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On the degree of singularity of one-dimensional analytically unramified noetherian local rings

By

Tadayuki MATSUOKA

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§1. The purpose of this paper is to generalize the result of our previous paper [2] to the analytically unramified local ring case. Let R be a one-dimensional analytically unramified noetherian local integral domain with maximal ideal $\mathfrak{m}$. Let K be the quotient field of R, $\bar{R}$ the integral closure of R in K and $\mathfrak{c}$ the conductor of R in $\bar{R}$. It is clear that the length $l(\bar{R}/R)$ of the R-module $\bar{R}/R$ is finite since $\bar{R}$ is a finitely generated R-module. The length $l(\bar{R}/R)$ is called the *degree of singularity* of R (cf. [3]). Since $\mathfrak{c}$ is a non-zero ideal in R, the length $l(\bar{R}/\mathfrak{c})$ of the R-module $\bar{R}/\mathfrak{c}$ is finite, and similarly $l(R/\mathfrak{c})$ is finite. Set $\delta=l(\bar{R}/R)$, $c=l(\bar{R}/\mathfrak{c})$ and $d=l(R/\mathfrak{c})$. Since R is a one-dimensional Macaulay local ring, the length $l(\mathrm{Ext}_R^1(R/\mathfrak{m}, R))$ of the R-module $\mathrm{Ext}_R^1(R/\mathfrak{m}, R)$ is an invariant of R and is called the *type* of R (cf. [1]). Set $\mu=l(\mathrm{Ext}_R^1(R/\mathfrak{m}, R))$.

We shall prove the following theorem which is a generalization of Theorem 2 in [2].

Theorem. *The assumptions and notations being as above, suppose furthermore that $\bar{R}/\mathfrak{M}=R/\mathfrak{m}$ for all maximal ideals $\mathfrak{M}$ of $\bar{R}$. Then the following inequalities hold*:

$$(1+1/\mu)\delta \leq c \leq 2\delta-\mu+1.$$

Corollary. *With the same assumptions as in the above theorem, R is a Gorenstein ring if and only if $c=2\delta$.*[1)]

§2. In this section we shall prove the Theorem and the Corollary in §1. We will use the same notation as in §1. Throughout this section R is a one-dimensional analytically unramified noetherian local integral domain such that $\bar{R}/\mathfrak{M}=R/\mathfrak{m}$ for all maximal ideals $\mathfrak{M}$ of $\bar{R}$. We say that an ideal in R is a contracted ideal if it is the contraction of an ideal of $\bar{R}$. We first show the following:

Lemma 1. *There exists a strictly descending chain of contracted ideals in R:*

$$R=\mathfrak{a}_0\supset\mathfrak{a}_1\supset\cdots\supset\mathfrak{a}_{d-1}\supset\mathfrak{a}_d=\mathfrak{c}$$

where $d=l(R/\mathfrak{c})$.

Proof. Let c' be the length of the $\bar{R}$-module $\bar{R}/\mathfrak{c}$. Then there is a strictly descending chain of ideals in $\bar{R}$:

$$\bar{R}=\mathfrak{B}_0\supset\mathfrak{B}_1\supset\cdots\supset\mathfrak{B}_{c'-1}\supset\mathfrak{B}_{c'}=\mathfrak{c}.$$

Since $\mathfrak{B}_{i-1}/\mathfrak{B}_i$ is a simple $\bar{R}$-module, $\mathfrak{B}_{i-1}/\mathfrak{B}_i\simeq\bar{R}/\mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of $\bar{R}$. Hence, by our assumption, $\mathfrak{B}_{i-1}/\mathfrak{B}_i\simeq R/\mathfrak{m}$. This shows that $\mathfrak{B}_{i-1}/\mathfrak{B}_i$ is a simple R-module, i.e., $l(\mathfrak{B}_{i-1}/\mathfrak{B}_i)=1$[2)] and that $c'=c(=l(\bar{R}/\mathfrak{c}))$. Hence the R-module $\mathfrak{B}_i+R\cap\mathfrak{B}_{i-1}$ coincides with $\mathfrak{B}_i$ or $\mathfrak{B}_{i-1}$. Consider the chain of R-modules

$$(*)\qquad R=R\cap\mathfrak{B}_0\supset R\cap\mathfrak{B}_1\supset\cdots\supset R\cap\mathfrak{B}_{c-1}\supset R\cap\mathfrak{B}_c=\mathfrak{c}.$$

Since $(R\cap\mathfrak{B}_{i-1})/(R\cap\mathfrak{B}_i)\simeq(\mathfrak{B}_i+R\cap\mathfrak{B}_{i-1})/\mathfrak{B}_i$, we have either $R\cap\mathfrak{B}_{i-1}=R\cap\mathfrak{B}_i$ or $l((R\cap\mathfrak{B}_{i-1})/(R\cap\mathfrak{B}_i))=1$. Therefore we have the

1) In case when R is a locality over an algebraically closed field Serre has proved this fact by using differentials (cf. [4]).

2) We denote by $l(M)$ the length of an R-module M.

required chain by deleting the superfluous terms of the chain (*).

q. e. d.

Let $\mathfrak{a}$ be an ideal in R. We denote by $\mathfrak{a}^{-1}$ the fractional ideal of R consisting of the elements x in K such that $x\mathfrak{a}\subset R$.

Lemma 2. $\mathfrak{c}^{-1}=\bar{R}$.

Proof.[3] Since $\bar{R}$ is a semi-local Dedekind domain, $\bar{R}$ is a principal ideal domain (cf. [5]). Let $\mathfrak{c}=a\bar{R}$ and let x be an element in $\mathfrak{c}^{-1}$. Then we have $xa\in\mathfrak{c}=a\bar{R}$ and whence $x\in\bar{R}$. This shows that $\mathfrak{c}^{-1}\subset\bar{R}$. The opposite inclusion is obvious.

q. e. d.

Consider a *strictly* descending chain of *contracted* ideals in R:

$$R=\mathfrak{a}_0\supset\mathfrak{a}_1\supset\cdots\supset\mathfrak{a}_{d-1}\supset\mathfrak{a}_d=\mathfrak{c}$$

where $d=l(R/\mathfrak{c})$. Then $l(\mathfrak{a}_{i-1}/\mathfrak{a}_i)=1$ for $i=1,\cdots, d$, and we have the following chain of (fractional) ideals of R:

$$\bar{R}=\mathfrak{a}_d^{-1}\supset\mathfrak{a}_{d-1}^{-1}\supset\cdots\supset\mathfrak{a}_1^{-1}\supset R\supset\mathfrak{a}_1\supset\cdots\supset\mathfrak{a}_{d-1}\supset\mathfrak{a}_d=\mathfrak{c}.$$

Set $\delta_i=l(\mathfrak{a}_i^{-1}/\mathfrak{a}_{i-1}^{-1})$. We have the equalities:

$$c=d+\delta \quad\text{and}\quad \delta=\sum_{i=1}^{d}\delta_i \tag{1}$$

where $c=l(\bar{R}/\mathfrak{c})$ and $\delta=l(\bar{R}/R)$.

Let us now prove the following inequalities:

$$1\leq\delta_i\leq\mu \quad\text{for}\quad i=1,\cdots, d \tag{2}$$

where μ is the type of R, i.e., $\mu=l(\mathrm{Ext}_R^1(R/\mathfrak{m}, R))$.

The first inequality $1\leq\delta_i$ is the direct consequence of the follow-

3) The author's original proof is not simpler than the present one which is due to the referee.

ing two lemmas.

Lemma 3. *Let $\mathfrak{a}$ be an ideal in R and $\mathfrak{b}$ a contracted ideal in R. If $\mathfrak{b}$ is properly contained in $\mathfrak{a}$, then the extended ideal $\mathfrak{b}\bar{R}$ is also properly contained in $\mathfrak{a}\bar{R}$.*

Proof. Suppose that $\mathfrak{a}\bar{R}=\mathfrak{b}\bar{R}$. Then $\mathfrak{a}\subset\mathfrak{a}\bar{R}\cap R=\mathfrak{b}\bar{R}\cap R=\mathfrak{b}$, and this is a contradiction. q.e.d.

Lemma 4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R such that $\mathfrak{c}\subset\mathfrak{b}\subset\mathfrak{a}$. If $\mathfrak{a}\bar{R}\neq\mathfrak{b}\bar{R}$, then $l(\mathfrak{b}^{-1}/\mathfrak{a}^{-1})\geq 1$.*

Proof. Let $\mathfrak{M}_1,\cdots,\mathfrak{M}_s$ be the maximal ideals of the semi-local Dedekind domain $\bar{R}$. Let $\mathfrak{c}=\mathfrak{M}_1^{c_1}\cdots\mathfrak{M}_s^{c_s}$, $\mathfrak{b}\bar{R}=\mathfrak{M}_1^{n_1}\cdots\mathfrak{M}_s^{n_s}$ and $\mathfrak{a}\bar{R}=\mathfrak{M}_1^{m_1}\cdots\mathfrak{M}_s^{m_s}$. Since $\mathfrak{c}\subset\mathfrak{b}\bar{R}\subset\mathfrak{a}\bar{R}$, $c_i\geq n_i\geq m_i$ for all i. Since $\mathfrak{b}\bar{R}$ is properly contained in $\mathfrak{a}\bar{R}$, we may assume that $n_1>m_1$. Let $\mathfrak{A}$ be the ideal $\mathfrak{M}_1^{c_1-1-m_1}\mathfrak{M}_2^{c_2}\cdots\mathfrak{M}_s^{c_s}$ in $\bar{R}$. We first show that $\mathfrak{A}\subset\mathfrak{b}^{-1}$. Since the product $\mathfrak{A}\mathfrak{b}\bar{R}$ is $\mathfrak{M}_1^{c_1-1-m_1+n_1}\mathfrak{M}_2^{c_2+n_2}\cdots\mathfrak{M}_s^{c_s+n_s}$ and since $c_1-1-m_1+n_1\geq c_1$, $\mathfrak{A}\mathfrak{b}\bar{R}\subset\mathfrak{c}$. Hence $\mathfrak{A}\mathfrak{b}\subset R$ and this shows that $\mathfrak{A}\subset\mathfrak{b}^{-1}$. Next we must show that $\mathfrak{A}\not\subset\mathfrak{a}^{-1}$. Suppose that $\mathfrak{A}\subset\mathfrak{a}^{-1}$. Since $\mathfrak{A}\mathfrak{a}(=\mathfrak{A}\mathfrak{a}\bar{R})$ is an ideal in R and is also an ideal in $\bar{R}$ and since the conductor $\mathfrak{c}$ is the largest ideal in R which is also an ideal in $\bar{R}$, we have $\mathfrak{A}\mathfrak{a}\subset\mathfrak{c}$. On the other hand, $\mathfrak{A}\mathfrak{a}=\mathfrak{A}\mathfrak{a}\bar{R}=\mathfrak{M}_1^{c_1-1}\mathfrak{M}_2^{c_2+m_2}\cdots\mathfrak{M}_s^{c_s+m_s}$. Hence we have $c_1-1\geq c_1$, and this is a contradiction. Thus we have $\mathfrak{A}\not\subset\mathfrak{a}^{-1}$. Therefore $\mathfrak{a}^{-1}$ is properly contained in $\mathfrak{b}^{-1}$ and hence $l(\mathfrak{b}^{-1}/\mathfrak{a}^{-1})\geq 1$. q.e.d.

The proof of the second inequality $\delta_i\leq\mu$ in (2) is entirely the same as that of Proposition 3 in [2] and we omit it.

Notice that $\mu=\delta_1$ since $\mathfrak{m}=\mathfrak{a}_1$ (cf. [2]). The Theorem in §1 follows directly from (1) and (2) (see Remark 3 below).

Remark 1. In Lemma 1 the ideals $\mathfrak{a}_i$ are divisorial, i.e., $\mathfrak{a}_i=(\mathfrak{a}_i^{-1})^{-1}$. In fact, since $\mathfrak{a}_1=\mathfrak{m}$, we see easily that $\mathfrak{a}_1$ is divisorial. Hence, for $i>1$ it is sufficient to see that if $\mathfrak{a}_{i-1}$ is divisorial, then so is $\mathfrak{a}_i$. Since $\mathfrak{a}_i \subset \mathfrak{a}_{i-1}$, we have $\mathfrak{a}_i \subset (\mathfrak{a}_i^{-1})^{-1} \subset (\mathfrak{a}_{i-1}^{-1})^{-1} = \mathfrak{a}_{i-1}$. Suppose that $(\mathfrak{a}_i^{-1})^{-1} = (\mathfrak{a}_{i-1}^{-1})^{-1}$. Then $\mathfrak{a}_i^{-1} = ((\mathfrak{a}_i^{-1})^{-1})^{-1} = ((\mathfrak{a}_{i-1}^{-1})^{-1})^{-1} = \mathfrak{a}_{i-1}^{-1}$. This contradicts $l(\mathfrak{a}_i^{-1}/\mathfrak{a}_{i-1}^{-1}) \geq 1$. Hence $(\mathfrak{a}_i^{-1})^{-1}$ is properly contained in $\mathfrak{a}_{i-1}$. Since $l(\mathfrak{a}_{i-1}/\mathfrak{a}_i) = 1$, we have $\mathfrak{a}_i = (\mathfrak{a}_i^{-1})^{-1}$.

Remark 2. Since $\mu = \delta_1 \leq \delta$, we have $\delta + 1 \leq (1 + 1/\mu)\delta$. Hence the inequalities in the Theorem in §1 are better than the inequalities $\delta + 1 \leq c \leq 2\delta$ in [4], and the lower bound $(1 + 1/\mu)$ for the ratio c/δ is obviously the best possible. It may happen that $(1 + 1/\mu)\delta = c < 2\delta$ (see Examples 1 and 2 in [2]).

Remark 3.[4)] Since $\delta - \mu = \sum_{i=2}^{d} \delta_i \geq d - 1$ by (2), we have the inequality $\mu \leq 2\delta - c + 1$ (cf. Bemerkung b) in §2, [6]).

Ehime University

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4) This remark was added on September, 1971 and the second inequality in the Theorem in §1 was amended to the present form.