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Generalized bilinear relations on open Riemann surfaces[1)]

By

Michio YOSHIDA

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Introduction

Let W be a Riemann surface of infinite genus. A connected subregion Ω of W will be called *normal* if it has positive finite genus and its relative boundary consists of a finite number of mutually disjoint dividing analytic Jordan curves. We stress that a normal subregion is not necessarily a relatively compact region. It may have, besides the relative boundary, the ideal boundary.

In §§2, 3, we introduce two intrinsic conformal invariants $\mu(\bar{\Omega})$ and $M(\Omega)$ for a normal subregion Ω.

Suppose that W is decomposed into a sequence $\{\Omega_k\}_{k=1}^{\infty}$ of normal subregions. Consider a canonical homology basis $\{A_j^k, B_j^k\}_{j=1}^{g_k}$ of Ω_k modulo dividing cycles. Set $D_n=\bigcup_{k=1}^{n}\Omega_k$. The main purpose of this paper is to establish the following evaluations which lead to a generalized bilinear relation:

Suppose $\mu(\bar{\Omega}_k)\leqq\mu$ (*resp.* $M(\Omega_k)\leqq M$), $k=1,2,\cdots$. *Then, we have*

$$|(\sigma, \omega^*)-\sum_{k=1}^{n}\sum_{j=1}^{g_k}\left(\int_{A_j^k}\sigma\int_{B_j^k}\bar{\omega}-\int_{B_j^k}\sigma\int_{A_j^k}\bar{\omega}\right)|$$
$$\leqq(1+\mu)\|\sigma\|_{W-D_n}\|\omega\|_{W-D_n}\ (\textit{resp. } 2M\|\sigma\|_{W-D_n}\|\omega\|_{W-D_n})$$

for all $\sigma, \omega^*\in\Gamma_{h0}(W)\cap\Gamma_{hse}^*(W)$.[2)]

1) This is a revised version of the contents of the talk given at a seminar held at Kyoto University on March 14, 1966.

2) In this paper, differentials are complex in general.

Most of the ideas in this work have their origins in R. Accola [1], [3], M. Mori [7] and K. Matsui [6]. In this note, as for the notation and terminology, we follow L. Ahlfors and L. Sario [4].

§1. Preliminaries

Periods reproducing differentials. We shall recall some of the basic properties of period reproducing differentials. A detailed explanation is found in M. Yoshida [13].

Let W be an arbitrary Riemann surface, and c be a finite 1-cycle on W. Then $\omega\rightarrow\int_c\omega$ $(\omega\in\Gamma_{hse}(W))$ is a continuous linear functional on $\Gamma_{hse}(W)$. By an elementary theorem in Hilbert space theory, there exists a unique $\tilde{\sigma}(c)^*\in\Gamma_{hse}(W)$ such that

$$\int_c\omega=(\omega, \tilde{\sigma}(c)^*) \qquad \text{for all } \omega\in\Gamma_{hse}(W).$$

When it is needed to indicate the basic surface W, $\tilde{\sigma}(c)$ is written as $\tilde{\sigma}_W(c)$. It is called the Γ_{hse}-*period reproducing differential for* c. It follows immediately from the definition that

$$\text{(i)} \qquad \int_c\tilde{\sigma}(c)^*=\|\tilde{\sigma}(c)\|^2.$$

The following properties of $\tilde{\sigma}(c)$ are well-known:

(ii) $\tilde{\sigma}(c)$ is a real distinguished differential in Ahlfors' sense.

(iii) $(\tilde{\sigma}(c), \tilde{\sigma}(d)^*)=c\times d$, i.e., $\int_d\tilde{\sigma}(c)=c\times d$ for any cycle d.

(iv) If c runs through all non-dividing cycles, then $\tilde{\sigma}(c)$ span $\Gamma_{h0}(W)\cap\Gamma_{hse}^*(W)$.

The following interesting fact was found by B. Rodin [6][3)]

$$\text{(v)} \qquad \|\tilde{\sigma}(c)\|^2=\lambda(\check{c})$$

where $\check{c}$ denotes the family of cycles which are homologous to o modulo dividing cycles, and $\lambda(\check{c})$ denotes the extremal length of $\check{c}$.

3) It is R. Accola [2] who first gave an extremal length interpretation to the norm of a Γ_h-period reproducing differential.

Approximation theorems. We shall state two lemmas which will be needed later.

Lemma 1. *Let Ω denote any regularly imbedded relatively compact subregion of W which contains c and each of whose contours is a dividing cycle of W. Then*

$$\|\tilde{\sigma}_{\Omega}(c)-\tilde{\sigma}_{W}(c)\|_{\Omega}\to 0 \quad \text{as } \Omega\uparrow W.$$

Proof. Let Ω' be another such region containing Ω. Then, the restriction of $\tilde{\sigma}_{\Omega'}(c)^*$ to Ω is semi-exact. Hence, by making use of (i), we have

$$(\tilde{\sigma}_{\Omega'}(c)^*,\ \tilde{\sigma}_{\Omega}(c)^*)_{\Omega}=\int_c \tilde{\sigma}_{\Omega'}(c)^*=\|\tilde{\sigma}_{\Omega'}(c)\|_{\Omega'}^2.$$

Hence

$$\begin{aligned}\|\tilde{\sigma}_{\Omega}(c)-\tilde{\sigma}_{\Omega'}(c)\|_{\Omega}^2&=\|\tilde{\sigma}_{\Omega}(c)\|_{\Omega}^2-2\|\tilde{\sigma}_{\Omega'}(c)\|_{\Omega'}^2+\|\tilde{\sigma}_{\Omega'}(c)\|_{\Omega}^2\\&\leqq\|\tilde{\sigma}_{\Omega}(c)\|_{\Omega}^2-\|\tilde{\sigma}_{\Omega'}(c)\|_{\Omega'}^2.\end{aligned}$$

Therefore $\|\tilde{\sigma}_{\Omega}(c)\|_{\Omega}$ decreases as $\Omega\uparrow W$. Hence, the above inequality implies

$$\|\tilde{\sigma}_{\Omega}(c)-\tilde{\sigma}_{\Omega'}(c)\|_{\Omega}\to 0 \quad \text{as } \Omega\uparrow W.$$

We conclude that $\tilde{\sigma}_{\Omega}(c)$ tends to a differential of $\Gamma_{hse}^*(W)$ which we denote, temporarily, by σ. Then it follows that

$$\|\tilde{\sigma}_{\Omega}(c)-\sigma\|\to 0 \quad \text{as } \Omega\uparrow W.$$

For any $\omega\in\Gamma_{hse}(W)$ we have

$$(\omega,\sigma^*)=\lim_{\Omega\uparrow W}(\omega,\sigma^*)_{\Omega}=\lim_{\Omega\uparrow W}(\omega,\tilde{\sigma}_{\Omega}(c)^*)_{\Omega}=\int_c\omega.$$

This shows that $\sigma=\tilde{\sigma}_W(c)$.

Consider the exterior V of a regularly imbedded relatively compact subregion of W. Let $C^{\omega}(\partial V)$ be the space of real analytic functions on ∂V, and $HD(\bar{V})$ the space of Dirichlet finite real harmonic functions on $\bar{V}$.

Sario's principal operator $L_{1,V}$ is a linear mapping of $C^{\omega}(\partial V)$ into $HD(\bar{V})$, which is characterized by the following properties (cf. [13]):

1) $L_{1,V}f=f$ on ∂V for all $f\in C^{\omega}(\partial V)$,

2) $(dL_{1,V}f, \omega^*)_V=-\int_{\partial V}f\bar{\omega}$ for any $\omega\in\Gamma_{hse}(W)$,

3) $\int_{\gamma}(dL_{1,V}f)^*=0$ for any dividing cycle γ of W contained in V.

If we adopt this characterization as the definition of L_1 as in [13], we need to give a proof of the following fact.[4)]

Lemma 2. *Let Ω be a regularly imbedded relatively compact subregion of W such that $\Omega\supset W-V$, and let $f\in C^{\omega}(\partial V)$. Then*

$$\|d(L_{1,V}f-L_{1,V\cap\Omega}f)\|_{V\cap\Omega}\to 0 \quad \textit{as } \Omega\uparrow W.$$

Proof. For the brevity of notation, put $L_{1,V\cap\Omega}f=u_{\Omega}$. Let Ω' be another regularly imbedded relatively compact subregion containing Ω. Then we have

$$(du_{\Omega}, du_{\Omega'})_{V\cap\Omega}=\int_{\partial V}u_{\Omega}(du_{\Omega'})^*=\|du_{\Omega'}\|^2_{V\cap\Omega'}$$

and

$$\begin{aligned}\|d(u_{\Omega}-u_{\Omega'})\|^2_{V\cap\Omega}&=\|du_{\Omega}\|^2_{V\cap\Omega}-2\|du_{\Omega'}\|^2_{V\cap\Omega'}+\|du_{\Omega'}\|^2_{V\cap\Omega}\\&\leqq\|du_{\Omega}\|^2_{V\cap\Omega}-\|du_{\Omega'}\|^2_{V\cap\Omega'}.\end{aligned}$$

We can infer, by a customary reasoning, that u_{Ω} converges to a harmonic function locally uniformly and in norm on $\bar{V}$. It is evident that this limit function satisfies the above conditions 1)−3) which characterize $L_{1,V}f$. We thus conclude that $L_{1,V\cap\Omega}f$ converges to $L_{1,V}f$ in norm.

Generalized bilinear relation. Now suppose W has infinite genus and let $\{A_j, B_j\}_{j=1}^{\infty}$ be a canonical homology basis of W modulo dividing cycles. $\{A_j, B_j\}$ has the following intersection property:

$$A_j\times A_k=B_j\times B_k=0 \quad\text{and}\quad A_j\times B_k=\delta_{jk} \quad\text{for all } j, k.$$

Let $\{p(n)\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers.

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4) Both in L. Ahlfors and L. Sario [4] and B. Rodin and L. Sario [9], the convergence of $L_{1,V\cap\Omega}f$ is first proved and then $L_{1,V}f$ is defined as the limit of $L_{1,V\cap\Omega}f$.

We will say that the generalized bilinear relation holds with respect to $\{A_j, B_j\}$ and $\{p(n)\}$ if we have

$$(1) \qquad (\sigma, \omega^*) = \lim_{n\to\infty} \sum_{j=1}^{p(n)} \left(\int_{A_j} \sigma \int_{B_j} \bar{\omega} - \int_{B_j} \sigma \int_{A_j} \bar{\omega} \right)$$

for all $\sigma \in \Gamma_{h0}(W)$ and $\omega \in \Gamma_{hse}(W)$.

Theorem of R. Accola and M. Mori. In the rest of this section we shall write Γ_{h0}, Γ_{hm}, etc. for $\Gamma_{h0}(W)$, $\Gamma_{hm}(W)$, etc. We use the orthogonal decompositions

$$\Gamma_{h0} = \Gamma_{hm} + (\Gamma_{h0} \cap \Gamma^*_{hse}),$$

$$\Gamma_{hse} = \Gamma_{he} + (\Gamma^*_{h0} \cap \Gamma_{hse})$$

to obtain

$$\sigma = \sigma_1 + \sigma_2, \quad \text{where } \sigma_1 \in \Gamma_{hm} \text{ and } \sigma_2 \in \Gamma_{h0} \cap \Gamma^*_{hse},$$

$$\omega = \omega_1 + \omega_2, \quad \text{where } \omega_1 \in \Gamma_{he} \text{ and } \omega_2 \in \Gamma^*_{h0} \cap \Gamma_{hse}.$$

Then, $(\sigma, \omega^*) = (\sigma_2, \omega_2^*)$ because of the orthogonalities: $\Gamma_{hm} \perp \Gamma^*_{hse}$ and $\Gamma_{h0} \perp \Gamma^*_{he}$. On the other hand, since σ_1 and ω_1 are exact, σ_2 and ω_2 have the same periods as σ and ω respectively. We have thus shown that (1) holds for all $\sigma \in \Gamma_{h0}$ and $\omega \in \Gamma_{hse}$ if and only if it holds for all $\sigma, \omega^* \in \Gamma_{h0} \cap \Gamma^*_{hse}$. The idea of this reduction process is due to M. Mori [7], p. 93.

Now, after K. Virtanen [12], R. Accola [1] and M. Mori [7], we introduce linear operators T_n on $\Gamma_{h0} \cap \Gamma^*_{hse}$ as follows:

$$T_n \sigma = \sum_{j=1}^{p(n)} \left\{ -\left(\int_{A_j} \sigma \right) \tilde{\sigma}(B_j) + \left(\int_{B_j} \sigma \right) \tilde{\sigma}(A_j) \right\} \quad \text{for } \sigma \in \Gamma_{h0} \cap \Gamma^*_{hse}.$$

This is the distinguished differential characterized by the property that it has the same periods as σ along $A_j, B_j, 1 \leq j \leq p(n)$, and has vanishing periods along $A_j, B_j, j > p(n)$. Since

$$(T_n \sigma, \omega^*) = \sum_{j=1}^{p(n)} \left(\int_{A_j} \sigma \int_{B_j} \bar{\omega} - \int_{B_j} \sigma \int_{A_j} \bar{\omega} \right) \quad \text{for all } \omega^* \in \Gamma_{h0} \cap \Gamma^*_{hse}$$

the generalized bilinear relation (1) holds for all $\sigma, \omega^* \in \Gamma_{h0} \cap \Gamma^*_{hse}$ if and only if $T_n \sigma$ converges to σ weakly for all $\sigma \in \Gamma_{h0} \cap \Gamma^*_{hse}$.

If $T_n\sigma$ converges weakly for a fixed σ, $\|T_n\sigma\|$ is bounded in virtue of the principle of uniform boundedness; see, for example, [10], p. 174, Problems 25. a. If $\|T_n\sigma\|$ is bounded for every $\sigma\in\Gamma_{h0}\cap\Gamma^*_{hse}$ (the bound may depend on σ), the same principle implies the boundedness of $\|T_n\|$; see [10], p. 171, Proposition. The validity of (1) thus implies the boundedness of $\|T_n\|$.

Conversely suppose $\|T_n\|\leq M$. As stated in (iv), $\{\tilde{\sigma}(c)\}$ span $\Gamma_{h0}\cap\Gamma^*_{hse}$. Hence given $\sigma\in\Gamma_{h0}\cap\Gamma^*_{hse}$ and $\varepsilon>0$, there exists a finite linear combination, say $\tilde{\sigma}_1$, of $\tilde{\sigma}(A_j)$ and $\tilde{\sigma}(B_j)$ such that $\|\sigma-\tilde{\sigma}_1\|<\varepsilon$. For a sufficiently large n, we have $T_n\tilde{\sigma}_1=\tilde{\sigma}_1$ and hence

$$\|\sigma-T_n\sigma\|\leq\|\sigma-\tilde{\sigma}_1\|+\|T_n(\sigma-\tilde{\sigma}_1)\|<(1+M)\varepsilon.$$

Therefore, $T_n\sigma$ converges to σ strongly. Consequently, the boundedness of $\|T_n\|$ implies the validity of (1). This completes a proof of the following theorem.

Theorem 1. (R. Accola [1], M. Mori [7]) *If $T_n\sigma$ converges to σ weakly for all $\sigma\in\Gamma_{h0}\cap\Gamma^*_{hse}$, $\|T_n\|$ is bounded. Conversely if $\|T_n\|$ is bounded, $T_n\sigma$ converges to σ strongly for all $\sigma\in\Gamma_{h0}\cap\Gamma^*_{hse}$. Thus the generalized bilinear relaiton* (1) *holds with respect to $\{A_j, B_j\}$ and $\{p(n)\}$ if and only if the norms $\|T_n\|$ of the linear operators T_n associated with $\{A_j, B_j\}$ and $\{p(n)\}$ are bounded.*

If the generalized bilinear relation (1) holds, we have $(\Gamma_{h0}\cap\Gamma_{he})\perp\Gamma^*_{hse}$ and hence $\Gamma_{h0}\cap\Gamma_{he}\subset\Gamma_{hm}$. On the other hand, it holds always that $\Gamma_{h0}\cap\Gamma_{he}\supset\Gamma_{hm}$. Thus the validity of (1) implies $\Gamma_{h0}\cap\Gamma_{he}=\Gamma_{hm}$.

However, R. Accola [1] showed that the relation $\Gamma_{h0}\cap\Gamma_{he}=\Gamma_{hm}$ is not always true. Therefore there exists a surface on which the generalized bilinear relation does not hold with respect to any $\{A_j, B_j\}$ and $\{p(n)\}$. In the following sections we shall introduce two conformal invariants of a normal subregion Ω of W in order to establish criteria which ensure the validity of (1).

§2. Conformal invariant $\mu(\overline{\Omega})$

Let Ω be a normal subregion of W. Let $\{A_j, B_j\}_{j=1}^g$ be a canonical homology basis of Ω modulo dividing cycles. We define a continuous linear operator $T_{\Omega,W}$ on $\Gamma_{h0}(W)\cap\Gamma_{hse}^*(W)$ by setting

$$T_{\Omega,W}\,\sigma=\sum_{j=1}^{g}\left\{-\left(\int_{A_j}\sigma\right)\tilde{\sigma}_W(B_j)+\left(\int_{B_j}\sigma\right)\tilde{\sigma}_W(A_j)\right\}$$

for $\sigma\in\Gamma_{h0}(W)\cap\Gamma_{hse}^*(W)$. We shall use a shorter notation T_Ω for $T_{\Omega,W}$ in case this abbreviation causes no ambiguity.

Observe that $T_\Omega\,\sigma$ is the regular distinguished differential which is characterized by the property that $\sigma-T_\Omega\,\sigma$ is exact in Ω and $T_\Omega\,\sigma$ is exact in $W-\Omega$. Hence, T_Ω is defined independently of the choice of $\{A_j, B_j\}$. Furthermore $T_\Omega=T_{\Omega'}$ if Ω and Ω' share the same canonical homology basis. In particular, if $\Omega'\subset\Omega$ and if Ω' has the same genus as Ω, then $T_\Omega=T_{\Omega'}$. We note

$$(T_\Omega\,\sigma,\,\omega^*)_W=\sum_{j=1}^{g}\left(\int_{A_j}\sigma\int_{B_j}\bar{\omega}-\int_{B_j}\sigma\int_{A_j}\bar{\omega}\right)\quad\text{for all }\omega\in\Gamma_{hse}.$$

We shall prove

Lemma 3. *Let $\sigma\in\Gamma_{h0}(W)$ and $\omega\in\Gamma_{hse}^*(W)$. Let V be a regularly imbedded planar subregion of W such that ∂V is compact. Let u be a harmonic function in a neighborhood of $\overline{V}$ such that $du=\sigma$.*[5] *Then*

$$(\sigma,\,\omega)_V=\int_{\partial V}u\bar{\omega}^*.$$

Proof. Let D be a regularly imbedded relatively compact subregion of W such that $D\supset\partial V$. Denote by σ_D the projection of the restriction $\sigma|D\in\Gamma_h(D)$ on the subspace $\Gamma_{h0}(D)$. Then it is well-known (see pp. 292–293 of [4]) that

$$\|\sigma-\sigma_D\|_D\rightarrow 0\quad\text{as } D\uparrow W.$$

Let u_D be the harmonic function on $\overline{V}\cap D$ such that $du_D=\sigma_D$ and

5) This is possible because $\Gamma_{h0}(W)\subset\Gamma_{hse}(W)$.

$u_D = u$ at a fixed point of ∂V. It follows that u_D converges to u uniformly on ∂V. Since $\sigma_D \in \Gamma_{h0}(D)$, u_D is constant on each contour of D contained in V. Hence,

$$(\sigma_D, \omega)_{V\cap D} = \int_{\partial V} u_D \bar{\omega}^*.$$

By letting $D \uparrow W$, we obtain the desired equality.

Lemma 4. *Let Ω be a normal subregion of W. Suppose that σ, $\omega^* \in \Gamma_{h0}(W) \cap \Gamma^*_{hse}(W)$. Then,*

$$(\sigma, \omega^*)_\Omega = (T_\Omega\, \sigma, \omega^*)_W - \int_{\partial\Omega} u\bar{\omega},$$

where the integration $\int_{\partial\Omega}$ is taken in the positive sense along $\partial\Omega$ and u is a harmonic function defined separately on each contour of $\partial\Omega$ such that $du = \sigma$.

Proof. Let D be a relatively compact region of W such that $\partial\Omega \subset D$ and $\Omega - D$ consists of a finite number of regions of the character described in Lemma 3. Denote by u a harmonic function on $\Omega - D$ such that $du = \sigma$. By Green's formula we have

$$(\sigma, \omega^*)_{\Omega\cap D} = (T_\Omega\, \sigma, \omega^*)_W - \int_{\partial\Omega} u\bar{\omega} + \int_{\partial D\cap\Omega} u\bar{\omega}.$$

Lemma 3 yields

$$(\sigma, \omega^*)_{\Omega - D} = -\int_{\partial D\cap\Omega} u\bar{\omega}$$

and our lemma is proved.

Next, after R. Accola [1] and [3], we introduce a conformal invariant for a family of subregions. Let R_i, $1 \leqq i \leqq k$, be regularly imbedded subregions of finite genus of W with compact relative boundaries ∂R_i such that the number of the components of ∂R_i is greater than one for each i. Suppose $\bar{R}_i$ are mutually disjoint. Let $\partial R_i = \alpha_i \cup \beta_i$ be a partition of ∂R_i into two disjoint non-empty cosets of contours. Let u_i be the harmonic function on $\bar{R}_i$ such that $u_i = 0$ on α_i, $u_i = 1$ on β_i and u_i has L_1-behavior along the ideal boundary of R_i. Denote the union of R_i by R. Setting $\alpha = \cup \alpha_i$, $\beta = \cup \beta_i$, define

$$\mu_R(\alpha, \beta) = \max_{1 \leq i \leq k} \|du_i\|^2.$$

We define a harmonic function u on R by $u = u_i$ on R_i and state

Lemma 5. *Suppose that the number of the components of the ideal boundary of R is finite. Let v and w be square integrable harmonic functions on $\overline{R}$. Then*

$$\left| \int_{u=t} v dw \right| \leq \mu_R(\alpha, \beta) \|dv\|_R \|dw\|_R$$

for some $t \in [0, 1]$.

Proof. Regard each R_i as the interior of a compact bordered surface. In other words, we may suppose that each component of the ideal boundary of R_i is realized as an analytic Jordan curve or as a point. Consequently we may assume that u_i is constant and has zero flux along each ideal contour of R_i, and that u_i is harmonic at each point-like component. The rest of the proof is similar to that of the lemma of R. Accola [3].

We now return to a normal subregion Ω of W and define an intrinsic conformal invariant $\mu(\overline{\Omega})$ of $\overline{\Omega}$ in the following manner. Consider a normal subregion Ω_0 of W such that $\overline{\Omega}_0 \subset \Omega$, that Ω_0 has the same genus as Ω and that the relative boundary of each component of $\Omega - \overline{\Omega}_0$ contains points of both $\partial\Omega$ and $\partial\Omega_0$. Such a subregion will be called admissible. We define

$$\mu(\overline{\Omega}) = \inf_{\Omega_0} \mu_{\Omega - \overline{\Omega}_0}(\partial\Omega_0, \partial\Omega)$$

where Ω_0 runs through admissible normal subregions of W.

With these definition and notation, we state

Theorem 2. *Let* $\sigma, \omega^* \in \Gamma_{h0}(W) \cap \Gamma^*_{hse}(W)$. *Then*

$$|(T_\Omega \sigma, \omega^*)_W| \leq \{1 + \mu(\overline{\Omega})\} \|\sigma\|_\Omega \|\omega\|_\Omega \tag{2}$$

and

$$\|T_\Omega\| \leq 1 + \mu(\overline{\Omega}). \tag{3}$$

Proof. We first treat the case where the number of the components of the ideal boundary of Ω is finite. For any $\varepsilon>0$ there exists a normal subregion $\Omega_\varepsilon\subset\Omega$ such that Ω_ε has the same genus as Ω and that $\mu_{\Omega-\bar{\Omega}_\varepsilon}(\partial\Omega_\varepsilon, \partial\Omega)<\mu(\bar{\Omega})+\varepsilon$. In $\Omega-\bar{\Omega}_\varepsilon$, σ and ω are exact, and hence can be expressed as $\sigma=dv$ and $\omega=dw$ respectively in $\Omega-\bar{\Omega}_\varepsilon$. Taking $\Omega-\bar{\Omega}_\varepsilon$ as R in Lemma 5, we find that there exists $t\in[0,1]$ such that the level curve $u=t$ is composed of a finite number of mutually disjoint analytic dividing Jordan curves and that

$$\left|\int_{u=t} vdw\right|\leqq\{\mu(\bar{\Omega})+\varepsilon\}\,\|\sigma\|_{\Omega-\bar{\Omega}_\varepsilon}\|\omega\|_{\Omega-\bar{\Omega}_\varepsilon}.$$

By Ω' we denote the normal subregion of W bounded by the level curve $u=t$. Then $\Omega_\varepsilon\subset\Omega'\subset\Omega$ and $T_{\Omega'}=T_\Omega$. It follows from Lemma 4 that

$$(T_\Omega\,\sigma, \omega^*)_W=(\sigma, \omega^*)_{\Omega'}+\int_{u=t} vd\bar{w}.$$

Hence

$$\begin{aligned}|(T_\Omega\,\sigma, \omega^*)_W|&\leqq|(\sigma, \omega^*)_{\Omega'}|+\{\mu(\bar{\Omega})+\varepsilon\}\,\|\sigma\|_{\Omega-\bar{\Omega}_\varepsilon}\|\omega\|_{\Omega-\bar{\Omega}_\varepsilon}\\&\leqq\{1+\mu(\bar{\Omega})+\varepsilon\}\,\|\sigma\|_\Omega\|\omega\|_\Omega.\end{aligned}$$

Next we treat the general case. Recall that $\{\tilde{\sigma}_W(c)\}$ span $\Gamma_{h0}(W)\cap\Gamma^*_{hse}(W)$ as stated in (iv). Therefore, in proving (2) we may assume that both σ and ω are expressed as finite linear combinations:

$$\sigma=\sum_{m=1}^{\mu}a_m\,\tilde{\sigma}_W(c_m)\quad\text{and}\quad\omega^*=\sum_{n=1}^{\nu}b_n\,\tilde{\sigma}_W(c'_n).$$

Let D be a relatively compact normal subregion of W such that D contains $\partial\Omega$, all c_m and c'_n and such that $\Omega\cap D$ is connected. Set

$$\sigma_D=\sum_{m=1}^{\mu}a_m\,\tilde{\sigma}_D(c_m)\quad\text{and}\quad\omega_D^*=\sum_{n=1}^{\nu}b_n\,\tilde{\sigma}_D(c'_n).$$

Then by Lemma 1

$$\|\sigma_D-\sigma\|_D\to0\quad\text{and}\quad\|\omega_D-\omega\|_D\to0\quad\text{as } D\uparrow W.$$

As we have already proved,

$$|(T_{\Omega\cap D,D}\sigma_D, \omega_D^*)_D|\leqq\{1+\mu(\bar{\Omega}\cap D)\}\,\|\sigma_D\|_{\Omega\cap D}\|\omega_D\|_{\Omega\cap D}. \tag{4}$$

To complete the proof, observe first that

$$\| T_{\Omega\cap D, D}\sigma_D - T_\Omega \sigma\|_D$$
$$= \|\sum_{j=1}^{g}\{-(\sigma_D,\ \tilde{\sigma}_D(A_j)^*)\tilde{\sigma}_D(B_j) + (\sigma_D,\ \tilde{\sigma}_D(B_j)^*)\tilde{\sigma}_D(A_j)\}$$
$$-\sum_{j=1}^{g}\{-(\sigma,\ \tilde{\sigma}_W(A_j)^*)\tilde{\sigma}_W(B_j) + (\sigma,\ \tilde{\sigma}_W(B_j)^*)\tilde{\sigma}_W(A_j)\}\|_D \to 0$$
$$\text{as } D \uparrow W.$$

We shall next show that $\mu(\overline{\Omega}\cap D) \downarrow \mu(\overline{\Omega})$ as $D\uparrow W$. It is easily seen that $\mu(\overline{\Omega}\cap D) \geqq \mu(\overline{\Omega})$ and $\mu(\overline{\Omega}\cap D)$ decreases as $D\uparrow W$. Consequently $\lim\limits_{D\uparrow W}\ \mu(\overline{\Omega}\cap D) \geqq \mu(\overline{\Omega})$. To establish the opposite inequality take an Ω_0 which is admissible in the sense described in the definition of $\mu(\overline{\Omega})$. We may assume $\partial\Omega_0 \subset D$. Let $\{R_i\}$ be the components of $\Omega - \overline{\Omega}_0$. Let u be the harmonic function in $\Omega - \overline{\Omega}_0$ which is equal to 0 on $\partial\Omega_0$ and 1 on $\partial\Omega$ and which has L_1-behavior along the ideal boundary of $\Omega - \overline{\Omega}_0$. Define u_D in $(\Omega - \overline{\Omega}_0)\cap D$ similarly so that $u_D = 0$ on $\partial\Omega_0$, $u_D = 1$ on $\partial\Omega$ and u_D has L_1-behavior along $\partial D\cap(\Omega - \overline{\Omega}_0)$. By Lemma 2 $\|d(u_D - u)\|_{(\Omega-\Omega_0)\cap D} \to 0$ as $D\uparrow W$.[6] Hence

$$\mu_{\Omega-\overline{\Omega}_0}(\partial\Omega_0,\ \partial\Omega) = \max_{1\leq i\leq k}\|du\|^2_{R_i} = \lim_{D\uparrow W}\max_{1\leq i\leq k}\|du_D\|^2_{R_i\cap D}$$
$$= \lim_{D\uparrow W}\mu_{(\overline{\Omega}-\Omega_0)\cap D}(\partial\Omega_0,\ \partial\Omega) \geqq \lim_{D\uparrow W}\mu(\overline{\Omega}\cap D).$$

On account of the arbitrariness of Ω_0, it follows that $\mu(\overline{\Omega}) \geqq \lim\limits_{D\uparrow W}\mu(\overline{\Omega}\cap D)$. Letting $D\uparrow W$ in (4), we obtain (2).

By setting $\omega^* = T_\Omega\sigma$, we derive (3) from (2).

Corollary 1. *Let $\{\Omega_k\}_{k=1}^{\infty}$ be a decomposition of W into normal subregions. Set $D_n = \bigcup_{k=1}^{n}\Omega_k$ and $T_n = \sum_{k=1}^{n} T_{\Omega_k}$. Suppose $\mu(\Omega_k) \leqq \mu$, $k=1, 2, \cdots$. Then*

$$\|T_n\| \leqq 1+\mu$$

and

$$|(\sigma,\ \omega^*)_W - (T_n\sigma,\ \omega^*)_W| \leqq (1+\mu)\|\sigma\|_{W-D_n}\|\omega\|_{W-D_n}$$

for any σ, $\omega^* \in \Gamma_{h0}(W)\cap\Gamma^*_{hse}(W)$.

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6) If W-Ω is not compact, we imbed Ω in another Riemann surface W' so that W'-Ω is compact and apply Lemma 2.

Proof. For $\sigma, \omega^* \in \Gamma_{h0}(W) \cap \Gamma^*_{hse}(W)$,

$$|(T_n\sigma, \omega^*)_W| \leqq \sum_{k=1}^{n} |(T_{\Omega_k}\sigma, \omega^*)_W| \leqq (1+\mu) \sum_{k=1}^{n} \|\sigma\|_{\Omega_k} \|\omega^*\|_{\Omega_k}$$
$$\leqq (1+\mu)\|\sigma\|_{D_n}\|\omega^*\|_{D_n} \leqq (1+\mu)\|\sigma\|_W \|\omega^*\|_W .$$

Substituting $T_n\sigma$ for ω^* in this inequality, we obtain $\|T_n\sigma\|^2_W \leqq (1+\mu)\|\sigma\|_W \|T_n\sigma\|_W$. Hence, $\|T_n\sigma\|_W \leqq (1+\mu)\|\sigma\|_W$ for any $\sigma \in \Gamma_{h0} \cap \Gamma^*_{hse}$, i.e.,

$$\|T_n\| \leqq 1+\mu.$$

Therefore, by Theorem 1, $T_n\sigma$ converges to σ strongly. We have

$$|(\sigma, \omega^*)_W - (T_n\sigma, \omega^*)_W| \leqq \sum_{k=n+1}^{\infty} |(T_{\Omega_k}\sigma, \omega^*)_W|$$
$$\leqq (1+\mu) \sum_{k=n+1}^{\infty} \|\sigma\|_{\Omega_k} \|\omega\|_{\Omega_k} \leqq (1+\mu)\|\sigma\|_{W-D_n} \|\omega\|_{W-D_n} .$$

This corollary, together with the reduction process given in §1, implies

Corollary 2. *Under the same condition as in the above corollary,*

$$(\sigma, \omega^*)_W = \sum_{k=1}^{\infty} \sum_{j=1}^{g_k} \left(\int_{A_j^k} \sigma \int_{B_j^k} \bar{\omega} - \int_{B_j^k} \sigma \int_{A_j^k} \bar{\omega} \right)$$

for any $\sigma \in \Gamma_{h0}(W)$ *and* $\omega \in \Gamma_{hse}(W)$.

§3. Conformal invariant $M(\Omega)$

Let Ω be an arbitrary Riemann surface of positive finite genus g, and $\{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis of Ω modulo dividing cycles. The well-known relation: $(\tilde{\sigma}(A_j), \tilde{\sigma}(B_k)^*) = A_j \times B_k$, together with Rodin's result mentioned in §1, implies

$$\lambda(\widetilde{A}_j)\lambda(\widetilde{B}_j) \geqq 1.$$

Set

$$M(\Omega) = \inf \sum_{j=1}^{g} \sqrt{\lambda(\widetilde{A}_j)\lambda(\widetilde{B}_j)} ,$$

where $\{A_j, B_j\}$ ranges over all canonical homology bases. Note that $M(\Omega) \geqq g$. The sum $\sum_{j=1}^{g} \sqrt{\lambda(\widetilde{A}_j)\lambda(\widetilde{B}_j)}$ was first introduced by

Y. Kusunoki [5], and then utilized by K. Matsui [6] although they did not consider its infimum.

For $\omega_1, \omega_2 \in \Gamma_{hse}(\Omega)$, we set

$$R(\omega_1, \omega_2) = \sum_{j=1}^{g} \left(\int_{A_j} \omega_1 \int_{B_j} \bar{\omega}_2 - \int_{B_j} \omega_1 \int_{A_j} \bar{\omega}_2 \right).$$

It should be remarked that this quantity does not depend on the particular choice of a homology basis $\{A_j, B_j\}$. This is seen from Green's formula as follows: Take a canonical subregion Ω_0 which contains all A_j, B_j. Then

$$(\omega_1, \omega_2^*)_{\Omega_0} = \sum_{j=1}^{g} \left(\int_{A_j} \omega_1 \int_{B_j} \bar{\omega}_2 - \int_{B_j} \omega_1 \int_{A_j} \bar{\omega}_2 \right) - \int_{\partial\Omega_0} u\bar{\omega}_2,$$

where u is a harmonic function defined separately on each contour of $\partial\Omega_0$ such that $du = \omega_1$.

We shall now give the key lemma.

Lemma 6. *Suppose* $\omega_1, \omega_2 \in \Gamma_{hse}(\Omega)$. *Then*

$$|R(\omega_1, \omega_2)| \leqq 2M(\Omega) \|\omega_1\|_\Omega \|\omega_2\|_\Omega.$$

Proof. It holds that $\int_{A_j} \omega_1 = (\omega_1, \tilde{\sigma}(A_j)^*)$, $\int_{B_j} \bar{\omega}_2 = (\bar{\omega}_2, \tilde{\sigma}(B_j)^*)$, $\|\tilde{\sigma}(A_j)\| = \sqrt{\lambda(\widetilde{A}_j)}$ and $\|\tilde{\sigma}(B_j)\| = \sqrt{\lambda(\widetilde{B}_j)}$. Hence, by the Schwarz inequality we have

$$\left| \int_{A_j} \omega_1 \int_{B_j} \bar{\omega}_2 \right| \leqq \|\omega_1\|_\Omega \|\omega_2\|_\Omega \sqrt{\lambda(\widetilde{A}_j)\lambda(\widetilde{B}_j)}.$$

Thus

$$|R(\omega_1, \omega_2)| \leqq 2\|\omega_1\|_\Omega \|\omega_2\|_\Omega \sum_{j=1}^{g} \sqrt{\lambda(\widetilde{A}_j)\lambda(\widetilde{B}_j)}.$$

Let W be a Riemann surface of infinite genus, and Ω be a normal subregion of W. With the same notation as in §2, the above lemma implies

$$|(\omega^*, T_\Omega\, \sigma)_W| \leqq 2M(\Omega) \|\omega\|_\Omega \|\sigma\|_\Omega$$

for any $\sigma, \omega^* \in \Gamma_{h0}(W) \cap \Gamma_{hse}^*(W)$. From this follows

Theorem 3. *Let* $\{\Omega_k\}_{k=1}^{\infty}$ *be a decomposition of* W *into normal* *subregions, and* $\{A_j^k, B_j^k\}_{j=1}^{g_k}$ *be a canonical homology basis of* Ω_k *modulo dividing cycles* ($k=1, 2, \cdots$). *Suppose* $M(\Omega_k)\leqq M$. *Then*

$$\left|(\sigma, \omega^*)_W-\sum_{k=1}^{n}\sum_{j=1}^{g_k}\left(\int_{A_j^k}\sigma\int_{B_j^k}\bar{\omega}-\int_{B_j^k}\sigma\int_{A_j^k}\bar{\omega}\right)\right|\leqq 2M\|\sigma\|_{W-D_n}\|\omega\|_{W-D_n}$$

for any $\sigma, \omega^*\in\Gamma_{h0}(W)\cap\Gamma_{hse}^*(W)$, *and*

$$(\sigma, \omega^*)_W=\sum_{k=1}^{\infty}\sum_{j=1}^{g_k}\left(\int_{A_j^k}\sigma\int_{B_j^k}\bar{\omega}-\int_{B_j^k}\sigma\int_{A_j^k}\bar{\omega}\right)$$

for any $\sigma\in\Gamma_{h0}(W)$ *and* $\omega\in\Gamma_{hse}(W)$.

§4. Conformal invariant $\mu(\partial\Omega)$

Let Ω be a normal subregion of W, and consider two normal subregions Ω_1 and Ω_2 of W such that $\bar{\Omega}_1\subset\Omega$, $\bar{\Omega}\subset\Omega_2$ and that these three regions have the same genus. We shall call such a pair (Ω_1, Ω_2) admissible. With the notation in §2, we define

$$\mu(\partial\Omega)=\inf\,\mu_{\Omega_2-\bar{\Omega}_1}(\partial\Omega_1, \partial\Omega_2),$$

where $\{\Omega_1, \Omega_2\}$ ranges over all admissible pairs of normal subregions.

By the same reasoning as in §2, we obtain

Theorem 4. $\|T_\Omega\|\leqq 1+\mu(\partial\Omega)$.

Corollary. *Let* $\{W_n\}$ *be an increasing sequence of normal subregions of* W *such that the genus of* W_n *is strictly smaller than the genus of* W_{n+1} *and* $W=\bigcup W_n$. *Let* $\{A_j, B_j\}_{j=1}^{\infty}$ *be a canonical homology basis of* W *modulo dividing cycles such that* $\{A_j, B_j\}_{j=1}^{p(n)}$ *forms a basis of* W_n. *If* $\mu(\partial W_n)$ *is bounded, then it holds that*

$$(\sigma, \omega^*)_W=\lim_{n\to\infty}\sum_{j=1}^{p(n)}\left\{\int_{A_j}\sigma\int_{B_j}\bar{\omega}-\int_{B_j}\sigma\int_{A_j}\bar{\omega}\right\}$$

for any $\sigma\in\Gamma_{h0}(W)$ *and* $\omega\in\Gamma_{hse}(W)$.

We have $\mu(\partial\Omega)\leqq\mu(\bar{\Omega})$. However, $\mu(\partial\Omega)$ is not an intrinsic invariant of $\bar{\Omega}$. It may be properly called a relative invariant. This corollary is a generalization of Theorem 2 of R. Accola [1], p. 155;

see also Remarks in R. Accola [3], p. 610, 5.

It is easy to find a Riemann surface and a decomposition into normal subregions $\{\Omega_k\}$ such that the genus g_k of Ω_k is not bounded while $\mu(\overline{\Omega}_k)$ is bounded. Since $M(\Omega_k)\geqq g_k$, $M(\Omega_k)$ is not bounded in this case. However, it is an open question whether there exists an example in which $M(\Omega_k)$ is bounded while $\mu(\overline{\Omega}_k)$ is not bounded.

§5. Special case where $M(\Omega_k)$ is bounded

Let $\{\Omega_k\}_{k=1}^{\infty}$ be a decomposition of W into normal subregions. In this section, we fix a canonical homology basis $\{A_j^k, B_j^k\}_{j=1}^{g_k}$ of Ω_k once for all. The notation $(\widetilde{A}_j^k)_{\Omega_k}((\widetilde{A}_j^k)_W$ resp.) will be used to indicate that the curves are in Ω_k (W resp.). The same remark applies to $(\widetilde{B}_j^k)_{\Omega_k}$ and $(\widetilde{B}_j^k)_W$. With K. Matsui [6] we consider the special case that $g_k\leqq g<\infty$, $\lambda((\widetilde{A}_j^k)_{\Omega_k})\leqq\lambda_0<\infty$ and $\lambda((\widetilde{B}_j^k)_{\Omega_k})\leqq\lambda_0$.

Theorem 5. *For any* $\omega\in\Gamma_{hse}(W)$,

$$\sum_k\sum_{j=1}^{g_k}(|a_{k,j}|^2+|b_{k,j}|^2)\leqq 2g\lambda_0\|\omega\|^2<\infty,$$

where $a_{k,j}=\int_{A_j^k}\omega$ *and* $b_{k,j}=\int_{B_j^k}\omega$.

Conversely, for any system of complex numbers $\{a_{k,j}, b_{k,j}\}$ *satisfying* $\sum_k\sum_{j=1}^{g_k}(|a_{k,j}|+|b_{k,j}|)<\infty$, *there exists a unique differential in* $\Gamma_{h0}(W)\cap\Gamma_{hse}^*(W)$, *which has* $a_{k,j}$ *and* $b_{k,j}$ *as* A_j^k*-and* B_j^k*-periods respectively.*

Proof. Since $a_{k,j}=(\omega, \tilde{\sigma}_{\Omega_k}(A_j^k)^*)_{\Omega_k}$ and $b_{k,j}=(\omega, \tilde{\sigma}_{\Omega_k}(B_j^k)^*)_{\Omega_k}$,

$$\sum_{j=1}^{g_k}(|a_{k,j}|^2+|b_{k,j}|^2)\leqq\sum_{j=1}^{g_k}\left\{\lambda((\widetilde{A}_j^k)_{\Omega_k})+\lambda((\widetilde{B}_j^k)_{\Omega_k})\right\}\|\omega\|_{\Omega_k}^2$$
$$\leqq 2g\lambda_0\|\omega\|_{\Omega_k}^2.$$

Hence

$$\sum_k\sum_{j=1}^{g_k}(|a_{k,j}|^2+|b_{k,j}|^2)\leqq 2g\lambda_0\sum_k\|\omega\|_{\Omega_k}^2=2g\lambda_0\|\omega\|^2.$$

In order to prove the converse, we note $\lambda((\widetilde{A}_j^k)_W)\leqq\lambda((\widetilde{A}_j^k)_{\Omega_k})$ and $\lambda((\widetilde{B}_j^k)_W)\leqq\lambda((\widetilde{B}_j^k)_{\Omega_k})$ and have

$$\|\sum_{j=1}^{g_k}\{-a_{k,j}\,\tilde{\sigma}_W(B_j^k)+b_{k,j}\,\tilde{\sigma}_W(A_j^k)\}\|\leqq\lambda_0^{\frac{1}{2}}\sum_{j=1}^{g_k}(|a_{k,j}|+|b_{k,j}|).$$

Therefore, $\sum_k\sum_{j=1}^{g_k}\{-a_{k,j}\,\tilde{\sigma}_W(B_j^k)+b_{k,j}\,\tilde{\sigma}_W(A_j^k)\}$ converges. It is easy to check that this differential has the required properties.

§6. Covering surfaces of Schottky type

In this last section, we shall give some examples of Riemann surfaces which admit such decompositions as described in §§3, 4.

Let R be a Riemann surface with finite genus $g\geqq 2$. We draw mutually disjoint non-dividing loops $c_1, \cdots, c_l$ on R, and cut R along $c_1, \cdots, c_l$. We denote by Ω the resulting surface, which we assume to be connected. Consequently $l\leqq g$ and the genus of Ω is $g-l$.

We take an infinite number of replicas of Ω, and glue them along opposite shores of $c_j(1\leqq j\leqq l)$ so that we obtain a regular covering surface W of R, on which all the inverse images of c_j are dividing curves. It is konwn that $W\notin O_{HD}$ if $l\geqq 2$; see [11].

Next we construct a little more general surfaces in the following manner. We consider a finite number of not necessarily compact bordered Riemann surfaces $\overline{\Omega}^{(1)}, \cdots, \overline{\Omega}^{(m)}$ of positive finite genus with compact borders and we prepare an infinite number of replicas of each of them. Glue them along contours so that on the surface W thus constructed all the joints are dividing curves. Write $W=\bigcup_{k=1}^{\infty}\overline{\Omega}_k$, where $\overline{\Omega}_k$ is one of the replicas of $\overline{\Omega}^{(1)}, \cdots, \overline{\Omega}^{(m)}$. Evidently both $\mu(\overline{\Omega}_k)$ and $M(\Omega_k)$ are bounded, and hence the generalized bilinear relation of the type described in the introduction holds for W.

Finally we remark the following fact: Suppose that $\overline{\Omega}^{(i)}$ are all compact and that in $W=\bigcup\overline{\Omega}_k$ no contour of Ω_k is left unglued. Then $W\in O_{KD}$[7]. ($W\in O_{KD}$ means $\Gamma_{he}(W)\cap\Gamma_{hse}^*(W)=\{0\}$, or equivalently $\Gamma_{he}(W)=\Gamma_{hm}(W)$.) To prove this, take any $dv\in\Gamma_{he}(W)\cap\Gamma_{hse}^*(W)$. We shall define $\{W_n\}$ by induction. Set $W_1=\Omega_1$, and denote by W_{n+1} the surface obtained by gluing all the adjacent Ω_k's

7) This class O_{KD} is denoted by O_{H_1D} in [4].

to W_n. By Lemma 5

$$\|dv\|^2_{W^{(n)}}=\left|\int_{u=t_n} v(dv)^*\right| \leq \mu_{W_{n+1}-\overline{W}_n}(\partial W_n, \partial W_{n+1})\|dv\|^2_{W_{n+1}-\overline{W}_n},$$

where u is the harmonic measure of ∂W_{n+1} with respect to $W_{n+1}-\overline{W}_n$, t_n is some value of $[0, 1]$ and $W^{(n)}$ is the part of W_{n+1} whose boundary is the level curve $u=t_n$. As $n\to\infty$, $\|dv\|^2_{W^{(n)}}$ tends to $\|dv\|^2_W$ and $\|dv\|^2_{W_{n+1}-\overline{W}_n}$ tends to 0 while $\mu_{W_{n+1}-\overline{W}_n}(\partial W_n, \partial W_{n+1})$ is bounded. This shows that $\|dv\|^2_W=0$.

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