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On the degree of singularity of one-dimensional analytically irreducible noetherian local rings

By

Tadayuki MATSUOKA

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Let R be the local ring of an algebraic curve at a point, $\bar{R}$ its integral closure and $\mathfrak{c}$ the conductor of R in $\bar{R}$. The length $l(\bar{R}/R)$ of the R-module $\bar{R}/R$ is called the *degree of singularity* of R, and it has been proved by many algebraic geometers that if R is a Gorenstein ring, then the following equality holds:

(1) $\qquad l(\bar{R}/R)=l(R/\mathfrak{c})$, or equivalently $2l(\bar{R}/R)=l(\bar{R}/\mathfrak{c})$

(cf. the Introduction in [2]). Recently, in the case where R is a one-dimensional analytically irreducible (not necessarily geometric) local ring such that R and $\bar{R}$ have the same residue field, E. Kunz has proved that R is a Gorenstein ring if and only if the value semigroup of R is symmetric. In the course of the proof it is implicitly demonstrated that R is a Gorenstein ring if and only if the equality (1) holds (cf. [4]).

In this paper, under the same assumption as in the above result of Kunz, we shall prove the following theorem which contains the above result as a special case:

If R is a Macaulay local ring of type μ (i.e., $MC\mu$ ring in the sence of H. Bass [1], cf. [5]), then the following inequalities hold:

$$d \leq \delta \leq \mu d, \text{ or equivalently } 2d \leq c \leq (\mu+1)d$$

where $\delta = l(\bar{R}/R)$, $d = l(R/\mathfrak{c})$ and $c = l(\bar{R}/\mathfrak{c})$ (Theorem 2 in § 3).

§ 1. Preliminary.

Let R be a one-dimensional analytically irreducible noetherian local ring with maximal ideal $\mathfrak{m}$. Let $\bar{R}$ be the integral closure of R in the quotient field K and $\mathfrak{c}$ the conductor of R in $\bar{R}$. It is known that $\bar{R}$ is a finitely generated R-module and is a discrete valuation ring (for instance, see Exercise 1, § 33, Chap. V, [6]). Let v be the valuation of K with the valuation ring $\bar{R}$. We will use the following notations: For a subset S of K, $v\{S\} = \{v(x) \mid x \in S - 0\}$ and $v(S) = \inf\{v(x) \mid x \in S\}$. For an ideal $\mathfrak{a}$ in R, $\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subseteq R\}$ a fractional ideal of R in K. For an R-module M, $l(M)$ = the length of M. For a finite set F, $\sharp F$ = the number of the elements of F.

Let A be an r-dimensional Macauly local ring with maximal ideal $\mathfrak{n}$. We say that A is a Macaulay local ring of type μ if $\mu = l(\mathrm{Ext}_A^r(A/\mathfrak{n}, A))$. Hence A is a Gorenstein local ring if and only if A is a Macaulay local ring of type one.

We shall use later the following Rees' theorem (cf. [2] or [7]):

Let A be a noetherian local ring and $\mathfrak{a}$ an ideal in A. Let $x_1, \cdots, x_n$ be an A-sequence in $\mathfrak{a}$ and $\mathfrak{b}$ the ideal generated by $x_1, \cdots, x_n$. Then:

$$\mathrm{Ext}_A^p(A/\mathfrak{a}, A) = \begin{cases} 0, & 0 \leq p < n \\ \mathrm{Hom}_A(A/\mathfrak{a}, A/\mathfrak{b}), & p = n . \end{cases}$$

Throughout this paper, R is a one-dimensional analytically irreducible noetherian local ring such that R and $\bar{R}$ have the same residue field, and we will use constantly the same notation as above.

§ 2. The type of R.

Although the following two lemmas are contained in [4], we give the proofs for the convenience of the reader.

Lemma 1. *If* $v(x)=v(y)$ *for* x *and* y *in* K, *then* $v(x-ay)>v(y)$ *for some unit* a *in* R.

Proof. Since x/y is a unit in $\bar{R}$ and since R and $\bar{R}$ have the same residue field, there is a unit a in R such that $x/y-a$ is in the maximal ideal of $\bar{R}$. Hence we have $v(x-ay)>v(y)$. q.e.d.

Lemma 2. *Set* $c=v(\mathfrak{c})$. *Then* $\mathfrak{c}=\{x\in K \mid v(x)\geq c\}$ *and* $v\{R\}\supseteq\{c+n \mid n=0, 1, 2, \cdots\}$. *Moreover,* $c-1$ *is the largest integer not belonging to* $v\{R\}$.

Proof. The proof of the first part is easy and we omit it. Suppose that $c-1\in v\{R\}$. Let t be an element in $\bar{R}$ such that $v(t)=1$. By Lemma 1 we have $v(x-at^{c-1})>c-1$ for some x in R and for some unit a in R. Hence we have $t^{c-1}\in R$. Let y be an element in $\bar{R}$. By Lemma 1 there is an element z in R such that $v(y-z)>0$. Then $(y-z)t^{c-1}\in R$ because $v((y-z)t^{c-1})\geq c$. Hence we have $yt^{c-1}=(y-z)t^{c-1}+zt^{c-1}\in R$. This shows that $t^{c-1}\in\mathfrak{c}$. Therefore $c-1\geq v(\mathfrak{c})=c$. This is a contradiction. q.e.d.

Lemma 3. *Let* M *be an* R*-module such that* $K\supseteq M\supseteq R$. *If* $v\{M\}=v\{R\}$, *then* $M=R$.

Proof. Let x be an element in M. Since $v(x)=v(y)$ for some y in R, by successive applications of Lemma 1 there is an element z in R such that $v(x-z)\geq c$ where $c=v(\mathfrak{c})$. Hence by Lemma 2 we have $x\in R$. q.e.d.

We remark that if M is a finitely generated R-module contained in K, then $v(M)$ is an integer. In this case $v\{M\}-v\{R\}$ is a finite set.

Lemma 4. *Let M be a finitely generated R-module such that $K\supseteq M\supseteq R$ and $M\neq R$. Let $v\{M\}-v\{R\}=\{m_1,\cdots,m_\lambda\}$, $m_1<\cdots<m_\lambda$, and set $M_i'=\{x\in M\mid v(x)\geq m_i\}$. Let M_i be the R-module generated by M_i' and R. Then $v\{M_i\}=\{m_i,\cdots,m_\lambda\}\cup v\{R\}$ and $M_1=M$.*

Proof. Let x be an element in M_i and $x=x'+y$, $x'\in M_i'$ and $y\in R$. Since $v(x)=v(y)$ or $v(x)\geq v(x')\geq m_i$, we have the first assertion. Let x be an element in M such that $v(x)<m_1$. Since $v(x)\in v\{R\}$, by Lemma 1 we have $v(x-y)\geq m_1$ for some y in R. Hence we have $x-y\in M_1'$ and therefore we have $x\in M_1$. q.e.d.

Proposition 1. *Let M be a finitely generated R-module such that $K\supseteq M\supseteq R$. Then $l(M/R)=\#(v\{M\}-v\{R\})$.*

Proof. Set $\lambda=\#(v\{M\}-v\{R\})$. We proceed by induction on λ. In case when $\lambda=0$, by Lemma 3 our assertion is obvious. Assume that $\lambda>0$. Let $v\{M\}-v\{R\}=\{m_1,\cdots,m_\lambda\}$, $m_1<\cdots<m_\lambda$, and let M_i be the R-module defined in Lemma 4. Let N be a submodule of M such that N contains M_2 properly. We first show that $v\{N\}=v\{M\}$. Let x be an element in N such that $x\notin M_2$. Suppose that $v(x-y)\neq m_1$ for any y in R. Since $v(x)\in v\{R\}$, by Lemma 1 we have $v(x-z)\geq m_2$ for some z in R. Hence we have $x\in M_2$. This is a contradiction. Therefore $v(x-y)=m_1$ for some y in R, and this shows that $v\{N\}=v\{M\}$. Next we show that $N=M$. Let x be an element in M such that $v(x)<m_2$. If $v(x-y)=m_1$ for some y in R, then there is an element z in N such that $v(x-y)=v(z)$. By Lemma 1 we have $v(x-u)\geq m_2$ for some u in N, and hence we have $x\in N$. If $v(x-y)\neq m_1$ for any y in R, then $v(x)\in v\{R\}$, and hence by Lemma 1 there is an element z in R such that $v(x-z)\geq m_2$. Whence we also have $x\in N$. This shows that $N=M$. Therefore we have $l(M/M_2)=1$. On the other hand, by Lemma 4 $v\{M_2\}-v\{R\}=\{m_2,\cdots,m_\lambda\}$. Hence, by our induction hypothesis, $l(M_2/R)=\lambda-1$. Therefore we have $l(M/R)=\lambda$. q.e.d.

Corollary. *Let M be a finitely generated R-module such that $K\supseteq M\supseteq R$. Then for every submodule N of M such that $N\supseteq R$, $l(M/N)=\sharp(v\{M\}-v\{N\})$.*

Remark. Let r and s be integers in $v\{R\}$ such that $r\leq s$. Set $\mathfrak{a}=\{x\in R\mid v(x)\geq r\}$ and $\mathfrak{b}=\{x\in R\mid v(x)\geq s\}$. For $\mathfrak{a}$ and $\mathfrak{b}$, by the same way as the proof of Proposition 1, we have $l(\mathfrak{a}/\mathfrak{b})=\sharp(v\{\mathfrak{a}\}-v\{\mathfrak{b}\})$.

Theorem 1. *R is a Macaulay local ring of type $\sharp(v\{\mathfrak{m}^{-1}\}-v\{R\})$.*

Proof. Since R is a one-dimensional noetherian local integral domain, R is a Macaulay local ring. Let a be a non-zero element in $\mathfrak{m}$. Then $(aR:\mathfrak{m})=a\mathfrak{m}^{-1}$ (cf. Rèchenregel 4, §1, [3]). Hence by Rees' theorem we have:

$$\mathrm{Ext}_R^1(R/\mathfrak{m},R)\simeq(aR:\mathfrak{m})/aR=a\mathfrak{m}^{-1}/aR\simeq\mathfrak{m}^{-1}/R.$$

This shows that $l(\mathrm{Ext}_R^1(R/\mathfrak{m},R))=l(\mathfrak{m}^{-1}/R)$. Therefore our assertion follows from Proposition 1. q.e.d.

§3. The degree of singularity.

Let $v\{R\}=\{v_0,v_1,\cdots,v_{d-1}\}\cup\{n\in Z\mid n\geq c\}$, $0=v_0<v_1<\cdots<v_{d-1}<c=v(\mathfrak{c})$, where Z is the set of integers. Set $\mathfrak{a}_i=\{x\in R\mid v(x)\geq v_i\}$ and $\mathfrak{a}_d=\mathfrak{c}$. Obviously $\mathfrak{a}_i$ is an ideal in R and $\mathfrak{a}_0=R$, $\mathfrak{a}_1=\mathfrak{m}$. Next we remark that $\mathfrak{c}^{-1}=\bar{R}$[1]. In fact, let x be an element in $\mathfrak{c}^{-1}$. Suppose that $x\notin\bar{R}$. Since $v(x)<0$, we have $c-1-v(x)\geq c$. Let t be an element in $\bar{R}$ such that $v(t)=1$. By Lemma 2 we have $z=t^{c-1-v(x)}\in\mathfrak{c}$. Since $x\mathfrak{c}\subseteq R$, we have $xz\in R$. Hence $c-1=v(x)+v(z)\in v\{R\}$. This contradicts Lemma 2. Therefore $\mathfrak{c}^{-1}\subseteq\bar{R}$. Since the opposite inclusion is obvious, we have $\mathfrak{c}^{-1}=\bar{R}$.

Consider the following ascending chain of (fractional) ideals:

$$\mathfrak{c}=\mathfrak{a}_d\subset\mathfrak{a}_{d-1}\subset\cdots\subset\mathfrak{a}_1\subset R\subset\mathfrak{a}_1^{-1}\subset\cdots\subset\mathfrak{a}_{d-1}^{-1}\subset\mathfrak{a}_d^{-1}=\bar{R}. \tag{2}$$

1) We can show that the ideal $\mathfrak{a}_i$ is divisorial, i.e., $\mathfrak{a}_i=(\mathfrak{a}_i^{-1})^{-1}$ for every i.

Since $v\{\mathfrak{a}_{i-1}\}-v\{\mathfrak{a}_i\}=\{v_{i-1}\}$, we have $l(\mathfrak{a}_{i-1}/\mathfrak{a}_i)=1$ by the remark after Corollary to Proposition 1. Hence we have $l(R/\mathfrak{c})=d$. We also remark that $l(\bar{R}/\mathfrak{c})=c$ because R and $\bar{R}$ have the same residue field. Set $\delta=l(\bar{R}/R)$ and $\delta_i=l(\mathfrak{a}_i^{-1}/\mathfrak{a}_{i-1}^{-1})$. By the above chain (2) the following equalities hold:

$$(3)\qquad c=d+\delta \quad \text{and} \quad \delta=\sum_{i=1}^{d}\delta_i.$$

Proposition 2. *With the same notation as above, $c-1-v_{i-1}$ is the largest integer belonging to $v\{\mathfrak{a}_i^{-1}\}-v\{\mathfrak{a}_{i-1}^{-1}\}$. Therefore $\delta_i\geq 1$.*

Proof, Set $w=c-1-v_{i-1}$. By Lemma 2 $w+v_{i-1}=c-1\notin v\{R\}$, and hence we have $w\notin v\{\mathfrak{a}_{i-1}^{-1}\}$. Let x be an element in K such that $v(x)=w$. Let y be an element in $\mathfrak{a}_i$. If $v(y)=v_j$ for some j, $i\leq j\leq d-1$, then $v(xy)=c-1+(v_j-v_{i-1})\geq c$. If $v(y)\geq c$, then obviously $v(xy)\geq c$. Hence by Lemma 2 we have $xy\in R$ and whence $x\in\mathfrak{a}_i^{-1}$. This shows that $w\in v\{\mathfrak{a}_i^{-1}\}$. Therefore we have $w\in v\{\mathfrak{a}_i^{-1}\}-v\{\mathfrak{a}_{i-1}^{-1}\}$. Next we have to show that if $n\in v\{\mathfrak{a}_i^{-1}\}-v\{\mathfrak{a}_{i-1}^{-1}\}$, then $n\leq w$. In order to see this, it is enough to show that if $n>w$, then $n\in v\{\mathfrak{a}_{i-1}^{-1}\}$. Assume that $n>w$. Let z be an element in K such that $v(z)=n$. Let y be an element in $\mathfrak{a}_{i-1}$. If $v(y)=v_j$ for some j, $i-1\leq j\leq d-1$, then $v(zy)=n+v_j>w+v_j\geq c-1$. If $v(y)\geq c$, then $v(zy)\geq c$. Hence by Lemma 2 we have $zy\in R$, and whence $z\in\mathfrak{a}_{i-1}^{-1}$. This shows that $n\in v\{\mathfrak{a}_{i-1}^{-1}\}$. The second part follows from the first part and Corollary to Proposition 1. q.e.d.

Corollary. *With the same notation as above, the integer $c-1$ belongs to $v\{\mathfrak{m}^{-1}\}-v\{R\}$. Moreover, R is a Gorenstein ring if and only if $v\{\mathfrak{m}^{-1}\}=\{c-1\}\cup v\{R\}$.*

Proof. The first part is a special case of Proposition 2. The second part follows from the first part and Theorem 1. q.e.d.

Proposition 3. *With the same notation as above, if R has type μ, then the inequality $\delta_i \leq \mu$ holds for every i.*

Proof. Since $l(\mathfrak{a}_{i-1}/\mathfrak{a}_i)=1$, $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is isomorphic to $R/\mathfrak{m}$ as R-modules. Hence we have the exact sequence

$$0 \rightarrow R/\mathfrak{m} \rightarrow R/\mathfrak{a}_i \rightarrow R/\mathfrak{a}_{i-1} \rightarrow 0.$$

This exact sequence gives us the following long exact sequence:

$$\cdots \rightarrow \mathrm{Hom}_R(R/\mathfrak{m},\ R) \rightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}_{i-1},\ R) \rightarrow$$
$$\mathrm{Ext}^1_R(R/\mathfrak{a}_i,\ R) \rightarrow \mathrm{Ext}^1_R(R/\mathfrak{m},\ R) \rightarrow \cdots.$$

Since the conductor $\mathfrak{c}$ contains a non-zero element a, by Rees' theorem we have

$$\mathrm{Ext}^1_R(R/\mathfrak{a}_j,\ R) \simeq (aR:\ \mathfrak{a}_j)/aR = a\mathfrak{a}_j^{-1}/aR \simeq \mathfrak{a}_j^{-1}/R$$

and $\mathrm{Hom}_R(R/\mathfrak{a}_j,\ R)=0$ for every j. Therefore the above long exact sequence is reduced to the following exact sequence:

$$0 \rightarrow \mathfrak{a}_{i-1}^{-1}/R \rightarrow \mathfrak{a}_i^{-1}/R \rightarrow \mathfrak{m}^{-1}/R \rightarrow \cdots.$$

This shows that $\mathfrak{a}_i^{-1}/\mathfrak{a}_{i-1}^{-1}$ is isomorphic to a submodule of $\mathfrak{m}^{-1}/R$.[2] Therefore we have $\delta_i \leq \mu$. q.e.d.

Theorem 2. *With the same notation as above, if R has type μ, then the following inequalities hold*:

$$2d \leq c \leq (\mu+1)d.$$

Proof. Since $1 \leq \delta_i \leq \mu$ by Propositions 2 and 3, our assertions follow directly from the equalities (3).[3] q.e.d.

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2) In the proof of Lemma 1 in [8] P. Roquette proved this fact by a method different from ours.

3) In [4] E. Kunz proved the inequality $2d \leq c$ by considering the value-semigroup of R, and in [9] P. Samuel had already pointed out that if the embedding dimension of R is greater than two, it may happen the strict inequality $2d < c$.

Corollary. *With the same notation as above, R is a Gorenstein ring if and only if the equality $2\delta=c$ holds.*

Proof. The only if part follows from Theorem 2 and (3). Assume that $2\delta=c$. By (3) we have $d=\sum_{i=1}^{d}\delta_i$. Since $\delta_i\geq 1$, we have $\delta_i=1$ for every i and hence $\mu=\delta_1=1$. q.e.d.

§4. Examples.

In general, the equality $c=(\mu+1)d$, or equivalently $\delta=\mu d$, in the second inequality in Theorem 2 does not hold. However, it may happen that the equality does hold even if R is not a Gorenstein ring. Note that $\delta=\mu d$ if and only if $\delta_i=\mu$ for all i. Hence, for instance, if $d=1$, then the equality $\delta=\mu d$ trivially holds. To see these facts we give the following examples.

Let k be an algebraically closed field and t an analytically independent element over k. Let $n_1, \cdots, n_q$ be positive integers such that $\gcd(n_1, \cdots, n_q)=1$. Set $x_i=t^{n_i}$, $i=1, \cdots, q$. Let C be the affine algebraic curve with generic point $(x_1, \cdots, x_q)$ over k and R the local ring of C at the origin. Then R is an analytically irreducible local ring and the integral closure $\bar{R}$ in the quotient field $K(=k(t))$ is the regular local ring $k[t]_{(t)}$. Hence R and $\bar{R}$ have the same residue field k. Let r be an integer in $v\{R\}$. Set $\mathfrak{a}=\{x\in R\mid v(x)\geq r\}$ and $\mathrm{I}(\mathfrak{a})=\{n\in Z\mid n+v\{\mathfrak{a}\}\subseteq v\{R\}\}$. We show that

$$v\{\mathfrak{a}^{-1}\}=\mathrm{I}(\mathfrak{a}). \tag{4}$$

Proof of (4): Let y be an element in $\mathfrak{a}$ and write $y=f/g$ where f and g are in $k[x_1, \cdots, x_q]$ and $g\neq 0$ at the origin. We first note that two monomials in $x_1, \cdots, x_q$ coincide with each other if and only if they have the same value. Hence we can write

$$f=b_1 m_{r_1}(x)+\cdots+b_s m_{r_s}(x)$$

where $b_i\in k$, $b_1\neq 0$, $m_{r_i}(x)$ is a monomial in $x_1, \cdots, x_q$ with value r_i, and $v(y)=r_1<\cdots<r_s$. Since the value of $m_{r_i}(x)$ is not less than

r, $m_{r_i}(x)\in\mathfrak{a}$, whence $r_1, \cdots, r_s$ are in $v\{\mathfrak{a}\}$. Let n be an integer in $\mathrm{I}(\mathfrak{a})$. Since $n+r_i\in v\{R\}$, $t^n m_{r_i}(x)=m_{n+r_i}(x)$. Therefore $t^n f=\sum b_i m_{n+r_i}(x)\in R$, that is, $t^n y\in R$. This shows that $t^n\in\mathfrak{a}^{-1}$, whence $n\in v\{\mathfrak{a}^{-1}\}$. Thus we have $\mathrm{I}(\mathfrak{a})\subseteq v\{\mathfrak{a}^{-1}\}$. Since the opposite inclusion is obvious, we have the assertion.

By using (4) we can compute δ_i and μ because $\delta_i=\#(v\{\mathfrak{a}_i^{-1}\}-v\{\mathfrak{a}_{i-1}^{-1}\})$.

Example 1. In case where $q=3$ and $n_1=3, n_2=4, n_3=5$, we have $v\{R\}=\{0\}\cup\{n\in Z\,|\,n\geq 3\}$, $c=3$, $d=1$, $\mu=\delta_1=2$.

Example 2. In case where $q=3$ and $n_1=3, n_2=10, n_3=11$, we have $v\{R\}=\{0, 3, 6\}\cup\{n\in Z\,|\,n\geq 9\}$, $c=9$, $d=3$, $\mu=\delta_i=2$ for $i=1, 2, 3$.

Example 3. In case where $q=4$ and $n_1=11, n_2=12, n_3=13, n_4=15$, we have
$v\{R\}=\{0, 11, 12, 13, 15, 22, 23, 24, 25, 26, 27, 28, 30\}\cup\{n\in Z\,|\,n\geq 33\}$, $c=33$, $d=13$, $\mu=\delta_1=\delta_5=3$, $\delta_4=\delta_{12}=\delta_{13}=2$ and the others δ_i are all equal to 1.

In Examples 1 and 2, the equality $c=(\mu+1)d$ holds, though R is not a Gorenstein ring. In Example 3, the equality does not hold. The author knows few of examples such that R is not a Gorenstein ring and $d>1$, $c=(\mu+1)d$.

Addendum: Theorem 2 and Corollary to Theorem 2 can be generalized to the analytically unramified local ring case. We shall give the proof in a forthcoming paper.

EHIME UNIVERSITY

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Added in Proof: During the symposium on algebraic geometry held on September 6-9, 1971, at Zaô, Yamagata, Mr. K. Watanabe kindly informed me that the results of the present paper are closely related to Herzog and Kunz's latest work, Die Wertehalbgruppe eines lokalen Rings der Dimension 1, S.-B. Heidelberger Akad. Wiss. Math. -naturw. 1971, 2 Abh.. And I found that my paper has some considerable overlap with theirs, but my investigation had been done independently of this Herzog-Kunz's paper.