J. Math. Kyoto Univ. (JMKYAZ)
11-3 (1971) 553-563

On the uniqueness of solutions of stochastic differential equations II

By

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(Received, May 11, 1971)

The present paper is a continuation of [1] in which we have discussed the uniqueness of solutions of stochastic differential equations. The condition of the pathwise uniqueness of solutions obtained in [1] is essentially in one-dimensional case and we shall investigate here the general multi-dimensional case.

Let $\sigma(t, x)=(\sigma_j^i(t, x))$, $i=1, \cdots, n$, $j=1, \cdots, r$, and $b(t, x)=(b^i(t, x))$, $i=1, \cdots, n$, be defined on $[0, \infty)\times R^n$, bounded and Borel measurable in (t, x) such that $\sigma(t, x)$ is an $n\times r$-matrix and $b(t, x)$ is an $n\times 1$-matrix. We consider the following Itô's stochastic differential equation;

$$dx_t=\sigma(t, x_t)\, dB_t+b(t, x_t)\, dt, \tag{1}$$

or, in component wise,

$$dx_t^i=\sum_{j=1}^{r}\sigma_j^i(t, x_t)\, dB_t^j+b^i(t, x_t)\, dt \qquad i=1, \cdots, n. \tag{1'}$$

A precise formulation is as follows; by a probability space $(\Omega, \mathscr{F}, P)$ with an increasing family of Borel fields $\mathscr{F}_t$, which is denoted as $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, we mean a standard probability space $(\Omega, \mathscr{F}, P)$ with a system $\{\mathscr{F}_t\}_{t\in[0,\infty)}$ of sub Borel-fields of $\mathscr{F}$ such that $\mathscr{F}_t\subset\mathscr{F}_s$ if $t<s$.

Definition 1. By a solution of the equation (1), we mean a family of stochastic processes $\mathfrak{X}=\{x_t=(x_t^1, \cdots, x_t^n), B_t=(B_t^1, \cdots, B_t^r)\}$ defined on a probability space with an increasing family of Borel fields $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ such that

(i) with probability one, x_t and B_t are continuous in t and $B_0=0$,

(ii) they are adapted to $\mathscr{F}_t$, i.e., for each t, x_t and B_t are $\mathscr{F}_t$-measurable,

(iii) B_t is a system of $\mathscr{F}_t$-martingales such that $<B_t^i, B_t^j>=\delta_{ij}\cdot t$ $i, j=1, \cdots, r$,

(iv) $\mathfrak{X}=\{x_t, B_t\}$ satisfies, with probability one,

$$x_t-x_0=\int_0^t\sigma(s, x_s)\, dB_s+\int_0^t b(s, x_s)\, ds, \tag{1$''$}$$

or, in component wise,

$$x_t^i-x_0^i=\sum_{j=1}^r\int_0^t\sigma_j^i(s, x_s)\, dB_s^j+\int_0^t b^i(s, x_s)ds, \qquad i=1, \cdots, n, \tag{1$'''$}$$

where the integral by dB_s is understood in the sense of the stochastic integral.

Definition 2. (Pathwise uniqueness)

We shall say that the pathwise uniqueness holds for (1) if, for any two solutions $\mathfrak{X}=(x_t, B_t)$ and $\mathfrak{X}'=(x_t', B_t')$ defied on a same probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, $x_0=x_0'$ and $B_t\equiv B_t'$ imply $x_t\equiv x_t'$.

We refer to [1] for some implications of the pathwise uniqueness. Now we shall obtain the condition of the pathwise uniqueness in the terms of the modulus of continuity of the coefficients.

First, we shall treat the case of equations without drift term, i.e.,

$$dx_t=\sigma(t, x_t)\, dB_t. \tag{2}$$

In the following, ρ is a function defined on some interval $[0, a)$ $(a>0)$ which is continuous, increasing and $\rho(0)=0$.

Theorem 1. *Let* ρ *satisfy*

$$\int_{0+}\rho^{-2}(\xi)\xi\, d\xi=+\infty, \tag{3}$$

$$\rho^2(\xi)\cdot\xi^{-1} \qquad \textit{is concave.} \tag{4}$$

Then, for every $n\times r$*-matrix* $\sigma(t, x)((t, x)\in[0, \infty)\times R^n)$ *such that*

$$\|\sigma(t, x)-\sigma(t, y)\|\leq\rho(|x-y|), \qquad x, y\in R^n,\ |x-y|<a, \tag{5}$$

the pathwise uniqueness of solutions of (2) *holds.*

Remark 1. For examples,

$$\rho(\xi)=\xi,\ \rho(\xi)=\xi\left(\log\frac{1}{\xi}\right)^{\frac{1}{2}},\ \rho(\xi)=\xi\left(\log\frac{1}{\xi}\right)^{\frac{1}{2}}\left(\log^{(2)}\frac{1}{\xi}\right)^{\frac{1}{2}}, \cdots \text{etc.,}$$

satisfy (3) and (4).

Proof. By extending ρ suitably, we may assume that ρ is defined everywhere on $[0, \infty)$ such that $\rho^2(\xi)\cdot\xi^{-1}$ is concave there and (5) holds for every $x, y\in R^n$.

Let $1=a_0>a_1>a_2>\cdots>a_m\rightarrow 0$ be defined by

$$\int_{a_m}^{a_{m-1}}\rho^{-2}(\xi)\cdot\xi\, d\xi=2 \qquad m=1, 2, \cdots.$$

Then, there exists a twice continuously differentiable function $\psi_m(\xi)$ on $[0, \infty)$ such that,

$$\psi'_m(\xi)=\begin{cases}0 & 0\leq\xi\leq a_m\\ \text{between 0 and 1,} & a_m<\xi<a_{m-1}\\ 1 & \xi\geq a_{m-1}\end{cases}$$

$$\psi''_m(\xi)=\begin{cases}0 & 0\leq\xi\leq a_m\\ \text{between 0 and } \rho^{-2}(\xi)\cdot\xi & a_m<\xi<a_{m-1}\\ 0. & \xi>a_{m-1}\end{cases}.$$

Let $f_m(x)=\psi_m(|x|)$ for $x\in R^n$. Then $f_m(x)$ is twice continuously differentiable and $f_m(x)\uparrow|x|$ as $m\to\infty$.

Now, let $\mathfrak{X}=(x_t, B_t)$, $\mathfrak{X}'=(x'_t, B'_t)$ be two solutions of (2) on the same probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ such that $x_0=x'_0$ and $B_t\equiv B'_t$.

Then, by Ito's formula,

$$f_m(x_t-x'_t)=a \text{ martingale}$$
$$+\frac{1}{2}\int_0^t\sum_{i,j=1}^n(f_m)_{x_ix_j}(x_s-x'_s)\Big\{\sum_{k=1}^r(\sigma^i_k(s, x_s)$$
$$-\sigma^i_k(s, x'_s))(\sigma^j_k(s, x_s)-\sigma^j_k(s, x'_s))\Big\}ds.$$

On the other hand,

$$(f_m)_{x_ix_j}(x)=\psi'_m(|x|)\frac{|x|^2\delta_{ij}-x_ix_j}{|x|^3}+\psi''_m(|x|)\frac{x_ix_j}{|x|^2}$$

and since ψ'_m is uniformly bounded,

$$|(f_m)_{x_ix_j}(x)|\leq K_1\frac{1}{|x|}\cdot I_{\{x\neq 0\}}+K_2\psi''_m(|x|)$$

where, K_1 and K_2 are some positive constants.
Then,

$$E[f_m(x_t-x'_t)]\leq K_3\cdot\int_0^t E[I_{\{x_s\neq x'_s\}}|x_s-x'_s|^{-1}$$
$$\times\rho^2(|x_s-x'_s|^{-1}\cdot\rho^2(|x_s-x'_s|)]\,ds$$
$$+K_4\int_0^t E[\psi''_m(|x_s-x'_s|)\cdot\rho^2(|x_s-x'_s|)]\,ds\equiv I_1+I_2, \quad \text{say.}$$

Since $\psi''_m(\xi)\leq\rho^{-2}(\xi)\cdot\xi$, we have for I_2,

$$0\leq I_2\leq K_4\int_0^t E[|x_s-x'_s|\cdot I_{\{a_m\leq|x_s-x'_s|\leq a_{m-1}\}}]\,ds\leq K_4\cdot a_{m-1}\cdot t\to 0$$
$$\text{as } m\to\infty.$$

Thus, we have

$$E\{|x_t-x'_t|\}\leq K_3\int_0^t E\{|x_s-x'_s|^{-1}\cdot\rho^2(|x_s-x'_s|)\}\,ds.$$

Let $G(\xi)=\rho^2(\xi)\cdot\xi^{-1}$. Since G is concave by assumption, we have, by Jensen's inequality, $E\{|x_t-x'_t|\}\leq K_3\int_0^t G(E|x_s-x'_s|)ds$.

Now, $\int_{0+} G^{-1}(\xi)\,d\xi=\infty$ implies $E|x_t-x'_t|\equiv 0$. Q.E.D.

Remark 2. The condition (3) in this theorem is, for $n\geq 3$, nearly best possible in the sense that, if $\int_{0+}\rho^{-2}(\xi)\xi\,d\xi<\infty$ and if ρ is subadditive, i.e., $\rho(\xi_1+\xi_2)\leq\rho(\xi_1)+\rho(\xi_2)$ $^\forall\xi_1, \xi_2\in[0, \infty)$, then, there exists σ satisfying (5) for which the pathwise uniqueness does not hold.

Indeed, let $\sigma_j^i(t, x)=\delta_{ij}\rho(|x|)$, $i, j=1, \cdots, n$, $x\in R^n$, $(n\geq 3)$. Then,

$$|\sigma_j^i(t, x)-\sigma_j^i(t, y)|\leq|\rho(|x|)-\rho(|y|)|\leq\rho(|x-y|).$$

Consider the equation

$$\begin{cases} dx_t=\sigma(x_t)\,dB_t \\ x_0=0. \end{cases} \tag{6}$$

Let an n-dimensional Brownian motion $\{\bar{B}_t, \mathscr{F}_t\}$ be given on a probability space $(\Omega, \mathscr{F}, P)$ such that $\bar{B}_0=0$. Let $A_t=\int_0^t\rho^{-2}(|\bar{B}_s|)ds$. A_t defines a continuous additive functional of $\bar{B}_t$ since,

$$E[A_t]=\omega_n\int_0^\infty\rho^{-2}(\xi)\cdot\left[\int_0^t\frac{1}{(2\pi s)^{\frac{n}{2}}}e^{-\frac{\xi^2}{2s}}ds\right]\xi^{n-1}d\xi$$

$$\leq K'\int_{0+}\rho^{-2}(\xi)\cdot\xi\,d\xi<\infty.$$

Now, $(\bar{B}_{A_t^{-1}}, \mathscr{F}_{A_t^{-1}})$ is a system of local martingales(*) such that

$$<\bar{B}^i_{A_t^{-1}}, \bar{B}^j_{A_t^{-1}}>=\delta_{ij}\cdot A_t^{-1}=\delta_{ij}\int_0^t\rho^2(\bar{B}_{A_s^{-1}})ds=\int_0^t(\sigma^t\sigma)_{ij}(\bar{B}_{A_s^{-1}})ds$$

and hence,

(*) A_t^{-1} is the inverse function of $t\rightsquigarrow A_t$.

$$\left(B_t\equiv\int_0^t\sigma^{-1}(\bar{B}_{A_s^{-1}})d\bar{B}_{A_s^{-1}},\ \mathscr{F}_t\equiv\bar{\mathscr{F}}_{A_t^{-1}}\right)$$

is an n-dimensional Brownian motion.

Thus $(x_t\equiv\bar{B}_{A_t^{-1}}, B_t)$ is a solution of (6) on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$. But $(x_t\equiv 0, B_t)$ is also a solution and thus, the pathwise uniqueness does not hold.

The theorem can be improved for some class of $\sigma(t, x)$ when $n=1$ or 2. The following theorem was essentially proved in [1].

Theorem 2. *Let* ρ *satisfy*

$$\int_{0+}\rho^{-2}(\xi)\,d\xi=+\infty. \tag{7}$$

Then, for every $1\times r$*-matrix* $\sigma(t, x)$, $(t, x)\in[0, \infty)\times R^1$, *such that*

$$\|\sigma(t, x)-\sigma(t, y)\|\leq\rho(|x-y|) \qquad {}^{\forall}x, y\in R^1,$$

the pathwise uniqueness of solutions of (2) *holds.*

Remark 3. Just as Remark 2, we see that the condition (7) is best possible.

We do not know whether Theorem 1 can be improved for the class of all $2\times r$-matrices. But, for a certain class of $2\times r$-matrices, it can really be improved as follows;

Theorem 3. *Let* ρ *satisfy*

$$\int_{0+}\rho^{-2}(\xi)\,\xi\log\frac{1}{\xi}\,d\xi=\infty, \tag{8}$$

$$G(\eta)=\eta^3 e^{\frac{2}{\eta}}\rho^2(e^{-\frac{1}{\eta}}) \qquad \textit{is concave on some interval } [0, a'). \tag{9}$$

Then, for every 2×2*-matrix* $\sigma(t, x)$, $(t, x)\in[0, \infty)\times R^2$, *of the form*

(10) $$\sigma^i_j(t, x)=\delta_{ij}a(t, x), \qquad i, j=1, 2,$$

such that

$$\|\sigma(t, x)-\sigma(t, y)\|\leq\rho(|x-y|),$$

the pathwise uniqueness of solutions of (2) *holds.*

Remark 4. For examples, $\rho(\xi)=\xi\cdot\left(\log\frac{1}{\xi}\right)$, $\rho(\xi)=\xi\cdot\left(\log\frac{1}{\xi}\right)\times\left[\log^{(2)}\frac{1}{\xi}\right]^{\frac{1}{2}}$,

$$\rho(\xi)=\xi\cdot\left(\log\frac{1}{\xi}\right)\cdot\left[\log^{(2)}\frac{1}{\xi}\right]^{\frac{1}{2}}\cdot\left[\log^{(3)}\frac{1}{\xi}\right]^{\frac{1}{2}}, \cdots \text{ etc.},$$

satisfy (8) and (9).

Proof. We may assume that ρ is defined on $[0, \infty)$ such that G is concave on $[0, \infty)$.

First, we note $\int_{0+}G^{-1}(\eta)\,d\eta=\int_{0+}\rho^{-2}(e^{-\frac{1}{\eta}})\,e^{-\frac{2}{\eta}}\eta^{-3}d\eta=\int_{0+}\rho^{-2}(\xi)\cdot\xi\times\log\frac{1}{\xi}\,d\xi=\infty$.

Let $1=a_0>a_1>a_2>\cdots>a_m\cdots\to 0$ be defined by $\int_{a_m}^{a_{m-1}}G^{-1}(\eta)\,d\eta=2$. Then, there exists a twice continuously differentiable function $\psi_m(\xi)$ on $(-\infty, \infty)$ such that

$$\psi'_m(\xi)=\begin{cases}0 & \xi\leq a_m\\ \text{between } 0 \text{ and } 1 & a_m<\xi<a_{m-1}\\ 1 & \xi\geq a_{m-1}\end{cases}$$

and

$$\psi''_m(\xi)=\begin{cases}0 & \xi\leq a_m\\ \text{between } 0 \text{ and } G^{-1}(\xi) & a_m<\xi<a_{m-1}\\ 0 & \xi\geq a_{m-1}.\end{cases}$$

Let $f_m(x)=\psi_m\left(\left[\log\frac{1}{|x|}\right]^{-1}\right)$ for $x\in R^2$. Then $f_m(x)$ is twice continuously differentiable and $f_m(x)\uparrow\left[\log^+\frac{1}{|x|}\right]^{-1(*)}$ as $m\to\infty$.

Now let $\mathfrak{X}=(x, B_t)$, $\mathfrak{X}'=(x_t', B_t')$ be two solutions of (2) on the same probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ such that $x_0=x_0'$ and $B_t\equiv B_t'$. Then, by Itô's formula,

$$f_m(x_t-x_t')=\text{a martingale}+\frac{1}{2}\int_0^t[(f_m)_{x_1x_1}(x_s-x_s')$$

$$+(f_m)_{x_2x_2}(x_s-x_s')][a(s, x_s)-a(s, x_s')]^2 ds.$$

On the other hand,

$$(f_m)_{x_1x_1}(x)+(f_m)_{x_2x_2}(x)$$

$$=\psi_m'\left(\left[\log^+\frac{1}{|x|}\right]^{-1}\right)\frac{1}{\left(\log^+\frac{1}{|x|}\right)^3}\frac{1}{|x|^2}$$

$$+\psi_m''\left(\left[\log^+\frac{1}{|x|}\right]^{-1}\right)\cdot\frac{1}{\left[\log^+\frac{1}{|x|}\right]^4}\cdot\frac{1}{|x|^2}\quad.$$

Since ψ_m' is uniformly bounded,

$$E[f_m(x_t-x_t')]$$

$$\leq K\int_0^t E\left[I_{\{x_s\neq x_s'\}}\left(\log^+\frac{1}{|x_s-x_s'|}\right)^{-3}\cdot|x_s-x_s'|^{-2}\cdot\rho^2(|x_s-x_s'|)\right]ds$$

$$+\int_0^t E\left[\psi_m''\left(\left[\log^+\frac{1}{|x_s-x_s'|}\right]^{-1}\right)\cdot\left(\log^+\frac{1}{|x_s-x_s'|}\right)^{-4}\right.$$

$$\left.\times|x_s-x_s'|^{-2}\cdot\rho^2(|x_s-x_s'|)\right]ds$$

$$\equiv I_1+I_2,\quad\text{say.}$$

Noting that $\psi_m''(\xi)\leq G^{-1}(\xi)$, we have,

$$0\leq I_2\leq\int_0^t E\left[\left(\log^+\frac{1}{|x_s-x_s'|}\right)^3|x_s-x_s'|^2\rho^{-2}(|x_s-x_s'|)\right.$$

(*) $\log^+ x=(\log x)\vee 0$, $x>0$.

$$\times\left(\log^{+}\frac{1}{|x_s-x'_s|}\right)^{-4}|x_s-x'_s|^{-2}\rho^2(|x_s-x'_s|)$$

$$\times I_{\left\{a_m\leq\left[\log^{+}\frac{1}{|x_s-x'_s|}\right]^{-1}\leq a_{m-1}\right\}}\Bigg]ds$$

$$\leq\int_0^t E\left[I_{\left\{a_m\leq\left[\log^{+}\frac{1}{|x_s-x'_s|}\right]^{-1}\leq a_{m-1}\right\}}\cdot\left[\log^{+}\frac{1}{|x_s-x'_s|}\right]^{-1}\right]ds$$

$$\leq t\cdot a_{m-1}\to 0 \qquad \text{as } m\to\infty.$$

Then, letting $m\to\infty$, we have,

$$E\left[\left[\log^{+}\frac{1}{|x_t-x'_t|}\right]^{-1}\right]\leq K\cdot\int_0^t E\left\{G\left(\left[\log^{+}\frac{1}{|x_s-x'_s|}\right]^{-1}\right)\right\}ds \qquad (*).$$

Noting $G(\xi)$ is concave, we have, by Jensen's inequality,

$$(*)\quad \leq K\int_0^t G\left(E\left(\left[\log^{+}\frac{1}{|x_s-x'_s|}\right]^{-1}\right)\right)ds.$$

Since $\int_{0+}\frac{d\xi}{G(\xi)}=\infty$, we have $E\left[\left(\log^{+}\frac{1}{|x_t-x'_t|}\right)^{-1}\right]=0.$

Thus $x_t\equiv x_{t'}$. Q.E.D.

Remark 5. The condition (8) of the theorem is nearly best possible, in the sense that, if ρ is subadditive and $\int_{0+}\rho^{-2}(\xi)\xi\log\frac{1}{\xi}\,d\xi<\infty$, then there exists a 2×2-matrix $\sigma(t, x)$ of the form (10) such that

$$\|\sigma(t, x)-\sigma(t, y)\|\leq\rho(|x-y|)$$

for which the pathwise uniqueness does not hold. This can be shown in the same way as Remark 2.

In the general case of equations with drift terms, i.e.

(1) $dx_t=\sigma(t, x_t)\,dB_t+b(t, x_t)\,dt$,

corresponding to Theorems 1 and 2, we have the following two theorems.

Theorem 4. *Let* ρ *and* $\bar{\rho}$ *satisfy*

(11) $$\int_{0+}[\rho^2(\xi)\cdot\xi^{-1}+\bar{\rho}(\xi)]^{-1}d\xi=\infty,$$

(12) $$\rho^2(\xi)\xi^{-1}+\bar{\rho}(\xi) \quad \text{is concave.}$$

Then, for every $n\times r$*-matrix* $\sigma(t, x)$ *and* $n\times 1$*-matrix* $b(t, x)$, $(t, x)\in[0, \infty)\times R^n$ *such that*

$$\|\sigma(t, x)-\sigma(t, y)\|\leq\rho(|x-y|)$$

and

$$|b(t, x)-b(t, y)|\leq\bar{\rho}(|x-y|),$$

the pathwise uniqueness of solutions of (1) *holds.*

Theorem 5. *Let* ρ *and* $\bar{\rho}$ *satisfy*

(13) $$\int_{0+}\rho^{-2}(\xi)\, d\xi=\infty,$$

(14) $$\int_{0+}\bar{\rho}^{-1}(\xi)d\xi=\infty,$$

(15) $$\bar{\rho} \quad \text{is concave.}$$

Then, for every $1\times r$*-matrix* $\sigma(t, x)$ *and function* $b(t, x)$, $(t, x)\in[0, \infty)\times R^1$, *such that*

$$\|\sigma(t, x)-\sigma(t, y)\|\leq\rho(|x-y|)$$

and

$$|b(t, x)-b(t, y)|\leq\bar{\rho}(|x-y|),$$

the pathwise uniqueness of solutions of (1) *holds.*

These theorems can be proved in a similar way as above.

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