ in Γ belongs to Γ', if and only if it is the form for some sets $\{m_j\}$ in M and $\{h_j\}$ in H,

(6. 22) $\gamma=n_1m_1h_1n_1^{-1}=\cdots=n_{l-2}m_{l-2}h_{l-2}n_{l-2}^{-1}=mh,$

but since MH is in the normalizer of N

(6. 23) $mh=m_jh_j((m_jh_j)^{-1}n_j(m_jh_j))n_j^{-1}, \quad (1\leq j\leq l-2).$

This results

(6. 24) $m=m_j,\ h=h_j,\ (m_jh_j)^{-1}n_j(m_jh_j)=n_j.$

That is, (mh) commutes with each n_j.

Conversely any element (mh) in the commutator of $\{n_j\}_j$ belongs to Γ'. Therefore Γ' is equal to the commutator of $\{n_j\}_j$ in MH.

Let

(6. 25) $n_j=\exp X_j, \quad X_j\in\mathfrak{n}(1\leqslant j\leqslant l-2),$

and

(6. 26) $X_j=\sum(X_j)_\lambda, \quad (X_j)_\lambda\in\mathfrak{n}_\lambda,$

then Γ' is characterized as a subgroup of MH as

$$\Gamma' = \{\gamma = \mathrm{mh} : \mathrm{ad}(\mathrm{mh}) X_j = X_j \ (1 \leqslant j \leqslant l-2)\}. \tag{6.27}$$

But since ad(mh) makes invariant each subspaces $\mathfrak{n}_\lambda$'s. So the above conditions in (6.27) are equal to

$$\mathrm{ad}(\mathrm{mh})(X_j)_\lambda = (X_j)_\lambda. \tag{6.28}$$

While ad m is an orthogonal transformation and $e^{\lambda(Y)}$ is real, so (6.28) means

$$\begin{cases} e^{\lambda(Y)}(X_j)_\lambda = (X_j)_\lambda, \\ (\mathrm{ad}\ m)(X_j)_\lambda = (X_j)_\lambda, \end{cases} \quad \text{for any } j \text{ and } \lambda. \tag{6.29}$$

This shows that for each $\gamma = \mathrm{mh}$ in Γ', its components m and h are in the commutator of $\{n_j\}$ separately. Conversely, if m and h commute with $\{n_j\}$, then $\gamma = \mathrm{mh}$ is in the commutator of $\{n_j\}$. Consequently Γ' is a direct product of the commutators M_0 and H_0 of $\{n_j\}$ in M and H respectively, which are both closed. Obviously H_0 must be a vector subgroup in H.

Now let a sequence

$$\{\mathrm{ad}(m_k h_k) X : m_k \in M_0,\ h_k \in H_0\}_k \tag{6.30}$$

in a Γ'-orbit converge to X_0, in which $\{\mathrm{ad}(m_k h_k) X\}_k$ and X_0 are contained in some $\Omega_J - \bigcup_{J_1 \geq J} \Omega_{J_1}$. This is equivalent to the convergence of each components $\{\mathrm{ad}(m_k h_k)(X)_{\lambda_j}\}$ to $(X_0)_{\lambda_j}$. From the definition of Ω_J, these components are not zero only for $j \notin J$.

This means that the sequence of their norms

$$\begin{aligned} \|\mathrm{ad}(m_k h_k)(X)_{\lambda_j}\| &= \|\mathrm{ad}(h_k)(X)_{\lambda_j}\| \\ &= e^{\lambda_j(Y_k)} \|(X)_{\lambda_j}\| \end{aligned} \tag{6.31}$$

converge to the non-zero value $\|(X_0)_{\lambda_j}\|$ for $j \notin J$. Since H_0 is a vector subgroup of H, so there exists a h_0 in H_0 such that

$$\|(\mathrm{ad}\, h_0)(X)_{\lambda_j}\| = \|(X_0)_{\lambda_j}\|, \quad \text{for any } j. \tag{6.32}$$

The compactness of M_0 assures the existence of subsequence of $\{(\mathrm{ad}\ m_k) X\}$ converging to some $\{(\mathrm{ad}\ m_0) X\}$. That is, there is a subsequence of $\{(\mathrm{ad}\gamma_k)X\}$ converging to $\mathrm{ad}(m_0h_0)X$ in this Γ'-orbit. Obviously the limit must coincide to X_0. That is, each Γ'-orbit is closed in $\Omega_J - \bigcup_{J_1 \geq J} \Omega_{J_1}$ which contains this orbit.

This completes the proof.

§7. Semi-direct product groups.

1. Let G be the semi-direct product of a separable closed abelian normal subgroup N and a closed subgroup K. In this §, we shall show that if G satisfies the assumptions $1\sim3$, which are given later, the duality of the first kind is valid for such a G.

As examples of groups of this type, one can quote the m-dimensional proper inhomogeneous Lorentz group, the motion group over m-dimensional Euclidean space.

Put the dual group $\widehat{N}$ of abelian group N, and consider an element g in G as a transformation on $\widehat{N}$ such as

$$g: \hat{n} \to g(\hat{n}), \tag{7.1}$$

where $g(\hat{n})$ is given by

$$\langle g(\hat{n}), n\rangle \equiv \langle \hat{n}, g^{-1}ng\rangle, \tag{7.2}$$

using the notation $\langle\ ,\ \rangle$ of ordinary dual relation between N and $\widehat{N}$.

We choose a representative $\hat{n}$ for each G-orbit in $\widehat{N}$, and denote the isotropy subgroup of $\hat{n}$ in G by $G(\hat{n})$. Since N is an abelian normal subgroup, so any element n fixes all $\hat{n}$. Then the isotropy group of $\hat{n}$ in K is given by

$$K(\hat{n}) \equiv G(\hat{n}) \cap K, \tag{7.3}$$

and $G(\hat{n})$ is the semi-direct product of N and $K(\hat{n})$.

Now we presuppose the assumptions.

[**Assumption 1**] *G is a regular semi-direct product in the sense of G.W. Mackey* [13].

[**Assumption 2**] *There exists an element $\hat{n}_0$ in $\widehat{N}$ such that* $K(\hat{n}_0)$ is separable compact.

For any finite set $F=\{\hat{n}_j\}$ $(1\leqq j\leqq t)$ in $\hat{N}$, we consider the continuous map φ_F of $(K(\hat{n}_1)\backslash K)\times\cdots\times(K(\hat{n}_t)\backslash K)$ into $\hat{N}$, defined by

$$(7.4)\qquad \varphi_F\colon(\tilde{k}_1,\cdots,\tilde{k}_t)\rightarrow\sum_{j=1}^{t}(k_j^{-1}(\hat{n}_j)),$$

where k_j shows a representative of a coset $\tilde{k}_j$ in $K(\hat{n}_j)\backslash K$.

[**Assumption 3**] *There exists a finite set $F=\{\hat{n}_j\}_{1\leq j\leq t}$ in $\hat{N}$ and a relative compact open neighborhood W of $(\tilde{e},\cdots,\tilde{e})$ in $(K(\hat{n}_1)\backslash K)\times\cdots\times(K(\hat{n}_t)\backslash K)$ such that,*

i) *$V\equiv\varphi_F(W)$ is a open neighborhood of $\sum_{j=1}^{t}\hat{n}_j$,*

ii) *the restriction on V of the Haar measure is absolutely continuous to the measure introduced by*

$$(7.5)\qquad \nu_0(E)=\nu_1\times\cdots\times\nu_t(W\cap\varphi_F^{-1}(E\cap V)),\quad \textit{for measurable } E \textit{ in } \hat{N},$$

where ν_j is a non-trivial quasi-invariant measure on $K(\hat{n}_j)\backslash K$.

Under these assumptions, we get the following.

Lemma 7.1. *For such a G, the regular representation R is countably generated.*

Therefore from the results of §4, the main proposition is deduced.

Proposition 7.1. *For a semi-direct product group G, satisfying the assumptions* 1～3, *the duality theorem of the first kind is valid.*

2. For such a group G, all irreducible representations are exhausted by the theory of induced representations given by G. W. Mackey [13].

At first, fix a $\hat{n}$ in $\hat{N}$ and take an irreducible representation $\tau\equiv\{W_k(\tau),\boldsymbol{H}(\tau)\}$ of $K(\hat{n})$. Then $\{\langle\hat{n},n\rangle W_k(\tau),\boldsymbol{H}(\tau)\}$ $(g=nk)$ gives an irreducible representation of $G(\hat{n})$. Put

$$(7.6)\qquad \mathfrak{D}(\hat{n},\tau)\equiv\underset{G(\hat{n})\rightarrow G}{Ind}\{\langle\hat{n},n\rangle W_k(\tau),\boldsymbol{H}(\tau)\}.$$

Then using the assumption 1, the following is valid.

Lemma 7.2. (*G. W. Mackey* [*13*]). *$\mathfrak{D}(\hat{n}, \tau)$ is irreducible and determined by the orbit containing $\hat{n}$ and the representation τ of $K(\hat{n})$ besides unitary equivalence.*

And arbitrary irreducible representation of G is equivalent to one of the representations of such a form.

3. To prove the lemma 7. 1, we consider a decomposition of the regular representation R of G.

For any f in $L^2(G)$, define a function $\tilde{f}$ on $\hat{N}\times G$ by

$$\tilde{f}(\hat{n}, g)\equiv\int f(ng)\overline{\langle\hat{n}, n\rangle}\, d\mu_N(n), \tag{7.7}$$

where μ_N is a Haar measure on N. Then by Plancherel's theorem on N, the integral converges for almost all $\hat{n}$ and g, and

$$\tilde{f}(\hat{n}, ng)=\langle\hat{n}, n\rangle\tilde{f}(\hat{n}, g), \tag{7.8}$$

$$\int_{\hat{N}}\{\int_K|\tilde{f}(\hat{n}, k)|^2 d\mu_K(k)\}\, d\mu_{\hat{N}}(\hat{n})=\|f\|^2<+\infty, \tag{7.9}$$

where $\mu_{\hat{N}}$ is a Haar measure on $\hat{N}$ (adequately normalized). And the operator R_{g_0} in the representation R corresponds to the map,

$$(U_{g_0}(\hat{n})\tilde{f})(\hat{n}, g)\equiv\tilde{f}(\hat{n}, gg_0). \tag{7.10}$$

That is, for fixed $\hat{n}$, $U_g(\hat{n})$ operates as in the representation $\underset{N\to G}{Ind}\langle\hat{n},n\rangle$. Consequently, we get the following.

Lemma 7.3. *R is decomposed as,*

$$R\sim\int_{\hat{N}}\underset{N\to G}{Ind}\langle\hat{n}, n\rangle d\mu_{\hat{N}}(\hat{n}). \tag{7.11}$$

To use the results of G. W. Mackey, the followings must be shown.

Lemma 7.4. *Let $G_j(0\leq j\leq s)$ be subgroups of the form NK_j in G, where K_j are closed subgroups of K. Then $G_0\times G_1\times\cdots\times G_s\backslash G^{s+1}/\tilde{G}_{s+1}$ is homeomorphic to the space $K_0\times K_1\times\cdots\times K_s\backslash K^{s+1}/\tilde{K}_{s+1}$ and to the space $K_1\times K_2\times\cdots\times K_s\diagdown K^s/(\tilde{K}_0)_s$.*

Proof. Since

$$(7.12)\qquad G_0\times G_1\times\cdots\times G_s=K_0N\times K_1N\times\cdots\times K_sN$$
$$=(K_0\times K_1\times\cdots\times K_s)\,N^{s+1},$$

and N^{s+1} is a normal subgroup of G^{s+1}, so

$$(7.13)\qquad G_0\times G_1\times\cdots\times G_s\backslash G^{s+1}=(K_0\times K_1\times\cdots\times K_s)\,N^{s+1}\backslash K^{s+1}N^{s+1}$$
$$\sim K_0\times K_1\times\cdots\times K_s\backslash K^{s+1}.$$

For $\widetilde{N}$ makes invariant any point of the last space, $\widetilde{G}_{s+1}=\widetilde{N}_{s+1}\widetilde{K}_{s+1}$ operates as same manner as $\widetilde{K}_{s+1}$ on this space, so

$$(7\text{-}14)\qquad G_0\times G_1\times\cdots\times G_s\backslash G^{s+1}/\widetilde{G}_{s+1}\sim G_0\times\cdots\times G_s\backslash G^{s+1}/\widetilde{K}_{s+1}$$
$$\sim K_0\times\cdots\times K_s\backslash K^{s+1}/\widetilde{K}_{s+1},$$

as a set. But the one-to-one correspondence of double cosetwise open sets is immediately.

Next, take the map

$$(7.15)\qquad (k_0, k_1, \cdots, k_s)\rightarrow(k_1k_0^{-1}, \cdots, k_sk_0^{-1}),$$

which is analogous to the map (6·7) given in §6, then this map gives the homeomorphism of

$$(7.16)\quad K_0\times K_1\times\cdots\times K_s/K^{s+1}/\widetilde{K}_{s+1}\sim K_1\times\cdots\times K_s\backslash\widetilde{K}^s/(K_0)_s.$$

Corollary. *If K_0 is compact, and $G_j=G(\hat{n}_j)$ for some $\{\hat{n}_j\}$ in $\widehat{N}(1\leqslant j\leqslant s)$, then two subgroups $G_0\times G_1\times\cdots\times G_s$ and $\widetilde{G}_{s+1}$ are regularly related in G^{s+1}, in the sense of G. W. Mackey.*

Especially, $N\times G(\hat{n}_1)\times\cdots\times G(\hat{n}_s)$ and $\widetilde{G}_{s+1}$ are regularly related in G^{s+1}.

Proof. It is easy to see, a orbits space by a compact subgroup over any separable homogeneous space is countably separated. While the separability of factor spaces $K(\hat{n}_j)\backslash K$ follows from the separability of $\widehat{N}$ and from the equivalence between the topologies of $K(\hat{n}_j)\backslash K$ and of the orbit containing $\hat{n}_j$ based on the assumption 1. This shows, $K_1\times\cdots\times K_s\diagdown K^s\diagup(\widetilde{K}_0)_s$ which is homeomorphic to $G_0\times G_1\times\cdots\times G_s\diagdown G^{s+1}\diagup(\widetilde{G})_{s+1}$, is countably separated.

The case of $G_0=N$ is the special case when $K_0=\{e\}$.

Now we can apply the decomposition theorem (lemma 1.23) given by G. W. Mackey, on the Kronecker product,

$$(7.17)\qquad \mathfrak{D}(\{\hat{n}_j\})\equiv(\underset{N\to G}{Ind}\langle\hat{n}_0, n\rangle)\otimes\mathfrak{D}(\hat{n}_1, 1)\otimes\cdots$$
$$\cdots\otimes\mathfrak{D}(\hat{n}_t, 1)\otimes\mathfrak{D}(\hat{n}_{t+1}, 1).$$

Such a representation is decomposed as an integral on the space

$$(7.18)\qquad S\equiv N\times G_1\times\cdots\times G_{t+1}\backslash G^{t+2}/(\widetilde{G})_{t+2}\sim$$
$$\sim K_1\times\cdots\times K_{t+1}\backslash K^{t+1}/\{e\}\sim K_1\times\cdots\times K_{t+1}\backslash K^{t+1}$$
$$\sim(K_1\backslash K)\times\cdots\times(K_{t+1}\backslash K).$$

And for representatives $\widetilde{k}\equiv(k_1, \cdots, k_{t+1})$ of each cosets, the component on this coset of the integral is given by

$$(7.19)\qquad \mathfrak{D}(\{\hat{n}_j\}, \{k_j^{-1}\})\equiv\underset{N\to G}{Ind}\langle\sum_{j=1}^{t+1}k_j^{-1}(\hat{n}_j)+\hat{n}_0, n\rangle,$$

because

$$(7.20)\qquad N=N\cap G(\hat{n}_1)\cap\cdots\cap G(\hat{n}_{t+1}),$$

$$(7.21)\qquad \langle\sum_{j=1}^{t+1}k_j^{-1}(\hat{n}_j)+\hat{n}_0, n\rangle=\langle\hat{n}_0, n\rangle\langle k_1^{-1}(\hat{n}_1), n\rangle\cdots$$
$$\cdots\langle k_{t+1}^{-1}(\hat{n}_{t+1}), n\rangle,$$

$$(7.22)\qquad 1\sim 1\otimes k_1^{-1}(1)\otimes\cdots\otimes k_{t+1}^{-1}(1).$$

The measure ν of decomposition over S is given by

$$(7.23)\qquad \nu\equiv\nu_1\times\cdots\times\nu_{t+1},$$

the product of quasi-invariant measures ν_j on $K(\hat{n}_j)\backslash K$ as a measure which has the same null sets as μ^{t+1}.

Lemma 7.5.

$$(7.24)\qquad \mathfrak{D}(\{\hat{n}_j\})\sim\int_{(K(\hat{n}_1)\backslash K)\times\cdots\times(K(\hat{n}_{t+1})\backslash K)}\mathfrak{D}(\{\hat{n}_j\}, \{k_j^{-1}\})\prod_{j=1}^{t+1}d\nu_j(\widetilde{k}_j).$$

Now we shall show the followings.

Lemma 7.6. *If the assumption* 3 *holds for some* $\{F=\{\hat{n}_j\}_{1\leq j\leq t}, W\}$ *then for any* $\hat{n}_{t+1}$ *in* $\widetilde{N}$, *there exists a relative compact open neighborhood* W' *of* $(\overbrace{\tilde{e}, \cdots, \tilde{e}}^{t+1})$ *in* $(K(\hat{n}_1)\backslash K)\times\cdots\times(K(\hat{n}_{t+1})\backslash K)$ *and* i), ii) *of the assumption 3 are valid for such a* $\{F'\equiv\{\hat{n}_{t+1}\}\cup F, W'\}$.

Proof. Take any relative compact open neighborhood W_0 of $\tilde{e}$ in $K(\hat{n}_{t+1})\backslash K$, and put

$$W' \equiv W \times W_0. \tag{7.25}$$

Then

$$\begin{aligned} V' \equiv \varphi_{F'}(W') &= \{\hat{n} = \sum_{j=1}^{t} k_j^{-1}(\hat{n}_j) + k_{t+1}^{-1}(\hat{n}_{t+1}) : (k_j) \in W'\} \\ &= \bigcup_{k_{t+1} \in W_0} [\varphi_F(W) + k_{t+1}^{-1}(\hat{n}_{t+1})] \end{aligned} \tag{7.26}$$

is a open set as a join of open sets, and contains $\sum_{j=1}^{t+1} \hat{n}_j$, that is, the property i).

Next, to prove the property ii), it is enough to show that for any measurable set E in $\widehat{N}$ satisfying

$$\mu_{\widehat{N}}(E \cap V') \neq 0, \tag{7.27}$$

the following inequation holds.

$$I_0 \equiv (\nu_1 \times \cdots \times \nu_{t+1})(W' \cap \varphi_{F'}^{-1}(E \cap V')) \neq 0. \tag{7.28}$$

But the value I_0 is representable as a double integral.

$$\begin{aligned} I_0 &= \int_{W_0 \times W} \chi_{\varphi_{F'}^{-1}(E \cap V')}(k_1, \cdots, k_{t+1}) d\nu_1(\widetilde{k}_1) \cdots d\nu_{t+1}(\widetilde{k}_{t+1}) \\ &= \int_{W_0} d\nu_{t+1}(\widetilde{k}_{t+1}) \left\{ \int_W \chi_{\{(k_1, \cdots, k_t) : \sum_{j=1}^{t} k_j^{-1}(\hat{u}_j) + k_{t+1}^{-1}(\hat{n}_{t+1}) \in E\}} d\nu_1 \cdots d\nu_t \right\} \\ &= \int_{W_0} d\nu_{t+1}(\widetilde{k}_{t+1}) \left\{ \int_W \chi_{\varphi_F^{-1}([E - k_{t+1}^{-1}(\hat{n}_{t+1})] \cap V)} d\nu_1 \cdots d\nu_t \right\} \\ &= \int_{W_0} \nu_0([E - k_{t+1}^{-1}(\hat{n}_{t+1})]) d\nu_{t+1}(k_{t+1}). \end{aligned} \tag{7.29}$$

While from the separability of $\widehat{N}$ and (7.26), one can choose a countable family $\{E_j = \varphi_F(W) + (k_{t+1}^j)^{-1}(\hat{n}_{t+1})\}$ such that V' is covered by $\bigcup_j E_j$. This shows the existence of k_{t+1} in W_0, from (7.27), such that

$$\mu_{\widehat{N}}([E - k_{t+1}^{-1}(\hat{n}_{t+1})] \cap \varphi_F(W)) \neq 0. \tag{7.30}$$

Since $\mu_{\widehat{N}}$ is a regular measure and $k_{t+1}^{-1}(\hat{n}_{t+1})$ is continuous on

$K(\hat{n}_{t+1})\diagdown K$, so there exists an open set $\mathfrak{O}$ in W_0 for any k_{t+1} in $\mathfrak{O}$, (7. 30) holds. But the assumption 3 asserts for such a k_{t+1},

$$\nu_0([E-k_{t+1}^{-1}(\hat{n}_{t+1})])\neq 0. \tag{7.31}$$

But with respect to ν_{t+1}, any open set has positive measure, so that from (7. 29) and (7. 31), (7. 28) is deduced. This is the proof of property ii).

Combining lemmata 7. 5 and 7. 6, a useful result is obtained.

Lemma 7.7. *For* $\{\hat{n}_j\}_{1\leq j\leq t}$ *given in the assumption 3, there exists a neighborhood* V *of* $\sum_{j=1}^{t}\hat{n}_j$ *in* $\widehat{N}$, *such that for any* $\hat{n}_0$ *and* $\hat{n}_{t+1}$ *in* $\widehat{N}$, $\mathfrak{D}(\{\hat{n}_j\}_{0\leq j\leq t+1})$ *which is given in lemma 7.5, contains a subrepresentation which is equivalent to*

$$\int_{V+\hat{n}_{t+1}} \underset{N\to G}{Ind}\langle\hat{n}+\hat{n}_0, n\rangle d\mu_{\widehat{N}}(\hat{n}). \tag{7.32}$$

Proof. Put $V\equiv\varphi_F(W)$ as in the assumption 3. From lemma 7. 5, obviously $\mathfrak{D}(\{\hat{n}_j\})$ contains

$$\begin{aligned}\int_{W'}\mathfrak{D}(\{\hat{n}_j\}, \{k_j^{-1}\})\prod_{j=1}^{t+1}d\nu_j(k_j)\\ =\int_{V'}\underset{N\to G}{Ind}\langle\hat{n}+\hat{n}_0, n\rangle d\nu_0'(\hat{n}),\end{aligned} \tag{7.33}$$

where W', V' are given in lemma 7. 6, and the measure ν_0' on $\widehat{N}$ is constructed for $\{F', W'\}$ as (7. 5) of the assumption 3.

But the absolute continuity of $\mu_{\widehat{N}}$ to ν_0' asserts that (7. 33) contains a subrepresentation which is equivalent to

$$\int_{V'}\underset{N\to G}{Ind}\langle\hat{n}+\hat{n}_0, n\rangle d\mu_{\widehat{N}}(\hat{n}). \tag{7.34}$$

Again, restrict the integral domain V' to the open domain $V+\hat{n}_{t+1}$ which is contained in V', the result is deduced.

4. Proof of lemma 7.1. At first, take a $\hat{n}_0$ as in the assumption 2. Since $K(\hat{n}_0)$ is separable compact, there exists an at most countable family $\{\tau_0^l\}$ of irreducible representations, such that, $\sum_l\oplus\tau_0^l$ is equivalent to the regular representation R_0 of the group $K(\hat{n}_0)$. That is,

$$(7.35)\qquad \sum_{l}\oplus\mathfrak{D}(\hat{n}_0,\tau_0^l)=\sum_{l=1}\oplus \operatorname*{Ind}_{NK(\hat{n}_0)\to G}\langle\hat{n}_0,n\rangle\tau_0^l\sim$$
$$\sim \operatorname*{Ind}_{NK(\hat{n}_0)\to G}\langle\hat{n}_0,n\rangle(\sum_{l=1}\oplus\tau_0^l)\sim \operatorname*{Ind}_{NK(\hat{n}_0)\to G}\langle\hat{n}_0,n\rangle R_0\sim$$
$$\sim \operatorname*{Ind}_{NK(\hat{n}_0)\to G}\langle\hat{n}_0,n\rangle\ \{\operatorname*{Ind}_{\{e\}\to K(\hat{n}_0)}1\}\sim$$
$$\sim \operatorname*{Ind}_{NK(\hat{n}_0)\to G}\{\operatorname*{Ind}_{N\to NK(\hat{n}_0)}\langle\hat{n}_0,n\rangle\}\sim \operatorname*{Ind}_{N\to G}\langle\hat{n}_0,n\rangle.$$

Next, we consider the Kronecker product of representations defined for the set $F=\{\hat{n}_j\}$ $(1\leqq j\leqq t)$ given in the assumption 3 and arbitrary given $\hat{n}$ in $\widehat{N}$,

$$(7.35)\qquad \mathfrak{D}(\hat{n})=\sum_{l}\oplus[\mathfrak{D}(\hat{n}_0,\tau_0^l)\otimes\mathfrak{D}(\hat{n}_1,1)\otimes\cdots$$
$$\cdots\otimes\mathfrak{D}(\hat{n}_t,1)\otimes\mathfrak{D}(\hat{n},1)]\sim$$
$$\sim[\sum_{l}\oplus\mathfrak{D}(\hat{n}_0,\tau_0^l)]\otimes\mathfrak{D}(\hat{n}_1,1)\otimes\cdots\otimes\mathfrak{D}(\hat{n}_t,1)\otimes\mathfrak{D}(\hat{n},1)\sim$$
$$\sim(\operatorname*{Ind}_{N\to G}\langle\hat{n}_0,n\rangle)\otimes(\operatorname*{Ind}_{NK(\hat{n}_1)\to G}\langle\hat{n}_1,n\rangle 1)\otimes\cdots$$
$$\cdots\otimes(\operatorname*{Ind}_{NK(\hat{n}_t)\to G}\langle\hat{n}_t,n\rangle 1)\otimes(\operatorname*{Ind}_{NK(\hat{n})\to G}\langle\hat{n},n\rangle\cdot 1).$$

Now we apply lemma 7.7 to (7.35) and get the following.

Lemma 7.8. *$\mathfrak{D}(\hat{n})$ contains a subrepresentation which is equivalent to*

$$(7.36)\qquad \int_{V+\hat{n}}\operatorname*{Ind}_{N\to G}\langle\hat{n}_1+\hat{n}_0,n\rangle d\mu_{\widehat{N}}(\hat{n}_1)\sim\int_{V+\hat{n}+\hat{n}_0}\operatorname*{Ind}_{N\to G}\langle\hat{n}_1,n\rangle d\mu_{\widehat{N}}(\hat{n}_1).$$

While the separability of N results the σ-compactness of $\widehat{N}$, and there exists an at most countable set $\{\hat{n}^j\}$ such that $\bigcup_j[V+\hat{n}^j+\hat{n}_0]$ covers whole $\widehat{N}$. And $\sum_j\oplus\mathfrak{D}(\hat{n}^j)$ contains a subrepresentation

$$(7.37)\qquad \int_{\widehat{N}}\operatorname*{Ind}_{N\to G}\langle\hat{n},n\rangle d\mu_{\widehat{N}}(\hat{n}),$$

which is equivalent to R, by the reason of lemma 7.3. This shows that, $\{\mathfrak{D}(\hat{n}_0,\tau_0^l),\mathfrak{D}(\hat{n}_j,1),\mathfrak{D}(\hat{n}^k,1)\}_{(l,j,k)}$ gives a countable family of generators of R.

This completes the proof.

5. Examples.

a) The motion group over m-dimensional Euclidean space.

$$(7.38)\qquad G=\left\{ g=\left.\left(\begin{array}{c|c} k_m & \mathfrak{n} \\ \hline 0,\ \cdots,\ 0 & 1 \end{array}\right)\right\} m+1 \right\}.$$

That is, G is the semi-direct product of closed normal subgroup

$$(7.39)\qquad N=\left\{ n=\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \mathfrak{n} \\ & & 1 & \\ \hline 0, & \cdots, & 0 & 1 \end{array}\right)\right\}$$

which is isomorphic to the additive group $\boldsymbol{R}^m$ with vectors $\mathfrak{n}$, and closed subgroup

$$(7.40)\qquad K=\left\{ k=\left(\begin{array}{c|c} & 0 \\ k_m & \vdots \\ & 0 \\ \hline 0,\ \cdots,\ 0 & 1 \end{array}\right)\right\}.$$

This group is isomorphic to the group of orthogonal matrices of degree m which is consisted of $\{k_m\}$.

The dual group $\widehat{N}$ of N is isomorphic to the additive group $\boldsymbol{R}^m$, and an element g in G operates on $\widehat{N}$ as

$$(7.41)\qquad \hat{n}\rightarrow k_m^{-1}(\hat{n}),$$

as an orthogonal matrix on the vector space $\boldsymbol{R}^m$. Then the G-orbits in $\widehat{N}$ are $(m-1)$-dimensional spheres $S(r)$ with the radius $r(0\leq r<\infty)$ and with the centre on origin. One can select a representative of orbit $S(r)$ as

$$(7.42)\qquad \hat{n}_r\equiv(r,0,\cdots,0),$$

then the isotropy group $K(r)$ of $\hat{n}_r$ in K is given by

$$(7.43)\qquad \begin{cases} \text{i)}\quad K(0)=K, \\ \text{ii)}\quad K(r)=K_0\equiv\left\{ k=\begin{pmatrix} 1, & 0, & \cdots, & 0, & 0 \\ 0, & & & & 0 \\ \vdots & & k_{m-1} & & \vdots \\ 0, & & & & 0 \\ 0, & 0, & \cdots, & 0, & 1 \end{pmatrix}\right\}. \end{cases}$$

K_0 is isomorphic to the group of orthogonal matrices of degree $m-1$

which is consisted of $\{k_{m-1}\}$.

From the compactness of K, the assumptions 1 and 2 are easily verified, therefore, if we show that the assumption 3 is fulfilled, then the duality theorem of G is obtained.

To show the assumption 3, put a pair $F=\{\hat{n}_1, \hat{n}_2\}=\{(1, 0, \cdots, 0), (-1/2, \sqrt{3}/2, 0, \cdots, 0)\}$ as the set $\{n_j\}$ $(t=2)$. We shall show the existence of a neighborhood $W_1\times W_2$ of $(\hat{n}_1, \hat{n}_2)$ in $S(1)\times S(1)$ which is homeomorphic to an open set of $(K(\hat{n}_1)\backslash K)\times(K(\hat{n}_2)\backslash K)$, and on which the measure ν_j are given as ordinary Lebesgue measure. That is, $W_1\times W_2$ satisfies the conditions of the assumption 3.

For this, let the coordinates of $\hat{n}_1'$ and $\hat{n}_2'$ on $S(1)$ be

$$\left\{\begin{array}{l}\hat{n}_1'=(x_1+1, x_2, \cdots, x_m),\\ \hat{n}_2'=(y_1-(1/2), y_2+(\sqrt{3}/2), y_3, \cdots, y_m),\end{array}\right. \tag{7.44}$$

then

$$\left\{\begin{array}{l}(x_1+1)^2+x_2^2+\cdots+x_m^2=1,\\ (y_1-(1/2))^2+(y_2+(\sqrt{3}/2))^2+y_3^2+\cdots+y_m^2=1.\end{array}\right. \tag{7.45}$$

A neighborhood $W_1(\varepsilon)\times W_2(\varepsilon)$ of $(\hat{n}_1, \hat{n}_2)$ is given by

$$|x_j|<\varepsilon, \text{ and } \quad |y_j|<\varepsilon \quad (1\leq j\leq m). \tag{7.46}$$

By the map $\varphi_F; (\hat{n}_1', \hat{n}_2')\rightarrow\hat{n}_1'+\hat{n}_2'$, the correspondence of coordinates are given by

$$\begin{array}{c}z_1=x_1+y_1+(1/2), \quad z_2=x_2+y_2+(\sqrt{3}/2),\\ z_j=x_j+y_j, \qquad (3\leq j\leq m).\end{array} \tag{7.47}$$

For fixed $x'=(x_3, \cdots, x_m)$, take the parametres $(x_2, y_2, \cdots, y_m)$ then for sufficiently near $(\hat{n}_1', \hat{n}_2')$ of $(\hat{n}_1, \hat{n}_2)$

$$\begin{array}{l}z_1=\sqrt{1-(x_2^2+\cdots+x_m^2)}\\ \qquad +\sqrt{1-\{(y_2+(\sqrt{3}/2))^2+y_3^2+\cdots+y_m^2\}}.\end{array} \tag{7.48}$$

Therefore,

$$\frac{\partial(z_1, \cdots, z_m)}{\partial(x_2, y_2, \cdots, y_m)} \tag{7.49}$$

$$=\begin{vmatrix} \frac{-x_2}{x_1+1}, & -\frac{y_2+(\sqrt{3}/2)}{y_1-(1/2)}, & \frac{-y_3}{y_1-(1/2)}, & \cdots, & \frac{-y_m}{y_1-(1/2)} \\ 1, & 1, & & & \\ 0, & & 1, & & \\ \vdots & & & \ddots & \\ 0, & & & & 1, \end{vmatrix}$$

$$=(-x_2)(x_1+1)^{-1}+(y_2+(\sqrt{3}/2))(y_1-(1/2))^{-1}.$$

This value is non-zero for instance for $(\hat{n}_1', \hat{n}_2')$ such that

$$\text{(7.50)} \qquad \mathrm{Max}(|x_1|, |x_2|, |y_1|, |y_2|)<1/3,$$

therefore on $W_1(1/3)\times W_2(1/3)$ too. Take this neighborhood of $(\hat{n}_1', \hat{n}_2')$ as the W of the assumption 3. Such a W satisfies the properties i), ii) of the assumption 3. In fact, let

$$\text{(7.51)} \qquad W_1'\equiv\{x'=(x_3, \cdots, x_m): (x_1, \cdots, x_m)\in W_1(1/3)\},$$

$$\text{(7.52)} \qquad W(x')\equiv\{x_2: (x_1, \cdots, x_m)\in W_1(1/3)\}, \qquad \text{for} \quad x'\in W_1'.$$

Then the map

$$\text{(7.53)} \qquad \varphi_{x'}': (x_2, y_2, \cdots, y_m)\rightarrow(z_1, z_2, \cdots, z_m),$$

is a regular map of $W(x')\times W_2(1/3)$ into $\hat{N}$ for any x' in W_1', because (7.49) does not take zero. But obviously this map $\varphi_{x'}'$ gives the restriction of φ_F of $W_1(1/3)\times W_2(1/3)$ onto a closed subspace $(W(x'), x')\times W_2(1/3)$ in $S(1)\times S(1)$. So that the image $\varphi_F(W_1(1/3)\times W_2(1/3))$ is open as a join of open sets $\{\varphi_{x'}'(W(x')\times W_2(1/3))\}_{x'}$ which are images of regular maps. This is the property i).

Moreover the regularity of $\varphi_{x'}'$ shows that the ordinary Lebesgue measure on $W(x')\times W_2(1/3)$ has same null set by the correspondence $\varphi_{x'}'$, as the measure $dz_1 dz_2\cdots dz_m$ on $\hat{N}$. The ordinary Lebesgue measure over $W_1(1/3)\times W_2(1/3)$ as an open subspace in $S(1)\times S(1)$, is decomposed to an integral over the space W_1', components of which are the Lebesgue measure on $(W(x'),x')\times W_2(1/3)$.

By the same reason as to prove the existence of k_{t+1} satisfying (7.30), for any given measurable E in $\hat{N}$ which satisfies $\mu_{\hat{N}}(E\cap$

$\varphi_F(W_1(1/3)\times W_2(1/3))\neq 0$, one can find x' such that $(\varphi'_{x'})^{-1}(E\cap \varphi'_{x'}(W(x')\times W_2(1/3)))$ has positive measure, and its value becomes a continuous function of x'. Consequently, for such a set E, $(\varphi_F)^{-1}(E\cap\varphi_F(W_1(1/3)\times W_2(1/3)))$ has positive measure on $S(1)\times S(1)$, value of which is the integral of continuous non-zero non-negative function of measure of $(\varphi'_{x'})^{-1}(E\cap\varphi'_{x'}(W(x')\times W_2(1/3))$ over W'_1. This proves the property ii).

b) m-dimensional proper inhomogeneous Lorentz group. For this group, the circumstance is analogous to the case of a), except the group K is isomorphic to the m-dimensional proper Lorentz group L_m instead of the orthogonal group.

$$G=\left\{ g=\left(\begin{array}{c|c} l_m & \mathfrak{n} \\ \hline 0,\ \cdots,\ 0, & 1 \end{array}\right) m+1 \right\}, \tag{7.54}$$

$$N=\left\{ n=\left(\begin{array}{ccc|c} 1, & & & \\ & \ddots & & \mathfrak{n} \\ & & 1, & \\ \hline 0, & \cdots, & 0, & 1 \end{array}\right) \right\}, \tag{7.55}$$

$$K=\left\{ l=\left(\begin{array}{c|c} l_m & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0,\cdots,\ 0, & 1 \end{array}\right) \right\}, \tag{7.56}$$

where l_m is a real matrix of degree m which makes invariant the quadratic form on $\boldsymbol{R}^m$,

$$q(\mathfrak{n})=x_0^2-(x_1^2+x_2^2+\cdots+x_{m-1}^2), \tag{7.57}$$

where $\mathfrak{n}\equiv(x_0, x_1, \cdots, x_{m-1})$, and in the connected component of e in this matrices group.

Thus G operates on $\widehat{N}\sim\boldsymbol{R}^m$ as hyperbolic rotations, and $\widehat{N}$ is separated to disjoint sum of the following G-orbitwise measurable sets.

$$(7.58)\quad \begin{cases} \text{i)} & \text{The origin } \{0\}, \\ \text{ii)} & \text{the light cone } \{\hat{n}: q(\hat{n})=0, \hat{n}\neq 0\}, \\ \text{iii)} & \text{the set of time like vectors } \{\hat{n}: q(\hat{n})<0\}, \\ \text{iv)} & \text{a) the set of positive space like vectors} \\ & \qquad \{\hat{n}: x_0>0, q(\hat{n})>0\}, \\ & \text{b) the set of negative space like vectors} \\ & \qquad \{\hat{n}: x_0<0, q(\hat{n})>0\}. \end{cases}$$

From the locally compactness of each orbits, using the results of J. Glimm [8], the assumption 1 is verified. As the vector $\hat{n}_0$ given in the assumption 2, one can select any space like vector. As wellknown, for such a G-orbit one can choose the representative of the form $\hat{n}_0=(r, 0, \cdots, 0)$, and the isotropy group of this representative in K is

$$(7.59)\qquad K(\hat{n}_0)=K_0\equiv\left\{ k=\begin{pmatrix} 1, & 0, & \cdots, & 0, & 0 \\ 0, & & & & 0 \\ \vdots & & k_{m-1} & & \vdots \\ 0, & & & & 0 \\ 0, & 0, & \cdots, & 0, & 1 \end{pmatrix}\right\},$$

where k_{m-1} is any orthogonal matrix of degree $m-1$, so $K(\hat{n}_0)$ is compact. This shows the assumption 2.

To prove the assumption 3, let for instance $\{\hat{n}_1, \hat{n}_2\}=\{(1, 0, \cdots, 0), (\sqrt{3}, \sqrt{2}, 0, \cdots, 0)\}$ as the set $\{\hat{n}_j\}$ $(t=2)$. On the G-orbit $H(1)$ passing through the both of $\hat{n}_1$, $\hat{n}_2$, we introduce coordinates,

$$(7.60)\qquad \begin{cases} \hat{n}_1'=(x_0+1, x_1, \cdots, x_{m-1}), \\ \hat{n}_2'=(y_0+\sqrt{3}, y_1+\sqrt{2}, y_2, \cdots, y_{m-1}), \end{cases}$$

where

$$(7.61)\qquad \begin{cases} (x_0+1)^2-(x_1^2+\cdots+x_{m-1}^2)=1, \\ (y_0+\sqrt{3})^2-\{(y_1+\sqrt{2})^2+y_2^2+\cdots+y_{m-1}^2\}=1. \end{cases}$$

A neighborhood $W_1(\varepsilon)\times W_2(\varepsilon)$ of $(\hat{n}_1, \hat{n}_2)$ is given by

$$(7.62)\qquad |x_j|<\varepsilon, \quad \text{and} \quad |y_i|<\varepsilon \ (1\leq j\leq m).$$

Put the coordinate of $\hat{n}_1'+\hat{n}_2'$, as $(z_0, z_1, \cdots, z_{m-1})$, and for fixed $x'\equiv(x_2, x_3, \cdots, x_{m-1})$, using the parameters $(x_1, y_1, \cdots, y_{m-1})$, we get,

(7.63)
$$\frac{\partial(z_0, z_1, \cdots, z_{m-1})}{\partial(x_1, y_1, \cdots, y_{m-1})}$$
$$= \begin{vmatrix} \frac{x_1}{x_0+1}, & \frac{y_1+\sqrt{2}}{y_0+\sqrt{3}}, & \frac{y_2}{y_0+\sqrt{3}}, & \cdots, & \frac{y_{m-1}}{y_0+\sqrt{3}} \\ 1, & 1, & & & \\ 0, & & \ddots & & 0 \\ \vdots & 0 & & \ddots & \\ 0, & & & & 1 \end{vmatrix}$$
$$= x_1(x_0+1)^{-1} - (y_1+\sqrt{2})(y_0+\sqrt{3})^{-1},$$

because of

(7.64)
$$z_0 = x_0 + y_0 + 1 + \sqrt{3}, \quad z_1 = x_1 + y_1 + \sqrt{2},$$
$$z_j = x_j + y_j, \quad (2 \leq j \leq m-1).$$

And the Jacobian (7.63) does not take zero for instance on $W_1(1/4) \times W_2(1/4)$.

The same arguments as in a) show that $W' = W_1(1/4) \times W_2(1/4)$ is the neighborhood required in the assumption 3.

Remark. In the previous paper [21] IV, the author used the different method to prove the duality theorem for some semi-direct product groups, for which the author did not set the measure theoretical assumption as stated in the assumption 3 of this paper. But the proof mentioned in the previous paper is not complete, and it seems us the measure theoretical assumption is necessary. Therefore the results of the previous paper are completely contained in the results of this paper.

The author does not know whether this assumption is necessary or not.

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