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Exponential decay of solutions for the wave equation in the exterior domain with spherical boundary

By

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§ 1. Introduction

The initial boundary value problem which we are concerned with is the following one:

$$w_{tt}-(w_{xx}+w_{yy}+w_{zz})=0, \text{ in } \Omega, \quad t>0 \tag{1. 1}$$

$$w(X, 0)=f_1(X), \quad w_t(X, 0)=f_2(X), \tag{1. 2}$$

$$\left(\frac{d}{dr}+\sigma\right)w(X, t)=0, \quad X\in\partial\Omega \tag{1. 3}$$

where $X=(x, y, z)\in R^3$, Ω is the exterior of the unit sphere, $\partial\Omega$ is its boundary and σ is a real constant. $f_1(X)$ and $f_2(X)$ have compact supports in $\partial\Omega\cap\Omega$ and are sufficiently smooth and satisfy the following compatibility conditions, $\left(\frac{d}{dr}+\sigma\right)f_1(X)=\left(\frac{d}{dr}+\sigma\right)f_2(X)=\left(\frac{d}{dr}+\sigma\right)\Delta f_1(X)=0$, $X\in\partial\Omega$.

Then the solution $w(X, t)$ of (1,1), (1,2) and (1,3) is written $w(X, t)=v(X, t)+u(X, t)$ where $v(X, t)$ is the solution of (1,1) and (1.2) is in R^3 instead of Ω. It is well-known that $v(X, t)=0$ for sufficiently large t. Hence the behavior of $w(X, t)$ for large t is decided by that of $u(X, t)$. Next $u(X, t)$ is the solution of

(1. 1)′ $u_{tt}-(u_{xx}+u_{yy}+u_{zz})=0$, in Ω, $t>0$,

(1. 4) $u(X, 0)=u_t(X, 0)=0$,

(1. 5) $\left(\frac{d}{dr}+\sigma\right)u(X, t)=-\left(\frac{d}{dr}+\sigma\right)v(X, t), \quad X\in\partial\Omega.$

The purpose of this paper is to show the exponential decay of $u(X, t)$ when t tends to infinity under the conditioned mentioned below.

Assumption 1. *σ is real and $\sigma<1$.*

Assumption 2. *We denote the right side of (1.5) by $f(\omega, t)$. Then compatibility conditions imply $f(\omega, 0)=f_t(\omega, 0)=f_{tt}(\omega, 0)=0$. We assume that $f(\omega, t)$ is sufficiently smooth, for example it suffices that $f(\omega, t)$ is of class C^8 on the product $\partial\Omega\times[0, \infty]$, and that there exists $T_0(>0)$ such that $f(\omega, t)=0$ for $t>T_0$.*

The above result was announced with a short proof by C. Wilcox [4] in the case of the first boundary value problem. Our method is essentially similar to that of [4] except for some additional considerations.

We state the Theorem and give its proof in § 2 assuming two Lemmas 1 and 2. These lemmas on the asymptotic behavior of zeros of modified Bessel functions of large order are established in §§ 3 and 4. In § 5 we apply this method to an equation of the fourth order and we show the exponential approach of the solution to a constant state by its explicit formula.

§ 2. Statement and proof of the Theorem

Theorem. *The solution $u(X, t)$ satisfying (1.1)′, (1.4) and (1.5) with the Assumptions 1 and 2 decays exponentially with t. Namely there exists a positive μ such that for fixed $X\in\Omega$,*

(2. 1) $$u(X, t)=0(e^{-\mu t}),$$

where μ is determined only by σ.

Proof. We begin with the construction of the formal solution. Introducing polar coordinates we expand $u(r, \omega, t)$ and $f(\omega, t)$ in spherical harmonics as follows

$$u(r, \omega, t)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} u_{n,m}(r, t)Y_{n,m}(\omega) \tag{2. 2}$$

$$f(\omega, t)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} f_{n,m}(t)Y_{n,m}(\omega), \tag{2. 3}$$

where $\{Y_{n,m}(\omega)\}_{m=1,2,\cdots,2n+1}$ are normalized spherical harmonics of order n. We remark that the series (2.3) is uniformly convergent with respect to (ω, t) because $f(\omega, t)$ is smooth. On the other hand the expansion (2.2) is formal.

Next we perform the Laplace transformation with respect to t,

$$\hat{u}_{n,m}(r, \lambda)=\int_0^{\infty} e^{-\lambda t}u_{n,m}(r, t)dt, \tag{2. 4}$$

$$\hat{f}_{n,m}(\lambda)=\int_0^{\infty} e^{-\lambda t}f_{n,m}(t)dt, \tag{2. 5}$$

where $R_e\, \lambda$ is positive.

An easy computation shows that

$$\hat{u}_{n,m}(r, \lambda)=f_{n,m}(\lambda)\Phi_n(r, \lambda), \tag{2. 6}$$ [1]

where

$$\Phi_n(r, \lambda)=\frac{(\lambda r)^{-\frac{1}{2}}K_{n+\frac{1}{2}}(\lambda r)}{\lambda^{\frac{1}{2}}K'_{n+\frac{1}{2}}(\lambda)+\left(\sigma-\frac{1}{2}\right)\lambda^{-\frac{1}{2}}K_{n+\frac{1}{2}}(\lambda)}, \tag{2. 7}$$

$$K_{n+\frac{1}{2}}(\lambda)=\left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}}e^{-\lambda}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!(2\lambda)^k}. \tag{2. 8}$$

1) Substituting (2.2) into (1.1)′ we obtain

$$\left\{\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}+\frac{n(n+1)}{r^2}\right\}u_{n,m}(r, t)=0,$$

$$\left(\frac{d}{dr}+\sigma\right)u_{n,m}(1, t)=f_{n,m}(t).$$

We remark that $\hat{u}_{n,m}(r, \lambda)$ has not the term of $(\lambda r)^{-\frac{1}{2}}I_{n+\frac{1}{2}}(\lambda r)$.

$K_{n+\frac{1}{2}}(\lambda)$ is a solution of the modified Bessel equation.

We consider the inverse Lapalace transform of $\hat{u}_{n,m}(r, \lambda)$. We first remark that $u(X, t)$ is identically zero if $t-r+1<0$. We consider the expression of $u(X, t)$ when $t-r+1\geq 0$. Fixing (r, t) in this region, let us divide (2.5) into two parts,

$$\hat{f}_{n,m}(\lambda)=\int_0^{t-r+1} e^{-\lambda\tau} f_{n,m}(\tau)d\tau+\int_{t-r+1}^{\infty} e^{-\lambda\tau} f_{n,m}(\tau)d\tau. \tag{2. 9}$$

We denote the first term by $\hat{f}^{(1)}_{n,m}(\lambda)$ and the second by $\hat{f}^{(2)}_{n,m}(\lambda)$. Then $\hat{f}^{(1)}_{n,m}(\lambda)$ and $\hat{f}^{(2)}_{n,m}(\lambda)$ are holomorphic in the whole domain and in $R_e\lambda>0$ respectively. The following estimates are obtained by means of integration by parts.

$$|\hat{f}^{(1)}_{n,m}(\lambda)|\leqq C|\lambda|^{-1}e^{-R_e\lambda(t-r+1)}, \text{ if } t-r+1\geqq 0,\ R_e\lambda<0. \tag{2. 10}$$

$$|\hat{f}^{(2)}_{n,m}(\lambda)|\leqq C|\lambda|^{-1}e^{-R_e\lambda(t-r+1)}, \text{ if } t-r+1\geqq 0,\ R_e\lambda>0. \tag{2. 11}$$

C is a positive constant.

We see that the number of the poles of $\Phi_n(r, \lambda)$ is at most $n+1$ from (2.7) and (2.8). We state two Lemmas 1 and 2 concerning these poles of $\Phi_n(r, \lambda)$ which will be proved in §§ 3 and 4.

Lemma 1. *$\Phi_n(r, \lambda)$ $(n=0, 1, 2, \ldots)$ is a meromorphic function of λ in the whole complex plane for fixed $r (r>1)$. All its poles lie in $R_e\lambda<0$ and they are at most of order 2 and simple if $n>-\sigma$.*

Lemma 2. *We have the following estimates for the poles $\lambda_n^s (s=1, 2, \ldots, n+1)$. If we choose n_0 sufficiently large there exists a constant $A, B (>0)$ such that*

$$R_e\lambda_n^s<-An^{\frac{1}{3}}, \tag{2. 12}$$

$$|\lambda_n^s|\leqq Bn, \tag{2. 13}$$

for $n\geqq n_0$ and $1\leqq s\leqq n+1$.

Lemma 1 and 2 give

Corollary. *There exists a positive number* μ *such that*

$$R_e\lambda_n^s<-\mu, \quad \text{for all } n \text{ and } s. \tag{2.14}$$

In order to calculate the inverse Laplace transform we divide the integral into two parts taking account of (2.9),

$$\begin{aligned} u_{n,m}(r,t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\hat{u}_{n,m}(r,\lambda)d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\hat{f}_{n,m}^{(1)}(\lambda)\Phi_n(r,\lambda)d\lambda \\ &+ \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\hat{f}_{n,m}^{(2)}(\lambda)\Phi_n(r,\lambda)d\lambda \\ &= u_{n,m}^{(1)}(r,t)+u_{n,m}^{(2)}(r,t) \end{aligned} \tag{2.15}$$

where γ is positive.

1) the case of $u_{n,m}^{(1)}(r,t)$: Let us replace the path of integration $R_e\lambda=\gamma(>0)$ by the line $R_e\lambda=-M$ with sufficiently large $M(>0)$. For this, take the rectangle R_L of vertices $\gamma\pm Li$, $-M\pm Li$ with a large $L(>0)$ such that all the poles of $\Phi_n(r,\lambda)$ are included in R_L. Then we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{R_L} e^{\lambda t}\hat{f}_{n,m}^{(1)}(\lambda)\Phi_n(r,\lambda)d\lambda \\ &= \sum_{s=1}^{n+1}\operatorname*{Res}_{\lambda=\lambda_n^s} e^{\lambda t}\hat{f}_{n,m}^{(1)}(\lambda)\Phi_n(r,\lambda). \end{aligned} \tag{2.16}$$

$\Phi_n(r,\lambda)$ is expanded in the neighborhood of $\lambda=\infty$ from (2.7) and (2.8).

$$\Phi_n(r,\lambda)=\frac{-e^{-\lambda(r-1)}}{r}\left\{\frac{1}{\lambda}+\frac{(\sigma-1)+2^{-1}n(n+1)(r-1)}{\lambda^2}+\dots\right\}. \tag{2.17}$$

(2.10) and (2.17) allow us to make L, $M\to+\infty$ in (2.16).

We have

$$u^{(1)}_{n,m}(r, t)=\sum_{s=1}^{n+1}\operatorname*{Res}_{\lambda=\lambda_n^s} e^{\lambda t}\hat{f}^{(1)}_{n,m}(\lambda)\Phi_n(r, \lambda). \tag{2. 18}$$

2) the case of $u^{(2)}_{n,m}(r, t)$: In this case we replace the path of integration $R_e\lambda=\gamma$ by $R_e\lambda=M(>\gamma)$. By Lemma 1 there is no poles of $\Phi_n(r, \lambda)$ between these two lines. Letting $M\to+\infty$, we have

$$u^{(2)}_{n,m}(r, t)=0. \tag{2. 19}$$

From (2.15), (2.18) and (2.19) we obtain

$$u_{n,m}(r, t)=\sum_{s=1}^{n+1}\operatorname*{Res}_{\lambda=\lambda_n^s} e^{\lambda t}\hat{f}^{(1)}_{n,m}(\lambda)\Phi_n(r, \lambda). \tag{2. 20}$$

Substituting this result in (2.2) we obtain

$$\begin{aligned} &u(r, \omega, t) \\ &=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(\sum_{s=1}^{n+1}\int_0^{t-r+1} e^{\lambda_n^s(t-\tau)}f_{n,m}(\tau)d\tau \operatorname*{Res}_{\lambda=\lambda_n^s}\Phi_n(r, \lambda)\right)Y_{n,m}(\omega)\end{aligned} \tag{2. 21}$$

for (r, ω, t) lying in the region $t-r+1\geqq 0$.

The series (2.21) and its derivatives up to order 2 all converge uniformly on any compact set of $\Omega\times[0, \infty]$ on account of Lemma 2, the smoothness of $f(\omega, t)$, and of

$$\left|\left(\frac{d}{dr}\right)^j \operatorname*{Res}_{\lambda=\lambda_n^s}\Phi_n(r, \lambda)\right|=0\,(n^{\frac{1}{2}+j}), \quad (j=0, 1, 2)\ ^{2)} \tag{2. 22}$$

$$\left|\left(\frac{\partial}{\partial\omega}\right)^{\nu} Y_{n,m}(\omega)\right|=0\,(n^{\frac{1}{2}+|\nu|}), \quad (|\nu|=0, 1, 2)\ ^{3)} \tag{2. 23}$$

Hence the series (2.21) is the genuine solution of (1,1)′, (1.4) and (1.5).

Now we show that $u(X, t)$ decays exponentially with $t(t\to+\infty)$.

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2) (2. 22) is derived by using the asymptotic expansion of $K_{n+\frac{1}{2}}(\lambda r)$ and its derivative. See §4.

3) See the paper by A.P. Calderon and A. Zygmund: Amer. J. Math. 79(1957) 901–921.

If we take t such that $t \geqq 2T_0 + r - 1$, then from Lemma 2

$$\int_0^{t-r+1} |e^{\lambda_n^s(t-\tau)} f_{n,m}(\tau)| d\tau$$

$$\leqq \left(\int_0^{T_0} e^{2R_e\lambda_n^s(t-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{T_0} |f_{n,m}(\tau)|^2 d\tau \right)^{\frac{1}{2}}$$

$$\leqq \left(\frac{e^{R_e\lambda_n^s t}}{2\mathrm{A}n^{\frac{1}{3}}} \right)^{\frac{1}{2}} \left(\int_0^{T_0} \int_{\partial\Omega} |f(\omega, \tau)|^2 d\omega d\tau \right)^{\frac{1}{2}}.$$

If A' is chosen in such a way that for sufficiently large n
$2\mu - An^{\frac{1}{3}} > -A'n^{\frac{1}{3}}$, then we have

$$\frac{e^{-2\mu t} e^{(2\mu - An^{\frac{1}{3}})t}}{2\mathrm{A}n^{\frac{1}{3}}} < \frac{e^{-2\mu t} e^{-A'n^{\frac{1}{3}}}}{2\mathrm{A}n^{\frac{1}{3}}}$$

Using these results we see that $|u(r, \omega, t)| = 0(e^{-\mu t})$, which proves the theorem.

Remark 2.1. We notice that the result of the theorem is still true when (1,1)′ and (1.5) are replaced by

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a^2} \Delta \right) u(X, t) = 0, \quad \text{in } r > \rho,$$

$$\left(\frac{d}{dr} + \sigma \right) u(X, t) = f(\rho\omega, t), \quad |X| = \rho.$$

In this case the condition on σ is $\sigma < \frac{1}{\rho}$.

Remark 2.2. One can prove in the similar way that all the first derivatives of $u(X, t)$ also decay exponentially with respect to t for fixed X.

§ 3. Proof of Lemma 1

We prove Lemma 1 by three steps. we remark that $\Phi_n(r, \lambda)$ is holomorphic at $\lambda=0$ because of $\sigma<1$ and $\Phi_n(r, 0)=\frac{1}{r^{n+1}(\sigma-n-1)}$. Hence it suffices to investigate the zeros of the denominator $D_n(\lambda)$ of $\Phi_n(r, \lambda)$,

$$D_n(\lambda)=\lambda K'_{n+\frac{1}{2}}(\lambda)+\left(\sigma-\frac{1}{2}\right)K_{n+\frac{1}{2}}(\lambda), \quad (n=0, 1, \ldots). \tag{3. 1}$$

1) *$D_n(\lambda)$ has no zero on the imaginary axis.*

Proof. Assume that $\lambda=is(s\neq 0$, real) is a zero of $D_n(\lambda)$. Remembering the relation

$$K_{n+\frac{1}{2}}(\lambda)=kH^{(1)}{}_{n+\frac{1}{2}}(i\lambda), \quad k=\frac{\pi i}{2}e^{\frac{1}{2}(n+\frac{1}{2})\pi i} \tag{3. 2}$$

we have

$$\begin{aligned} &isK'_{n+\frac{1}{2}}(is)+\left(\sigma-\frac{1}{2}\right)K_{n+\frac{1}{2}}(is) \\ &=k\left[\left(\sigma-\frac{1}{2}\right)J_{n+\frac{1}{2}}(-s)-sJ'_{n+\frac{1}{2}}(-s)+\right. \\ &\left. i\left\{\left(\sigma-\frac{1}{2}\right)Y_{n+\frac{1}{2}}(-s)-sY'_{n+\frac{1}{2}}(-s)\right\}\right]=0. \end{aligned} \tag{3. 3}$$

The following expression is well-known as Lommel's formula,

$$J_{n+\frac{1}{2}}(-s)Y'_{n+\frac{1}{2}}(-s)-J'_{n+\frac{1}{2}}(-s)Y_{n+\frac{1}{2}}(-s)=\frac{2}{\pi(-s)}\neq 0. \tag{3. 4}$$

From (3.3) and (3.4) we derive $s=0$.

2) *$D_n(\lambda)$ has no zero in $R_e\lambda>0$.*

Proof. Assume that $\lambda=\lambda_0(R_e\lambda_0>0)$ satisfies

$$D_n(\lambda_0)=0. \tag{3. 5}$$

Let us define the function $L_n(x)$ by

$$L_n(x)=K_{n+\frac{1}{2}}(\lambda_0 x). \tag{3. 6}$$

$L_n(x)$ satisfies the following differential equation

$$(3.\ 7)\qquad \frac{d^2}{dx^2}L_n(x)+\frac{1}{x}\frac{d}{dx}L_n(x)-\left\{\lambda_0^2+\frac{\left(n+\frac{1}{2}\right)^2}{x^2}\right\}L_n(x)=0,$$

and the boundary condition (from (3.5))

$$(3.\ 8)\qquad L'_n(1)=-\left(\sigma-\frac{1}{2}\right)L_n(1).$$

Reducing $\int_1^\infty x(L''_n(x)\overline{L_n(x)}-\overline{L''_n(x)}L_n(x))dx$ by means of integration by parts and by (3.7), we obtain from (3.8)

$$(3.\ 9)\qquad (\lambda_0^2-\bar{\lambda}_0^2)\int_1^\infty x|L_n(x)|^2dx=0.$$

Since $L_n(x)$ is a solution of (3.7), $L_n(x)\not\equiv 0$. Consequently we obtain $\lambda_0=\bar{\lambda}_0$. From this we see that $D_n(\lambda)$ has no zero in $R_e\lambda>0$ except on the real axis.

Next we assume (3.5) for a positive λ_0. In the similar way as above, we have

$$(3.\ 10)\qquad \int_1^\infty\left\{\lambda_0^2xL_n(x)^2+\frac{\left(n+\frac{1}{2}\right)^2}{x}L_n(x)^2+xL'_n(x)^2\right\}dx$$
$$=\left(\sigma-\frac{1}{2}\right)L_n(1)^2.$$

On the other hand, we have by Schwartz' inequality

$$(3.\ 11)\qquad L_n(1)^2=-\int_1^\infty(L_n(x)^2)'dx\leqq 2\int_1^\infty\left(xL'_n(x)^2+\frac{\frac{1}{4}}{x}L_n(x)^2\right)dx.$$

(3.10) and (3.11) lead us to $1<\sigma$. This contradicts to the Assumptin 1.

3) *The zeros of* $D_n(\lambda)$ *are at most of order 2. They are simple for* $-\sigma<n$.

Proof. Assume that $\lambda=\lambda_0(R_e\lambda_0<0)$ is the zero of order$\geqq 3$. That is

$$D_n(\lambda_0)=D'_n(\lambda_0)=D''_n(\lambda_0)=0. \tag{3. 12}$$

Since $K_{n+\frac{1}{2}}(\lambda)$ is a solution of the modified Bessel equation, we can eliminate $K'''_{n+\frac{1}{2}}(\lambda_0)$ and $K''_{n+\frac{1}{2}}(\lambda_0)$ in (3.12). In order that we have $K_{n+\frac{1}{2}}(\lambda_0)\neq 0$ or $K'_{n+\frac{1}{2}}(\lambda_0)\neq 0$, it is necessary and sufficient that

$$\lambda_0^2=\left(n+\frac{1}{2}\right)^2-\left(\sigma-\frac{1}{2}\right)^2 \quad \text{and} \tag{3. 13}$$

$$\lambda_0^2=\left(\sigma-\frac{1}{2}\right)^2-\left(n+\frac{1}{2}\right)^2. \tag{3. 14}$$

Hence we have $\lambda_0=0$ contradicting to Re $\lambda_0<0$. This proves the first half of 3).

If $-\sigma<n$, from (3.14) we have $\lambda_0^2<0$. This means that λ_0 lies on the imaginary axis contradicting to 1).

Summarizing 1), 2) and 3) we obtain finally Lemma 1.

§ 4. Proof of Lemma 2

In this section we investigate the distribution of zeros of $D_n(\lambda)$ when n is large. Rewriting $D_n(\lambda)$ by (3.2) we have

$$D_n(\lambda)=i\lambda kH^{(1)\prime}{}_{n+\frac{1}{2}}(i\lambda)+\left(\sigma-\frac{1}{2}\right)kH^{(1)}{}_{n+\frac{1}{2}}(i\lambda). \tag{4. 1}$$

If we put

$$F_{n+\frac{1}{2}}(\lambda)=\lambda H^{(1)\prime}{}_{n+\frac{1}{2}}(\lambda)+\left(\sigma-\frac{1}{2}\right)H^{(1)}{}_{n+\frac{1}{2}}(\lambda), \tag{4. 2}$$

the zeros of (4.1) are equal to those of $F_{n+\frac{1}{2}}(\lambda)$ multiplied by i^{-1}. Therefore we investigate the zeros of $F_{n+\frac{1}{2}}(\lambda)$. We remark that the zeros of $F_{n+\frac{1}{2}}(\lambda)$ are situated symmetrically with respect to the imaginary axis because of

$$\lambda e^{\pm\pi i}H^{(1)\prime}_{n+\frac{1}{2}}(\lambda e^{\pm\pi i})+\left(\sigma-\frac{1}{2}\right)H^{(1)}_{n+\frac{1}{2}}(\lambda e^{\pm\pi i})$$
$$=\pm(-1)^{n}i\left\{\bar{\lambda}H^{(1)\prime}_{n+\frac{1}{2}}(\bar{\lambda})+\left(\sigma-\frac{1}{2}\right)H_{n+\frac{1}{2}}(\bar{\lambda})\right\}.$$

From this we know that it suffices to investigate $\left[\frac{n+1}{2}\right]+1$ zeros lying in the right half plane.

Let us replace $n+\frac{1}{2}$ by n and λ by nz in $F_{n+\frac{1}{2}}(\lambda)$. We construct the asymptotic expansions of $F_n(nz)$ with respect to positive parameter n using the results of *F.W.* Olver [2], [3]. We take only the first term of the asymptotic expansion with respect to n and obtain the following

$$F_n(nz)=nzH^{(1)\prime}_{n}(nz)+\left(\sigma-\frac{1}{2}\right)H^{(1)}_{n}(nz) \tag{4.3}$$
$$=\Phi(\zeta, n, \sigma)(A'(\eta)-n^{-\frac{2}{3}}A(\eta)\psi(\zeta, n)),$$

where the relation between z, ζ and η are

$$\zeta\left(\frac{d\zeta}{dz}\right)^2=\frac{1-z^2}{z^2}, \qquad \eta=e^{\frac{2}{3}\pi i}n^{\frac{2}{3}}\zeta. \tag{4.4}$$

We can take a positive constant n_1, such that for $n\geqq n_1$, $\Phi(\zeta, n, \sigma)\neq 0$ and

$$|\psi(\zeta, n)|\leqq C(1+|\zeta|^{\frac{1}{2}}).\text{[4)]} \tag{4.5}$$

$A(\eta)$ is a solution of the following differential equation

$$\frac{d^2A(\eta)}{d\eta^2}=\eta A(\eta). \tag{4.6}$$

This is the Airy function and denoted as $A_i(\eta)$ in [2], [3]. The problem is reduced to investigate the zeros of $f_n(\eta)$ for $n\geqq n_1$, where $f_n(\eta)$ represents the second factor of (4.3),

4) We use the symbol C in order to represent positive constants.

(4. 7) $$f_n(\eta)=A'(\eta)-n^{-\frac{2}{3}}A(\eta)\psi(\zeta, n).$$

The following properties are known (see [2]).

All the zeros of $A'(\eta)$ are real, negative and simple. If we denote the sth zero by η'_s, we have

(4. 8) $$\eta'_s=-\left\{\frac{3}{8}\pi(4s-3)\right\}^{\frac{2}{3}}+0(s^{-\frac{4}{3}})$$

for sufficiently large s.

From (4.8) we obtain

(4. 9) $$\eta'_{[\frac{n+1}{2}]+1}\sim-\left(\frac{3}{4}\pi\right)^{\frac{2}{3}}n^{\frac{2}{3}}.$$

Since it is shown in the following that the zeros of $f_n(\eta)$ approach to those of $A'(\eta)$, it sufficies to discuss in the disk of the radius $K\, n^{\frac{2}{3}}$ where K is a opsitive number larger than $\left(\frac{3}{4}\pi\right)^{\frac{2}{3}}$.
That is, $K_{\eta,n}=\{\eta;\ |\eta|<Kn^{\frac{2}{3}}\}$. The corresponding domain in the ζ-plane is $K_{\zeta,n}=\{\zeta;\ |\zeta|<K\}$.
When $\zeta\in K_{\zeta,n}$ we obtain by (4.5)

(4. 10) $$|\psi(\zeta, n)|<C, \quad \text{for} \quad n\geqq n_1.$$

We define $g_n(\eta)$ as follows $f_n(\eta)=A'(\eta)\,(1-g_n(\eta))$ where

(4. 11) $$g_n(\eta)=\frac{A(\eta)}{n^{\frac{2}{3}}A'(\eta)}\psi(\zeta, n).$$

Look at any fixed zero $\eta=\eta'_s$ of $A'(\eta)$. Then there exist holomorphic $A_1(\eta)$, $A_2(\eta)$ such that

(4. 12) $$A(\eta)=A(\eta'_s)+A_1(\eta)\,(\eta-\eta'_s)^2,$$

(4. 13) $$A'(\eta)=\eta'_s A(\eta'_s)\,(\eta-\eta'_s)+A_2(\eta)\,(\eta-\eta'_s)^2.$$

Substituting (4.10), (4.12) and (4.13) in (4.11) we obtain

(4. 14) $$|g_n(\eta)|\leqq\frac{C}{n^{\frac{2}{3}}|\eta_s'(\eta-\eta_s')|}|h(\eta)|,$$

where

(4. 15) $$h(\eta)=\frac{1+A(\eta_s')^{-1}A_1(\eta)(\eta-\eta_s')^2}{1+A(\eta_s')^{-1}A_2(\eta)\eta_s'^{-1}(\eta-\eta_s')}.$$

Next we estimate $h(\eta)$ on a circle.
From (4.8) we have

(4. 16) $$|\eta_{s+1}'-\eta_s'|\leqq Cs^{-\frac{1}{3}},\quad s\geqq s_1.$$

we have from Cauchy's integral formula

(4. 17) $$|A_1(\eta)|\leqq\frac{1}{2\pi}\int_{|t-\eta_s'|=\rho_s}\frac{|A(t)|}{|t-\eta_s'|^2|t-\eta|}|dt|$$
$$\leqq\frac{\sup_{|\eta-\eta_s'|=\rho_s}|A(\eta)|}{\rho_s(\rho_s-|\eta-\eta_s'|)},$$

where ρ_s represents $C's^{-\frac{1}{6}}$, $(C'<C)$.
Using the asymptotic expansion of $A(\eta)$ (see [3], p. 364), we obtain

(4. 18) $$C_0s^{-\frac{1}{6}}\leqq\sup_{|\eta-\eta_s|\leqq\rho_s}|A(\eta)|\leqq C_1s^{-\frac{1}{6}},\quad s\geqq s_1.$$

From (4.17) and (4.18) we have if η belongs to the disk $|\eta-\eta_s'|\leqq n^{-\frac{1}{3}}\rho_s$

(4. 19) $$|A_1(\eta)|\leqq Cs^{\frac{1}{2}},\quad s\geqq s_1\quad n\geqq n_1.$$

similarly we have

(4. 20) $$|A_2(\eta)|\leqq Cs^{\frac{5}{6}},\quad s\geqq s_1,\quad n\geqq n_1.$$

Substituting (4.8), (4.18), (4.19) and (4.20) in (4.15) we see that

(4. 21) $$|h(\eta)|\leqq C,\ |\eta-\eta_s'|\leqq n^{-\frac{1}{3}}\rho_s,\quad s\geqq s_1,\quad n\geqq n_1.$$

Thus we obtain from (4.14) and (4.21)

$$(4.22)\qquad |g_n(\eta)|<1,\quad |\eta-\eta'_s|=n^{-\frac{1}{3}}\rho_s,\quad s\geqq s_1,\quad n\geqq n_2.$$

Rouche's theorem is now applicable to compare the zeros of $f_n(\eta)$ and those of $A'(\eta)$. It follows that $f_n(\eta)$ has only one zero in the disk $|\eta-\eta'_s|<n^{-\frac{1}{3}}\rho_s$. Let us denote it by $\eta'_s(n)$, then we have

$$(4.23)\qquad |\eta'_s(n)-\eta'_s|<n^{-\frac{1}{3}}\rho_s,\quad s\geqq s_1,\quad n\geqq n_2.$$

Let us consider the case $1\leqq s<s_1$.
We denote $2\epsilon=\min_{1\leq s<s_1}\{|\eta'_{s+1}-\eta'_s|\}$, $(\epsilon>0)$. In this case it is easily seen that $h(\eta)$ is bounded. We have again by Rouche's theorem

$$(4.24)\qquad |\eta'_s(n)-\eta'_s|\leqq \epsilon n^{-\frac{1}{3}},\quad 1\leqq s<s_1,\quad n\geqq n_3.$$

When we denote $\zeta'_s(n)=e^{-\frac{2}{3}\pi i}n^{-\frac{2}{3}}\eta'_s(n)$, we obtain from (4.23) and (4.24).

$$(4.25)\qquad \zeta'_s(n)=e^{-\frac{2}{3}\pi i}n^{-\frac{2}{3}}\eta'_s+0(n^{-1}),$$

where $s=1, 2, \ldots, \left[\frac{n+1}{2}\right]+1$ and $n>\max(n_2, n_3)$. From (4.25) each $\zeta'_s(n)$ approaches to the fixed segment $\left\{L=\{\zeta;\ \zeta=re^{\frac{1}{3}\pi i},\ 0\leqq r\leqq\left(\frac{3}{4}\pi\right)^{\frac{2}{3}}\right\}$ when n tends to infinity. On account of the boundedness of the image of L transformed into z plane there exists a positive constant B such that

$$(4.26)\qquad |nz(\zeta'_s(n))|\leqq Bn.$$

If $|\zeta'_s(n)|$ is small enough, we have by changing ζ into z (see [3], p. 336)

$$(4.27)\qquad nz(\zeta'_s(n))=n-2^{-\frac{1}{3}}n^{\frac{1}{3}}e^{-\frac{2}{3}\pi i}\eta'_s+0(1).$$

From this there exists a constant A (>0) such that

(4. 28) $$\operatorname{Im}\ nz(\zeta_s'(n)) < -An^{\frac{1}{3}}.$$

Writing $\lambda_n^s = nz(\zeta_s'(n))$, we have $\operatorname{Im}\ \lambda_n^s < -An^{\frac{1}{3}}$ and $|\lambda_n^s| \leqq Bn$. Thus the Lemma 2 is established.

§ 5. Extension to a fourth order equation

In this section we treat the product of the wave operator of the following form:

(5. 1) $$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_1^2}\Delta\right)\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_2^2}\Delta\right)w(X, t) = 0, \text{ in } \Omega,\ t > 0$$

(5. 2) $$\left(\frac{\partial}{\partial t}\right)^j w(X, 0) = f_j(X), \text{ for } j = 0, 1, 2, 3$$

(5. 3) $$w(X, t) = \Delta w(X, t) = 0,\ X \in \partial\Omega$$

where a_1 and a_2 are real $(0 < a_1 < a_2)$. The notations are the same as in § 1. In this case the solution $v(X, t)$ of the Cauchy problem in R^3 equals to a constant d for large t where $d = \frac{a_1^2 a_2^2}{4\pi(a_1 + a_2)} \int_{R^3} f_3(X)dx$. In order to investigate the behavior of $w(X, t)$ for large t our interest goes to the following equation:

(5. 1)′ $$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_1^2}\Delta\right)\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_2^2}\Delta\right)u(X, t) = 0, \text{ in } \Omega,\ t > 0$$

(5. 4) $$\left(\frac{\partial}{\partial t}\right)^j u(X, 0) = 0, \quad \text{for } j = 0, 1, 2, 3$$

(5. 5) $$u(X, t) = -v(X, t),\ \Delta u(X, t) = -\Delta v(X, t),\ X \in \partial\Omega.$$

When we denote the right sides of (5.5) by $f(\omega, t)$ and $g(\omega, t)$ respectively, we put the following assumption.

Assumption 1. *$f(\omega, t)$ and $g(\omega, t)$ are sufficiently smooth and*

$$\left(\frac{\partial}{\partial t}\right)^j f(\omega, 0)=\left(\frac{\partial}{\partial t}\right)^j g(\omega, 0)=0, \quad j=0, 1, 2, 3$$

Furthermore there is a positive constant T_0 *such that for* $t>T_0$, $f(\omega, t)=d$, *and* $g(\omega, t)=0$.

Theorem. *The solution* $u(X, t)$ *satisfying* (5.1)$'$, (5.4) *and* (5.5) *with the Assumption* 1 *approaches exponentially to a constant* dr^{-1} *for fixed* X *when* t *tends to infinity.*

Proof. Since the method of the proof is quite similar to that of the preceding problem, we state the proof briefly. As the differential operator of (5.1)$'$ is commutative $\hat{u}_{n,m}(r, \lambda)$ corresponding to (2.6) is represented as follows

$$(5.6)\qquad \hat{u}_{n,m}(r, \lambda)=\frac{\hat{g}_{n,m}(\lambda)}{\lambda^2(a_1^2-a_2^2)r^{\frac{1}{2}}}\left\{\frac{K_{n+\frac{1}{2}}(\lambda a_1 r)}{K_{n+\frac{1}{2}}(\lambda a_1)}-\frac{K_{n+\frac{1}{2}}(\lambda a_2 r)}{K_{n+\frac{1}{2}}(\lambda a_2)}\right\}$$
$$-\frac{\hat{f}_{n,m}(\lambda)}{(a_1^2-a_2^2)r^{\frac{1}{2}}}\left\{a_2^2\frac{K_{n+\frac{1}{2}}(\lambda a_1 r)}{K_{n+\frac{1}{2}}(\lambda a_1)}-a_1^2\frac{K_{n+\frac{1}{2}}(\lambda a_2 r)}{K_{n+\frac{1}{2}}(\lambda a_2)}\right\}.$$

$\dfrac{K_{n+\frac{1}{2}}(\lambda a_i r)}{K_{n+\frac{1}{2}}(\lambda a_i)}$ $(i=1, 2)$ has only simple poles and its poles lie in Re $\lambda<0$ for fixed $r(>1)$, more precisely the results of Lemma 1 and 2 are true in this case. We obtain by using the inverse Laplace transformation

$$(5.7)\qquad u_{n,m}(r, t)$$
$$=\frac{a_2^2}{a_2^2-a_1^2}\left\{\frac{f_{n,m}(t-a_1(r-1))}{r}\right.$$
$$\left.+\sum_{s=1}^{n}\int_0^{t-a_1(r-1)} e^{\lambda_n^s(t-\tau)}f_{n,m}(\tau)d\tau\frac{k_n(\lambda_n^s a_1 r)}{a_1 k_n'(\lambda_n^s a_1)}\right\}$$
$$+\frac{a_1^2}{a_1^2-a_2^2}\left\{\frac{f_{n,m}(t-a_2(r-1))}{r}\right.$$
$$\left.+\sum_{s=1}^{n}\int_0^{t-a_2(r-1)} e^{\mu_n^s(t-\tau)}f_{n,m}(\tau)d\tau\frac{k_n(\mu_n^s a_2 r)}{a_2 k'(\mu_n^s a_2)}\right\}$$

$$+\frac{1}{a_1^2-a_2^2}\left\{\frac{1}{r^{n+1}}\int_0^{t-a_1(r-1)}(t-\tau)g_{n,m}(\tau)d\tau\right.$$
$$\left.+\sum_{s=1}^{n}\int_0^{t-a_1(r-1)}(\lambda_n^s)^{-2}e^{\lambda_n^s(t-\tau)}g_{n,m}(\tau)d\tau\frac{k_n(\lambda_n^s a_1 r)}{a_1 k_n'(\lambda_n^s a_1)}\right\}$$
$$+\frac{1}{a_2^2-a_1^2}\left\{\frac{1}{r^{n+1}}\int_0^{t-a_2(r-1)}(t-\tau)g_{n,m}(\tau)d\tau\right.$$
$$\left.+\sum_{s=1}^{n}\int_0^{t-a_2(r-1)}(\mu_n^s)^{-2}e^{\mu_n^s(t-\tau)}g_{n,m}(\tau)d\tau\frac{k_n(\mu_n^s a_2 r)}{a_2 k_n'(\mu_n^s a_2)}\right\}$$

where $\lambda=\lambda_n^s$ and $\mu=\mu_n^s$ are the zeros of $k_n(\lambda a_1)$ and $k_n(\mu a_2)$. We have put $k_n(z)=\left(\frac{\pi}{2z}\right)^{\frac{1}{2}}K_{n+\frac{1}{2}}(z)$.

By the Assumption 1 we have for sufficiently large t if r is fixed

$$(5.\ 8)\qquad f_{n,m}(t-a_i(r-1))=\begin{cases} d/(4\pi)^{\frac{1}{2}}, & \text{for } n=0, \quad i=1,2 \\ 0, & \text{for } n\geqq 1, \quad i=1,2 \end{cases}$$

$$(5.\ 9)\qquad \frac{1}{(a_1^2-a_2^2)r^{n+1}}\left\{\int_0^{t-a_1(r-1)}(t-\tau)g_{n,m}(\tau)d\tau\right.$$
$$\left.-\int_0^{t-a_2(r-1)}(t-\tau)g_{n,m}(\tau)d\tau\right\}=0.$$

Consequently we see that $u(X, t)$ for fixed X approaches exponentially to dr^{-1} when t tends to infinity.

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References

[1] C. Morawetz: The decay of solutions of the exterior initial-boundary value problem for the wave equation. Comm. Pure Appl. Math. **14** (1961). 561–569.

[2] F. W. Olver: The asymptotic solution of linear differential equations of the second order for large values of a parameter. Philos. Trans. Roy. Soc. London Ser. A, Vol. **247**. 307–327.

[3] F. W. Olver: The asymptotic expansion of Bessel functions of large order. ibid. 328–368.

[4] C. Wilcox: The initial boundary value problem for the wave equation in an exterior domain with spherical boundary. Notices. A. M. S. Vol. 6, No. **7** (1959) 869–870.
[5] G. Watson: Theory of Bessel functions. 1958. Cambridge University Press.