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A conjecture of Nagata on graded Cohen-Macaulay rings

By

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Let $R=\sum_{n\geq 0} R_n$ be a commutative Noetherian graded ring. Nagata conjectured in [3] that if the localization $R_{\mathfrak{m}}$ is Cohen-Macaulay for every graded maximal ideal $\mathfrak{m}$ of R, then R is itself Cohen-Macaulay. We note that if $\mathfrak{p}$ is any graded prime ideal of R, then $\mathfrak{p}$ is contained in a graded maximal ideal $\mathfrak{m}$, and $R_{\mathfrak{p}}$ is a localization of $R_{\mathfrak{m}}$, so if $R_{\mathfrak{m}}$ is Cohen-Macaulay, then $R_{\mathfrak{p}}$ is as well. This remark enables us to state a slight generalization of the conjecture; that is, from now on we will allow graded rings to have elements of *negative degree.* We prove the following:

Theorem. *If R is a graded ring and $R_{\mathfrak{p}}$ is Cohen-Macaulay for every graded prime ideal $\mathfrak{p}$ of R, then R is Cohen-Macaulay.*

Let $\mathfrak{p}$ be a non-graded prime ideal of R, and let $\mathfrak{p}^*$ be the ideal generated by homogeneous elements of $\mathfrak{p}$; thus $\mathfrak{p}^*=\{\sum r_i \mid r_i\in\mathfrak{p}, r_i \text{ homogeneous of degree } i\}$. Then $\mathfrak{p}^*$ is a graded prime ideal and is the largest graded ideal contained in $\mathfrak{p}$. Let S be the multiplicative set of homogeneous elements of R not in $\mathfrak{p}$. The localization R_S is a graded ring whose subring of elements of degree zero is local (the maximal ideal consists of those x/s where x and s are homogeneous of the same degree and x is in $\mathfrak{p}$), and the quotient $R_S/\mathfrak{p}^*R_S$ is the localization of $R/\mathfrak{p}^*$ at the multiplicative set of all non-zero homogeneous elements. Thus if k is the residue field of $(R_S)_0$, $R_S/\mathfrak{p}^*R_S$

is isomorphic to the graded ring $k[X, X^{-1}]$, where X has degree equal to the greatest common divisor of degrees of elements of S. We note that since $\mathfrak{p}$ is not graded, S contains elements of non-zero degree. It follows that $\mathfrak{p}R_S$ is generated by $\mathfrak{p}^*R_S$ and one other element, and that there are no prime ideals properly between $\mathfrak{p}^*$ and $\mathfrak{p}$.

Lemma 1. height $(\mathfrak{p})=$ height $(\mathfrak{p}^*)+1$.

Proof. We prove this by induction on $n=$ height $(\mathfrak{p})$, and since the case $n=1$ is trivial, we can assume $n\geqslant 2$. Let $\mathfrak{q}$ be a prime contained in $\mathfrak{p}$ of height $n-1$. If $\mathfrak{q}$ is graded, then $\mathfrak{q}$ must be $\mathfrak{p}^*$ and we are done. Otherwise, $\mathfrak{q}^*\neq\mathfrak{q}$, and by induction $\mathfrak{q}^*$ has height $n-2$. Since $\mathfrak{p}^*$ is between $\mathfrak{q}^*$ and $\mathfrak{p}$, $\mathfrak{p}^*$ could only fail to have height $n-1$ if $\mathfrak{p}^*=\mathfrak{q}^*$. However, in that case $\mathfrak{q}$ would be a prime ideal properly between $\mathfrak{p}^*$ and $\mathfrak{p}$, and this is impossible.

We wish to prove that the localization $R_{\mathfrak{p}}$ is Cohen-Macaulay. The aim is now to find a regular sequence of homogeneous elements in $\mathfrak{p}^*$ and reduce to the case where $\mathfrak{p}^*$ is minimal. If x is a regular homogeneous element of $\mathfrak{p}^*$, then we can indeed replace R by R/xR and $\mathfrak{p}$ by $\mathfrak{p}/xR$; then $(\mathfrak{p}/xR)^*$ is just $\mathfrak{p}^*/xR$ and, since it suffices to show that $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$ is Cohen-Macaulay, we have reduced the height of $\mathfrak{p}^*$ by one.

We will assume, as we can do with no problem, that R is equal to the localization R_S described above. If $\mathfrak{p}^*$ consists of zero-divisors, then it annihilates some non-zero homogeneous element x of R; the annihilator of x is graded and must thus be precisely $\mathfrak{p}^*$. Thus x does not go to zero in $R_{\mathfrak{p}^*}$, so, since $R_{\mathfrak{p}^*}$ is Cohen-Macaulay, $\mathfrak{p}^*$ must be minimal. Hence if height $\mathfrak{p}^*>0$, there are non-zero-divisors in $\mathfrak{p}^*$, and we use the following variation of a well-known fact to show that we can find one which is homogeneous.

Lemma 2. *Let $\mathfrak{a}$ be a graded ideal of R, and let $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ be graded prime ideals which do not contain all elements of R*

of positive degree. If the set of homogeneous elements of $\mathfrak{a}$ *is contained in* $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, *then* $\mathfrak{a}$ *is contained in some* $\mathfrak{p}_i$.

Proof. Assume $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for all i. We can assume that no inclusion exists among the $\mathfrak{p}_i$, and, by induction, that the homogeneous elements of $\mathfrak{a}$ are not contained in the union $\bigcup_{i \neq j} \mathfrak{p}_i$ for any j. Thus, for each j, we can find a homogeneous element a_j in $\mathfrak{a} \cap \mathfrak{p}_j$ but in no other $\mathfrak{p}_i$. Let $y_j = \prod_{i \neq j} a_i$; then y_j is an element of $\mathfrak{a}$ and of every $\mathfrak{p}_i$ except $\mathfrak{p}_j$. We then multiply y_j by a homogeneous element of positive degree not in $\mathfrak{p}_j$ to insure that y_j itself has positive degree, and we can then take powers of the y_j's to produce a set of elements with the same properties as the y_j's and all of the same degree. Adding them up then gives a homogeneous element of $\mathfrak{a}$ which is not in $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$.

In our case, $\mathfrak{a}$ is $\mathfrak{p}^*$ and the $\mathfrak{p}_i$ are the associated primes of $R = R_S$. Since each $\mathfrak{p}_i$ is graded (cf. Bourbaki [1], section 3.1, Proposition 1), and since each $\mathfrak{p}_i$ is contained in $\mathfrak{p}^*$, the conditions of the lemma are satisfied. Thus we can find a homogeneous non-zero-divisor and reduce the height of $\mathfrak{p}^*$. When $\mathfrak{p}^*$ is minimal, $\mathfrak{p}$ has height 1 by Lemma 1, and it is enough to show that $\mathfrak{p}$ contains a non-zero-divisor. We take a non-zero element in $\mathfrak{p}/\mathfrak{p}^* \subseteq R/\mathfrak{p}^* \cong k[X, X^{-1}]$, and we can lift this back to an element of $\mathfrak{p}$ whose term of highest degree is a unit. Such an element cannot be a zero-divisor, so the proof is complete. Hence $R_{\mathfrak{p}}$ is Cohen-Macaulay for every prime ideal $\mathfrak{p}$ of R, and R is a Cohen-Macaulay ring.

Remarks. 1. There is another method of producing homogeneous non-zero-divisors which involves replacing R by the polynomial ring $R[T]$, where T is an indeterminate of degree one. If $\sum a_n$ is a non-zero-divisor in $\mathfrak{p}^*$, where a_n is homogeneous of degree n, then $\sum a_n T^{k-n}$ is a homogeneous non-zero-divisor in $\mathfrak{p}^* R[T]$ for k large enough, and R and $\mathfrak{p}$ can be replaced by $R[T]$ and $\mathfrak{p}R[T]$.

2. When $R = R_S$ as above, it is actually possible to find a regular

sequence contained in R_0. If $\mathfrak{m}$ is the maximal ideal of R_0, it is fairly easy to show that $\mathfrak{p}^*$ is nilpotent modulo $\mathfrak{m}R$, and from the fact that $R_{\mathfrak{p}^*}$ is Cohen-Macaulay, one deduces that if $\mathfrak{p}^*$ is not minimal, there is a non-zero-divisor in $\mathfrak{m}$. Furthermore, if $\mathfrak{p}^*$ is minimal, then if R_0 containes a prime $\mathfrak{p}_0$ different from $\mathfrak{m}$, the nilradical of $\mathfrak{p}_0R$ would be a prime ideal of R properly contained in $\mathfrak{p}^*$, so R_0 must be Artinian, and $\mathfrak{p}^*$ must be nilpotent. These facts could be used to replace the lemmas above and give a somewhat different proof of the theorem.

3. It has been brought to our attention that a different proof of the original conjecture is to appear in Hochster and Ratliff ([2]). They also raise the possibility that a proof might already exist somewhere in the literature.

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