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A remark on an iterative infinite higher derivation

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I. Let K be a field of arbitrary characteristic p and $D=\{D_0=id, D_1, D_2, \cdots\}$ be an infinite higher derivation in K with the subfield of D-constants k over which K is finitely generated (for the definition, see [3]). For D, define ring homomorphisms

$$K \underset{\varepsilon}{\overset{\widetilde{D}}{\rightleftarrows}} k[[t]]\widehat{\otimes}_k K \cong K[[t]]$$

by $\widetilde{D}(a)=a+t\otimes D_1(a)+\cdots+t^i\otimes D_i(a)+\cdots$, for $a\in K$ and by $\varepsilon(t)=0$, $\varepsilon(\lambda)=\lambda$, $\lambda\in K$, where $K[[t]]$ is the ring of formal power series of one variable over K. Then, $\varepsilon\cdot\widetilde{D}=id_K$. Conversely, if we have a ring homomorphism $\widetilde{D}: K\to K[[t]]$ such that $\varepsilon\cdot\widetilde{D}=id_K$ and if we define a family of maps in K, $\{D_i\}_{i=0,1,2,\cdots}$, by $\widetilde{D}(a)=D_0(a)+t\otimes D_1(a)+\cdots+t^i\otimes D_i(a)+\cdots$, $D=\{D_0=id, D_1, \cdots\}$ forms an infinite higher derivation in K. An infinite higher derivation D in K is called *iterative* if $D_iD_j=\binom{i+j}{j}D_{i+j}$ for all $i, j\geqq 0$. This is equivalent to say that if we define a ring homomorphism $\Delta: k[[t]]\to k[[t]]\widehat{\otimes}_k k[[t]]$ by $\Delta(t)=t\otimes 1+1\otimes t$, $\widetilde{D}$ commutes a diagram,

$$\begin{array}{ccc} K & \xrightarrow{\widetilde{D}} & k[[t]]\widehat{\otimes}_k K \\ {\scriptstyle\widetilde{D}}\downarrow & & \downarrow{\scriptstyle\Delta\widehat{\otimes} id} \\ k[[t]]\widehat{\otimes}_k K & \xrightarrow[id\widehat{\otimes}\widetilde{D}]{} & k[[t]]\widehat{\otimes}_k k[[t]]\widehat{\otimes}_k K. \end{array}$$

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In the following, an infinite higher derivation D in K is always assumed to be iterative.

Let R be a set of such elements a in K that $D_j(a)=0$ for all j large enough. Then, since $R=\varepsilon(\widetilde{D}(K)\cap K[t])$, R is a ring over k. We shall prove

Theorem. *The situation being as above, R is either isomorphic to the polynomial ring of one variable over k or reduced to k.*

Proof. Our proof consists of several steps.

1) We can suppose $R\neq k$. Let R' be a finitely generated subring of R, $(a_1, \cdots, a_r)$ be its generating basis, $A=k[(D_j(a_i))_{1\leqslant i\leqslant r, j=0,1,2,\cdots}]$ and $X=\mathrm{Spec}(A)$. Then $\widetilde{D}$ sends A into $k[t]\otimes_k A$ and the equalities $(\varDelta\otimes id)\widetilde{D}=(id\otimes\widetilde{D})\widetilde{D}$, $\varepsilon\widetilde{D}=id$ show that $\widetilde{D}$ defines an operation $\sigma: G_a\otimes_k X\rightarrow X$ of the additive group G_a on X. We shall show that X is a homogeneous space of G_a defined over k. Take a point x of X which is algebraic over k and let W be the k-closure of the orbit of x. If $W\neq X$, consider the non-zero ideal $\mathfrak{a}$ of A which defines W in X. Since W is stable under the operation σ of G_a, $\mathfrak{a}$ satisfies $D(\mathfrak{a})\subset k[t]\otimes_k\mathfrak{a}$. Take a non-zero element b ot $\mathfrak{a}$. Then there exists a positive integer l such that $D_l(b)\neq 0$ and $D_j(b)=0$ for all $j>l$. Using the iterativeness of D, we know that $D_l(b)\in k$, hence a unit in A. This contradicts the above inclusion. Therefore X is a homogeneous space of G_a defined over k.

2) X is an affine curve defined over k, hence obtained from the projective line $\boldsymbol{P}^1$, extracting some points. However, since X is a homogeneous space of G_a defined over k, hence has a k-rational point (cf. [2] p. 425), X is of the form $\boldsymbol{P}^1-$(the point at infinity). Thus X is considered as the affine line, i.e. A is isomorphic to the polynomial ring of one variable over k.

3) Suppose $A\neq R$. Then, adding a new element a of R (not in A), we construct $A'=A[(D_j(a))_{j=0,1,2,\cdots}]$. The above argument shows that A' is isomorphic to the polynomial ring of one variable over k. Suppose that we had an infinite series of such subrings in

R, $A_1 \subset A_2 \subset \cdots$. Consider the corresponding series of the quotient fields $Q(A_1) \subset Q(A_2) \subset \cdots$. Note that this ascending chain of subfields in K must stop because K is finitely generated over k and that if $Q(A_\lambda)=Q(A_\mu)$ for two members of the chain, then $A_\lambda=A_\mu$ because A_λ and A_μ are isomorphic to the polynomial ring of one variable over k. Therefore R is finitely generated over k. Applying again the argument of 2) to R, we know that R is isomorphic to the polynomial ring of one variable over k. q.e.d.

Corollary. *If K is algebraic over R, K is then an algebraic function field of one variable over k.*

II. Remarks. 1) Suppose $R \neq k$. Then for any element b of R not in k, we can find a non-negative integer h such that $D_h(b) \notin k$, $D_{h+1}(b), \cdots, D_l(b) \in k$ and $D_j(b)=0$ for all $j>l$, where l is as in the proof 1) of Theorem. Therefore there exists an element y in R of the form

$$\widetilde{D}(y)=y+\alpha t^r+\cdots+\gamma t^s, \quad \text{where } \alpha, \cdots, \gamma \in k.$$

2) There is no element in K which is purely inseparable over the quotient field $Q(R)$ of R.

Proof. Suppose the contrary. Let y be an element not in $Q(R)$ such that $y^{p^n}=g\cdot f^{-1} \in Q(R)$, $f, g \in R$. Then $(fy)^{p^n}=f^{p^n-1}g \in R$ and $\widetilde{D}((fy)^{p^n})=(\widetilde{D}(f)\widetilde{D}(y))^{p^n} \in k[t]\otimes_k R$. Hence $\widetilde{D}(f)\widetilde{D}(y) \in k[t]\otimes_k K$, i.e. $fy \in R$, hence $y \in Q(R)$. q.e.d.

3) It occurs that $R=k$. For example, let $K=k(x)$ a purely transcendental extension of one variable over k, k being algebraically closed. Define an iterative infinite higher derivation D in K by $\widetilde{D}(x)=x+t+t^p+\cdots+t^{p^i}+\cdots$. Then $R=k$.

Proof. It is easy to see that k is the field of D-constants. Suppose $R \neq k$. Then by Remark 1), there exists an element f in $k(x)$ such that $\widetilde{D}(f)=f+t^r+\cdots+\gamma t^s$. Write $f=\alpha\left(\prod_{i=1}^{u}(x-\alpha_i)^{m_i}\right) \times \left(\prod_{j=1}^{v}(x-\beta_j)^{n_j}\right)^{-1}$, where α_i, β_j belong to k and are mutually distinct.

Applying $\widetilde{D}$, we have

$$\alpha \prod_{i=1}^{u}(x-\alpha_i+t+t^p+\cdots)^{m_i}\prod_{j=1}^{v}(x-\beta_j)^{n_j}$$
$$=\prod_{j=1}^{v}(x-\beta_j+t+t^p+\cdots)^{n_j}\left\{\alpha\prod_{i=1}^{u}(x-\alpha_i)^{m_i}+\prod_{j=1}^{v}(x-\beta_j)^{n_j}(t^r+\cdots+\gamma t^s)\right\}.$$

The coefficients of t of both sides of the equality are

$$\alpha\prod_{i=1}^{u}(x-\alpha_i)^{m_i}\prod_{j=1}^{v}(x-\beta_j)^{n_j}\left(\sum_{i=1}^{u}m_i(x-\alpha_i)^{-1}\right)$$

and

$$\prod_{j=1}^{v}(x-\beta_j)^{n_j}\left\{\alpha\prod_{i=1}^{u}(x-\alpha_i)^{m_i}+\delta_{1r}\prod_{j=1}^{v}(x-\beta_j)^{n_j}\right\}\left(\sum_{j=1}^{v}n_j(x-\beta_j)^{-1}\right),$$

where δ_{1r} is Kronecker's delta. We can assume that some m_i or n_j is not 0 modulo p. If so, we have only to replace $(x_i, \alpha_i, \beta_j, t)$ by $(x^p, \alpha_i^p, \beta_j^p, t^p)$ and so on. Then it is easy to draw out a contradiction by equating the above two terms. q.e.d.

4) K is not necessarily algebraic over R. For example, let $\alpha_1, \alpha_2, \cdots$ be an infinite number of variables, k be an algebraic closure of the field $F_p(\alpha_1, \alpha_2, \cdots)$ which is generated by $\alpha_1, \alpha_2, \cdots$ over the prime field F_p and $L=k((x))$ be the quotient field of the formal power series ring of one variable x over k. Define an iterative infinite higher derivation D in L by $\widetilde{D}(x)=x+t$. Let $y=x+\alpha_1 x^p+\alpha_2 x^{p^2}+\cdots+\alpha_i x^{p^i}+\cdots$ and $K=k(x, y)$. Consider the induced higher derivation D in K. It is then easy to see that k is the field of D-constants in K, $R=k[x]$ and that K is not algebraic over $k(x)$.

5) Finally, we shall note that a separably generated algebraic function field K of one variable over k has an iterative infinite higher derivation with the constant field k. To see this fact, take a separating variable x in K and define an iterative infinite higher derivation D in $k(x)$ by $\widetilde{D}(x)=x+t$. Then k is the field of D-constants and D can be extended uniquely to K because K is algebraic and separable over $k(x)$ (cf. [1]). Moreover, we have $R=k[x]$.

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