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Dirichlet problem for elliptic equations of the second order in a singular domain of $\mathbf{R}^2$

by

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1. Introduction.

In this paper we treat the regularity of solutions of the Dirichlet problem for elliptic equations of the second order in a domain with edges.

In the case where the boundary of a domain is smooth, we know well the regularity of solutions of the Dirichlet problem.

T. Carleman [1] had studied the boundary value problem of the Laplace equation for a domain with edges. M. Š. Birman and G. E. Skvortsov [2] dealt with a kind of regularity of solutions of the Dirichlet problem in the case where the boundary of a bounded domain in R^2 consists of a finite number of three times continuously differentiable curves, which meet with the angles different from 0 or 2π. V. A. Kondrat'ev [3] studied the general boundary value problem for a domain with conical or angular points in R^n.

We shall extend the result of M. Š. Birman and G. E. Skvortsov. Let Ω be a bounded domain in R^2 and let the boundary of Ω consist of a finite number of three times continuously differentiable curves, which may meet even with the angles 0 or 2π, but they have not contact of order ∞.

Consider an elliptic differential operator of the second order:

$$(1.1) \qquad Lu = -\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} a_i(x)\frac{\partial u}{\partial x_i} + a(x)u$$

where the coefficients $a_{ij}(x)$ are real functions continuous on $\bar{\Omega}$ having the bounded first generalized derivatives, and $a_i(x)$ and $a(x)$ are real bounded measurable.

We set

$$D(L;\ \Omega)=\{u\in\mathscr{D}^1_{L^2}(\Omega);\ Lu\in L^2(\Omega)\}.^{1)}$$

Then our main theorem is the following:

Theorem 1. *The co-dimension of* $\mathscr{D}^1_{L^2}(\Omega)\cap\mathscr{E}^2_{L^2}(\Omega)$ *in* $D(L;\Omega)$ *is equal to the number of the edges with the angles larger than* π.

In section 5 this theorem will be proved with help of the result of M. Š. Birman and G. E. Skvortsov and of the following theorem.

Theorem 2. *Suppose that*

$$\Omega=\{(x_1,x_2);\ 0<x_1<d,\ 0<x_2<x_1^{\alpha}\},$$

where $\alpha>1$. *Then the solution* $u(x)\in\mathscr{D}^1_{L^2}(\Omega)$ *of the equation* $Lu=f$ *belongs to* $\mathscr{E}^2_{L^2}(\Omega)$, *if* $f\in L^2(\Omega)$.

We prove Theorem 2 in Section 2. In Sections 3 and 4, we prove the propositions which are needed in Section 2. Finally we prove Theorem 1 in Section 5.

2. Proof of Theorem 2.

In this section we prove Theorem 2 with help of three propositions below, which will be proved in Sections 3 and 4.

Proposition 1. *In the case where* $L=-\Delta$, *Theorem* 2 *is true, that is to say, for all* $f(x)\in L^2(\Omega)$ *the solution* $u(x)\in\mathscr{D}^1_{L^2}(\Omega)$ *of the equ-*

1) Throughout this paper, $\mathscr{E}^k_{L^2}(\Omega)$ denotes the Hilbert space of all functions $u(x)\in L^2(\Omega)$ whose derivatives (in the sense of distributions) up to order k belong to $L^2(\Omega)$, and $\mathscr{D}^k_{L^2}$ is the closure of $\mathscr{D}(\Omega)$ (in Schwartz' notation) in $\mathscr{E}^k_{L^2}(\Omega)$. For $u(x)\in\mathscr{E}^k_{L^2}(\Omega)$, we denote its norm by $\|u(x)\|_{k,L^2}$. Namely,

$$\|u(x)\|^2_{k,L^2}=\sum_{|\nu|\leq k}\int_{\Omega}|D^{\nu}u(x)|^2dx.$$

ation $\Delta u=f$ *belongs to* $\mathscr{E}^2_{L^2}(\Omega)$.

By definition, the set $C^k_0(\bar{\Omega})$ consists of all functions which are k-times continuously differentiable in $\bar{\Omega}$ and vanish at the boundary of Ω.

Proposition 2. $C^3_0(\bar{\Omega})$ *is dense in* $\mathscr{D}^1_{L^2}(\Omega)\cap\mathscr{E}^2_{L^2}(\Omega)$.

Proposition 3. *Let* L_1 *and* L_2 *be two elliptic operators of the form* (1.1). *If the infimums of the coefficients* $a(x)$ *are sufficiently large, we have*

$$(2.1)\qquad |(L_1u,\ L_2u)|\geqq c\|u\|^2_{2,L^2}\quad \textit{for}\quad u\in C^3_0(\bar{\Omega}),$$

where c *is a positive constant independent of* $u(x)$.

From Proposition 2, the estimate (2.1) also holds for $u(x)\in\mathscr{D}^1_{L^2}(\Omega)\cap\mathscr{E}^2_{L^2}(\Omega)$.

Proof of Theorem 2. Set $L_1=L+\lambda$ and $L_2=-\Delta+\lambda$, where λ is a sufficiently large positive number. It is easily checked that

$$R(L_1)=\{L_1u;\ u\in\mathscr{D}^1_{L^2}(\Omega)\cap\mathscr{E}^2_{L^2}(\Omega)\}$$

is a closed subspace of $L^2(\Omega)$ by setting $L_2=L_1$ in Proposition 3. On the other hand, we see, from Proposition 1, that

$$(2.2)\qquad R(L_2)=R(-\Delta+\lambda)=L^2(\Omega).$$

In order to show that $R(L_1)=L^2(\Omega)$, we have only to verify that $R(L_1)$ is dense in $L^2(\Omega)$. In fact, suppose that there exists a $g\in L^2(\Omega)$ such that $(g,L_1u)=0$ for all $u\in\mathscr{D}^1_{L^2}(\Omega)\cap\mathscr{E}^2_{L^2}(\Omega)$. Then there exists a solution $w(x)\in\mathscr{D}^1_{L^2}(\Omega)\cap\mathscr{E}^2_{L^2}(\Omega)$ of $L_2w=g$ owing to (2.2) and $(L_2w,L_1w)=0$. By virtue of Proposition 3, $w=0$, therefore $g=0$. Accordingly $R(L_1)$ is dense in $L^2(\Omega)$.

Now, let us take the solution in question $u\in\mathscr{D}^1_{L^2}(\Omega)$ of $Lu=f$. This equation can be rewritten as

$$L_1u=Lu+\lambda u=f+\lambda u.$$

The right hand side $f+\lambda u\equiv h(\in L^2(\Omega))$ is the image of a $v\in\mathscr{D}^1_{L^2}(\Omega)\cap$

$\mathscr{E}^2_{L^2}(\Omega)$ by L_1 as is shown just above. Hence, we have two equations $L_1u=h$ and $L_1v=h$. By the uniqueness of the solution in $\mathscr{D}^1_{L^2}(\Omega)$, we have $u=v$, that is, u itself belongs to $\mathscr{E}^2_{L^2}(\Omega)$. Thus, our Theorem 2 is proved assuming Propositions 1, 2 and 3.

3. Proof of Propositions 1 and 2.

Remark at first that we may suppose that d is sufficiently small, because if necessary, we may take a C^∞-function $\varphi(x)$ with a small compact support which is equal to 1 near the origin, and we may consider φu in place of u;

$$L(\varphi u)=\varphi Lu+uL\varphi+\sum_{i,j=1}^{2}a_{ij}(x)\left(\frac{\partial\varphi}{\partial x_i}\frac{\partial u}{\partial x_j}+\frac{\partial\varphi}{\partial x_j}\frac{\partial u}{\partial x_i}\right)$$
$$-a(x)\varphi u\in L^2(\Omega)$$

We begin with the change of variable $(x_1, x_2)\to(\xi, \eta)$

$$(3.1)\qquad \begin{cases}\xi=(\beta-1)x_1^{-\frac{1}{\beta-1}}\\ \eta=x_1^{-\frac{\beta}{\beta-1}}\cdot x_2\end{cases}\quad\text{or}\quad\begin{cases}x_1=(\beta-1)^{\beta-1}\cdot\xi^{1-\beta}\\ x_2=(\beta-1)^{\beta}\xi^{-\beta}\cdot\eta,\end{cases}$$

where $1/\alpha+1/\beta=1$. Then the domain Ω is mapped homeomorphically onto the domain:

$$\hat{\Omega}=\{(\xi, \eta)\in R^2;\ \xi>A,\ 0<\eta<1\}$$

where $A=(\beta-1)d^{-\frac{1}{\beta-1}}$. We obtain the following rules of calculus:

$$(3.2)\qquad\begin{cases}\dfrac{\partial u}{\partial x_1}=-(\beta-1)^{-\beta}\xi^{\beta}\dfrac{\partial u}{\partial\xi}-\beta(\beta-1)^{-\beta}\xi^{\beta-1}\eta\dfrac{\partial u}{\partial\eta}\\[2ex] \dfrac{\partial u}{\partial x_2}=(\beta-1)^{-\beta}\xi^{\beta}\dfrac{\partial u}{\partial\eta}\\[2ex] \dfrac{\partial^2 u}{\partial x_1^2}=(\beta-1)^{-2\beta}\xi^{2\beta}\dfrac{\partial^2 u}{\partial\xi^2}+2\beta(\beta-1)^{-2\beta}\xi^{2\beta-1}\eta\dfrac{\partial^2 u}{\partial\xi\partial\eta}\\ \qquad\qquad+\beta^2(\beta-1)^{-2\beta}\xi^{2\beta-2}\eta^2\dfrac{\partial^2 u}{\partial\eta^2}\end{cases}$$

$$
\left\{
\begin{aligned}
&+\beta(\beta-1)^{-2\beta}\xi^{2\beta-1}\frac{\partial u}{\partial\xi}+(2\beta^2-\beta)(\beta-1)^{-2\beta}\xi^{2\beta-2}\eta\frac{\partial u}{\partial\eta}\\
&\frac{\partial^2 u}{\partial x_1\partial x_2}=-(\beta-1)^{-2\beta}\xi^{2\beta}\frac{\partial^2 u}{\partial\xi\partial\eta}-\beta(\beta-1)^{-2\beta}\xi^{2\beta-1}\eta\frac{\partial^2 u}{\partial\eta^2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\beta(\beta-1)^{-2\beta}\xi^{2\beta-1}\frac{\partial u}{\partial\eta}\\
&\frac{\partial^2 u}{\partial x_2^2}=(\beta-1)^{-2\beta}\xi^{2\beta}\frac{\partial^2 u}{\partial\eta^2}
\end{aligned}
\right.
$$

$$
\frac{\partial(x_1, x_2)}{\partial(\xi, \eta)}=-(\beta-1)^{2\beta}\xi^{-2\beta}. \tag{3.3}
$$

Hence, the relation between the Laplacians corresponding to two systems of variables can be expressed as

$$
\begin{aligned}
\Delta_{x_1,x_2}=(\beta-1)^{-2\beta}\xi^{2\beta}\Big\{&\Delta_{\xi,\eta}+2\beta\xi^{-1}\eta\frac{\partial^2}{\partial\xi\partial\eta}+\beta^2\xi^{-2}\eta^2\frac{\partial^2}{\partial\eta^2}\\
&+\beta\xi^{-1}\frac{\partial}{\partial\xi}+(2\beta^2-\beta)\xi^{-2}\eta\frac{\partial}{\partial\eta}\Big\}.
\end{aligned} \tag{3.4}
$$

The formulae (3.2) and (3.3) lead us to the following;

Lemma 1. 1) *$u(x_1, x_2)\in L^2_x$, if and only if $\xi^{-\beta}u(\xi, \eta)\in L^2_{\xi,\eta}$;*
2) *$u(x_1, x_2)\in\mathscr{E}^1_{L^2_x}$, if and only if $\xi^{-\beta}u(\xi, \eta)$, $\frac{\partial u}{\partial\xi}$ and $\frac{\partial u}{\partial\eta}\in L^2_{\xi,\eta}$;*
3) *$u(x_1, x_2)\in\mathscr{E}^2_{L^2_x}$, if $\xi^{-\beta}u$, $\xi^{\beta-1}\frac{\partial u}{\partial\xi}$, $\xi^{\beta-1}\frac{\partial u}{\partial\eta}$, $\xi^{\beta}\frac{\partial^2 u}{\partial\xi^2}$, $\xi^{\beta}\frac{\partial^2 u}{\partial\xi\partial\eta}$ and $\xi^{\beta}\frac{\partial^2 u}{\partial\eta^2}\in L^2_{\xi,\eta}$;*
where L^2_x and $L^2_{\xi,\eta}$ are understood as L^2-spaces with respect to the usual Lebesgue measures dx_1dx_2 and $d\xi d\eta$ respectively.

We introduce some functional spaces, at first define

$$
\mathscr{E}^k_{\xi,\eta}=\{u(\xi, \eta);\ \xi^{\beta}D^{\nu}_{\xi,\eta}u(\xi, \eta)\in L^2_{\xi,\eta}\quad \text{for } |\nu|\leqq k\}
$$

with the norms

$$\|u(\xi, \eta)\|^2_{\mathscr{E}^k_{\xi,\eta}} = \sum_{|\mu|\leq k} \|\xi^\beta D^\mu_{\xi,\eta} u(\xi, \eta)\|^2_{L^2_{\xi,\eta}}.$$

Because $\{\sqrt{2} \sin n\pi\eta\}_{n=1}^\infty$ is a complete orthonormal system in $L^2_\eta(0, 1)$, we can develop every function $u(\xi, \eta) \in \mathscr{E}^0_{\xi,\eta}$ in a Fourier series with respect to η;

$$u(\xi, \eta) = \sum_{n=1}^\infty u_n(\xi) \sin n\pi\eta, \tag{3.5}$$

with coefficients $u_n(\xi)$ such that $\xi^\beta u_n(\xi) \in L^2_\xi(A, \infty)$.

We denote by $\mathscr{D}_{\xi,\eta}$ the set of all functions $u(\xi, \eta)$ in $\mathscr{E}^2_{\xi,\eta}$ which satisfy the following two conditions

$$u(A, \eta) = 0 \tag{3.6}$$

and

$$\sum_{n=1}^\infty \{n^4 \|\xi^\beta u_n\|^2_{L^2_\xi} + n^2 \|\xi^\beta \frac{\partial u_n}{\partial \xi}\|^2_{L^2_\xi} + \|\xi^\beta \frac{\partial^2 u_n}{\partial \xi^2}\|^2_{L^2_\xi}\} < \infty. \tag{3.7}$$

The norm of u in $\mathscr{D}_{\xi,\eta}$ is defined as the square root of the left hand side of (3.7). Then, for every function belonging to $\mathscr{D}_{\xi,\eta}$, the norms in $\mathscr{E}^2_{\xi,\eta}$ and in $\mathscr{D}_{\xi,\eta}$ are equivalent. The condition (3.7) implies a boundary condition at $\eta = 0$ and $\eta = 1$ to each element of $\mathscr{D}_{\xi,\eta}$.

It is easy to see that $\Delta_{\xi,\eta}$ is a bounded linear operator from $\mathscr{D}_{\xi,\eta}$ to $\mathscr{E}^0_{\xi,\eta}$. Our first step is to construct an operator $G: \mathscr{E}^0_{\xi,\eta} \to \mathscr{D}_{\xi,\eta}$ which is the inverse of $\Delta_{\xi,\eta}$.

For $g(\xi, \eta) \in \mathscr{E}^0_{\xi,\eta}$, we develop $g(\xi, \eta)$ in a Fourier series with respect to η;

$$g(\xi, \eta) = \sum_{n=1}^\infty g_n(\xi) \sin n\pi\eta. \tag{3.8}$$

We define $u_n(\xi)$ by

$$u_n(\xi) = \int_A^\infty \frac{1}{2n\pi} \{-e^{-n\pi|\xi - s|} + e^{n\pi(2A - \xi - s)}\} g_n(s) ds \tag{3.9}$$

In fact $u_n(\xi)$ is the solution of the ordinary differential equation:

$$\frac{d^2 u_n(\xi)}{d\xi^2} = (n\pi)^2 u_n + g_n \tag{3.10}$$

with the boundary conditions $u_n(A)=0$ and $u_n(\infty)=0$. Set

$$Gg(\xi, \eta)=\sum_{n=1}^{\infty} u_n(\xi) \sin n\pi\eta. \tag{3.11}$$

This is the operator which we have looked for. We are going to check up (3.6), (3.7) for $u=Gg$ and $\Delta_{\xi,\eta}Gg=g$ on $\mathscr{E}^0_{\xi,\eta}$. Assuming the condition (3.7) for $u=Gg$, the condition (3.6) and the equality $\Delta_{\xi,\eta}Gg=g$ are easily verified from (3.9) and (3.10). We now show (3.7).

Define two operators;

$$\begin{cases} K_{1,n}g(\xi)=\int_A^{\infty} e^{-n\pi|\xi-s|}g(s)ds \\ K_{2,n}g(\xi)=\int_A^{\infty} e^{n\pi(2A-\xi-s)}g(s)ds, \end{cases} \tag{3.12}$$

and the space;

$$\mathscr{E}^0_{\xi}=\{u(\xi);\ \xi^{\beta}u(\xi)\in L^2(A, \infty)\}$$

with the norm:

$$\|u(\xi)\|_{\mathscr{E}^0_{\xi}}=\|\xi^{\beta}u(\xi)\|_{L^2_{\xi}}.$$

Then the following lemma holds.

Lemma 2. *$K_{1,n}$ and $K_{2,n}$ are bounded operators on $\mathscr{E}^0_{\xi}$, moreover*

$$\|K_{i,n}\|_{\mathscr{E}^0_{\xi}\to\mathscr{E}^0_{\xi}}\leqq\frac{c}{n} \qquad (i=1, 2) \tag{3.13}$$

where c is independent of A for sufficiently large A.

Proof We are going to show the inequalities:

$$\int_A^{\infty} e^{-n\pi|\xi-s|}ds \leqq \frac{c}{n} \tag{3.14}$$

$$(3.15)\qquad \int_A^\infty e^{-n\pi|\xi-s|}\left(\frac{\xi}{s}\right)^{2\beta}d\xi \leqq \frac{c}{n},$$

$$(3.16)\qquad \int_A^\infty e^{n\pi(2A-\xi-s)}ds \leqq \frac{c}{n},$$

$$(3.17)\qquad \int_A^\infty e^{n\pi(2A-\xi-s)}\left(\frac{\xi}{s}\right)^{2\beta}d\xi \leqq \frac{c}{n},$$

where c is independent of $n \geqq 1$ and $s, \xi \geqq A$.

(3.14) and (3.16) are trivial. For (3.15), we can write

$$\int_A^\infty e^{-n\pi|\xi-s|}\left(\frac{\xi}{s}\right)^{2\beta}d\xi = \int_A^s e^{n\pi(\xi-s)}\left(\frac{\xi}{s}\right)^{2\beta}d\xi + \int_s^\infty e^{-n\pi(\xi-s)}\left(\frac{\xi}{s}\right)^{2\beta}d\xi,$$

the first term of the right hand side $\leqq \int_A^s e^{n\pi(\xi-s)}d\xi < \frac{1}{n\pi}$. In order to estimate the last term, we set $\xi = st$ and integrate by parts,

$$\text{the last term} = s\int_1^\infty e^{-n\pi s(t-1)}t^{2\beta}dt$$

$$= \frac{1}{n\pi} + \frac{2\beta}{n\pi}\int_1^\infty e^{-n\pi s(t-1)}t^{2\beta-1}dt$$

$$< \frac{c}{n}, \text{ because } s \geqq A.$$

We have then proved (3.15). (3.17) is simpler than (3.15). From (3.14) and Schwarz' inequality,

$$|K_{1,n}g(\xi)|^2 \leqq \int_A^\infty e^{-n\pi|\xi-s|}ds \cdot \int_A^\infty e^{-n\pi|\xi-s|}\cdot|g(s)|^2 ds$$

$$\leqq \frac{c}{n}\int_A^\infty e^{-n\pi|\xi-s|}\cdot|g(s)|^2 ds.$$

Moreover from (3.15) and Fubini's theorem, we have

$$\int_A^\infty \xi^{2\beta}\cdot|K_{1,n}g(\xi)|^2 d\xi \leqq \frac{c}{n}\int_A^\infty \xi^{2\beta}d\xi \cdot \int_A^\infty e^{-n\pi|\xi-s|}\cdot|g(s)|^2 ds$$

$$\leqq \frac{c}{n}\int_A^\infty s^{2\beta}|g(s)|^2 ds \int_A^\infty e^{-n\pi|\xi-s|}\left(\frac{\xi}{s}\right)^{2\beta}d\xi$$

$$\leqq\left(\frac{c}{n}\right)^2\|g(\xi)\|^2_{\mathscr{E}^0_\xi},$$

which is precisely the estimate (3.13) for $i=1$. Similarly, we can treat the case of $K_{2,n}$. (Q.E.D.)

By definition of $K_{1,n}$ and $K_{2,n}$, we can write

$$u_n(\xi)=-\frac{1}{2n\pi}\{K_{1,n}g_n(\xi)-K_{2,n}g_n(\xi)\}. \tag{3.18}$$

From Lemma 2,

$$\sum_{n=1}^{\infty}n^4\|\xi^\beta u_n(\xi)\|^2_{L^2_\xi}\leqq\sum_{n=1}^{\infty}c\|\xi^\beta g_n(\xi)\|^2_{L^2_\xi}\leqq c'\|g(\xi,\eta)\|^2_{\mathscr{E}^0_{\xi,\eta}} \tag{3.19}$$

Differentiate the both sides of (3.9), then

$$\frac{du_n}{d\xi}=\frac{1}{2}\int_A^\infty sgn(\xi-s)e^{-n\pi|\xi-s|}g_n(s)ds-\frac{1}{2}\int_A^\infty e^{n\pi(2A-\xi-s)}g_n(s)ds. \tag{3.20}$$

Therefore

$$\sum_{n=1}^{\infty}n^2\left\|\xi^\beta\frac{du_n}{d\xi}\right\|^2_{L^2_\xi}\leqq\sum_{n=1}^{\infty}n^2\{\|\xi^\beta K_{1,n}(|g_n|)\|^2_{L^2_\xi}+\|\xi^\beta K_{2,n}g_n\|^2_{L^2_\xi}\} \tag{3.21}$$

$$\leqq c\|g(\xi,\eta)\|^2_{\mathscr{E}^0_{\xi,\eta}}$$

And from (3.10) and (3.19),

$$\sum_{n=1}^{\infty}\left\|\xi^\beta\frac{d^2u_n}{d\xi^2}\right\|^2_{L^2_\xi}\leqq c\|g(\xi,\eta)\|^2_{\mathscr{E}^0_{\xi,\eta}} \tag{3.22}$$

By (3.19), (3.21) and (3.22), which together mean (3.7), we find that G is a bounded operator from $\mathscr{E}^0_{\xi,\eta}$ to $\mathscr{D}_{\xi,\eta}$.

Proof of Proposition 1. By (3.4), the equation $\Delta_{x_1,x_2}u=f$ turns out to be

$$\Delta_{\xi,\eta}u+2\beta\xi^{-1}\eta\frac{\partial^2u}{\partial\xi\partial\eta}+\beta^2\xi^{-2}\eta^2\frac{\partial^2u}{\partial\eta^2}+\beta\xi^{-1}\frac{\partial u}{\partial\xi}+(2\beta^2-\beta)\xi^{-2}\eta\frac{\partial u}{\partial\eta} \tag{3.23}$$

$$=(\beta-1)^{2\beta}\xi^{-2\beta}f.$$

Denote the right hand side of (3.23) by $h(\xi, \eta)$, then $h(\xi, \eta)$ belongs to $\mathscr{E}^0_{\xi,\eta}$. Look for the solution u of the form $u=Gg$ with some $g\in \mathscr{E}^0_{\xi,\eta}$, then, g must satisfy

$$g+Tg=h, \tag{3.24}$$

where

$$Tg=2\beta\xi^{-1}\eta\frac{\partial^2 Gg}{\partial\xi\partial\eta}+\beta^2\xi^{-2}\eta^2\frac{\partial^2 Gg}{\partial\eta^2}+\beta\xi^{-1}\frac{\partial Gg}{\partial\xi} + (2\beta^2-\beta)\xi^{-2}\eta\frac{\partial Gg}{\partial\eta}. \tag{3.25}$$

Since G is a bounded linear operator from $\mathscr{E}^0_{\xi,\eta}$ to $\mathscr{D}_{\xi,\eta}$, T is a bounded linear operator on $\mathscr{E}^0_{\xi,\eta}$, and if we take sufficiently large A, the operator norm of T may be smaller than 1. Therefore $I+T$ has the inverse $(I+T)^{-1}$. Since the equation $\Delta_{x_1,x_2}u=f$ has a unique solution in $\mathscr{D}^1_{L^2_x}(\Omega)$, if we know that $\mathscr{D}_{\xi,\eta}$ is included in $\mathscr{D}^1_{L^2_x}(\Omega)$, $u=G(I+T)^{-1}h$ is the unique solution of the equation $\Delta_{x_1,x_2}u=f$ and belongs to $\mathscr{D}_{\xi,\eta}$, a fortiori to $\mathscr{E}^2_{L^2_x}(\Omega)$.

Let us show the inclusion $\mathscr{D}_{\xi,\eta}\subset\mathscr{D}^1_{L^2_x}(\Omega)$. Take an infinitely differentiable function $\zeta(\xi)$ such that $\zeta(\xi)\equiv 1$ on $\xi<2A$ and $\zeta(\xi)\equiv 0$ on $\xi>3A$, and define

$$\zeta_M(\xi)=\zeta\left(\frac{\xi}{M}\right).$$

For every $u(\xi, \eta)=\sum_{n=1}^{\infty} u_n(\xi)\sin n\pi\eta$ in $\mathscr{D}_{\xi,\eta}$, we set

$$u_{N,M}(\xi, \eta)=\sum_{n=1}^{N}\zeta_M(\xi)u_n(\xi)\sin n\pi\eta. \tag{3.26}$$

As $M\to\infty$, $u_{N,M}(\xi, \eta)$ tends to $u_N(\xi, \eta)\equiv\sum_{n=1}^{N}u_n(\xi)\sin n\pi\eta$ in $\mathscr{D}_{\xi,\eta}$, and as $N\to\infty$, $u_N(\xi, \eta)$ tends to $u(\xi, \eta)$ in $\mathscr{D}_{\xi,\eta}$. Because the topology of $\mathscr{D}_{\xi,\eta}$ is stronger than that of $\mathscr{E}^1_{L^2_x}(\Omega)$ and because the left hand side of (3.26) belongs to $\mathscr{D}^1_{L^2_x}(\Omega)$, $u(\xi, \eta)$ belongs to $\mathscr{D}^1_{L^2_x}(\Omega)$. Thus we see that $\mathscr{D}_{\xi,\eta}\subset\mathscr{D}^1_{L^2_x}(\Omega)$. Proposition 1 is proved.

Remark *The three transformations in the diagram*

$$\mathscr{D}^1_{L^2_x}(\Omega)\cap\mathscr{E}^2_{L^2_x}(\Omega)\xrightarrow{\Delta x_1,x_2}L^2_x(\Omega)\xrightarrow{(\beta-1)^{2\beta}\xi^{-2\beta x}}\mathscr{E}^0_{\xi,\eta}\xrightarrow{G(1+T)^{-1}}\mathscr{D}_{\xi,\eta}$$

are continuous and the composition of them is identity mapping on $\mathscr{D}^1_{L^2_x}(\Omega)\cap\mathscr{E}^2_{L^2_x}(\Omega)$, *therefore the norm in* $\mathscr{E}^2_{L^2_x}(\Omega)$ *and the norm in* $\mathscr{D}_{\xi,\eta}$ *are equivalent on* $\mathscr{D}^1_{L^2_x}(\Omega)\cap\mathscr{E}^2_{L^2_x}(\Omega)$.

Proof of Proposition 2. By the above Remark, we have only to show that $C^3_0(\bar{\Omega})$ is dense in $\mathscr{D}_{\xi,\eta}$. For every $u(\xi,\eta)=\sum_{n=1}^{\infty}u_n(\xi)\sin n\pi\eta$ in $\mathscr{D}_{\xi,\eta}$, we extend $u_n(\xi)$ to $\xi<A$ in such a way that

$$u_n(\xi)=\begin{cases} u_n(\xi) & \text{if } \xi\geqq A\\ -u_n(2A-\xi) & \text{if } \xi<A.\end{cases}\tag{3.27}$$

$$u_{N,M,\varepsilon}(\xi,\eta)=\sum_{n=1}^{N}\{\rho_\varepsilon(\xi)*(\zeta_M(\xi)u_n(\xi))\}\sin n\pi\eta,\tag{3.28}$$

where $\rho_\varepsilon(\xi)*$ is Friedrichs' mollifier and $\rho_\varepsilon(\xi)$ is an even function of ξ, then $u_{N,M,\varepsilon}(\xi,\eta)$ belongs to $C^\infty_0(\bar{\Omega})$.[2] As $\varepsilon\to+0$, $u_{N,M,\varepsilon}(\xi,\eta)$ tends to $u_{N,M}(\xi,\eta)$ in $\mathscr{D}_{\xi,\eta}$, and as $M\to\infty$, $u_{N,M}(\xi,\eta)$ tends to $u_N(\xi,\eta)$. Finally $u_N(\xi,\eta)$ tends to $u(\xi,\eta)$ in $\mathscr{D}_{\xi,\eta}$ as $N\to\infty$. Thus $C^\infty_0(\bar{\Omega})$ is dense in $\mathscr{D}_{\xi,\eta}$. (Q.E.D.)

4. Proof of Proposition 3.

In the case where the boundary of a domain is of class C^3, Proposition 3 has already been proved by Ladyzhenskaya [4]. If the boundary is piece-wise smooth and has no edge with the angle 0, Proposition 3 also holds.

In this section, for convenience, L_1 and L_2 are written as L and M, respectively, and (x_1, x_2) as (x, y). Denote several positive constants by c_i.

Now we divide L and M as

$$L=L_0+L'+\lambda,$$

$$M=M_0+M'+\lambda,$$

2) $C^\infty_0(\bar{\Omega})$ denotes $\bigcap_{k=0}^{\infty}C^k_0(\bar{\Omega})$.

where L_0 and M_0 are homogeneous parts of the second order of L and M, L' and M' are lower order parts, and λ is the positive number which will be determined later.

Lemma 3. *If $d>0$ is sufficiently small, we have*

(4.1) $\mathrm{Re}\,(L_0u, M_0u)\geqq c_1\|u\|^2_{2,L^2}-c_2\|u\|^2_{1,L^2}$ *for* $u\in C^3_0(\bar{\Omega})$

Proof Denote

$$L_0u=au_{xx}+2bu_{xy}+cu_{yy}$$

(4.2)

$$M_0u=a'u_{xx}+2b'u_{xy}+c'u_{yy}$$

and

(4.3) $$v=u_x,\qquad w=u_y,$$

then we can write

$$L_0u=av_x+bw_x+bv_y+cw_y,$$

$$M_0u=a'v_x+b'w_x+b'v_y+c'w_y.$$

Now set

$$L_0u\cdot\overline{M_0u}=J_0+J_1,$$

where

(4.4) $$J_0=(av_x+bw_x)(a'\bar{v}_x+b'\bar{w}_x)+(bv_y+cw_y)(b'\bar{v}_y+c'\bar{w}_y)$$
$$+(av_y+bw_y)(b'\bar{v}_x+c'\bar{w}_x)+(bv_x+cw_x)(a'\bar{v}_y+b'\bar{w}_y)$$

and

(4.5) $$J_1=(av_x+bw_x)(b'\bar{v}_y+c'\bar{w}_y)-(av_y+bw_y)(b'\bar{v}_x+c'\bar{w}_x)$$
$$+(bv_y+cw_y)(a'\bar{v}_x+b'\bar{w}_x)-(bv_x+cw_x)(a'\bar{v}_y+b'\bar{w}_y).$$

Since $v_y = w_x = u_{xy}$, we have $J_0 \geqq c_3(|u_{xx}|^2 + 2|u_{xy}|^2 + |u_{yy}|^2)$ with c_3 independent of u and of (x, y), because of the ellipticity of L and M.[3]
Accordingly

$$(4.6) \qquad \iint_\Omega J_0 dxdy \geqq c_3(\|u\|^2_{2,L^2} - \|u\|^2_{1,L^2}).$$

Next, we look at J_1. Because $\mathrm{Re}(v_x\bar{v}_y - v_y\bar{v}_x) = \mathrm{Re}(w_x\bar{w}_y - w_y\bar{w}_x) = 0$, we have

$$\mathrm{Re}\, J_1 = F(v_x\bar{w}_y - v_y\bar{w}_x - w_x\bar{v}_y + w_y\bar{v}_x)$$

where $F = \frac{1}{2}(ac' + ca' - 2bb')$. Furthermore,

$$\begin{aligned}\mathrm{Re}\, J_1 dxdy &= F(dv \wedge d\bar{w} - dw \wedge d\bar{v}) \\ &= d\{F(vd\bar{w} - wd\bar{v})\} + (vd\bar{w} - wd\bar{v}) \wedge dF \\ &= d\omega + J_2 dxdy\end{aligned}$$

where $\omega = F(vd\bar{w} - wd\bar{v})$, and J_2 is a sum of products of the first derivatives of F, the first derivatives of u and the second derivatives of u. Therefore for an arbitrary small positive number ε,

3) Using the equality $v_y = w_x$, we obtain

$$\begin{aligned} J_0 &= (av_x + bv_y)(a'\bar{v}_x + b'\bar{w}_x) + (bw_x + cw_y)(b'\bar{v}_y + c'\bar{w}_y) + \\ &\quad + (aw_x + bw_y)(b'\bar{v}_x + c'\bar{w}_x) + (bv_x + cv_y)(a'\bar{v}_y + b'\bar{w}_y) \\ &= a'\{\bar{v}_x(av_x + bv_y) + \bar{v}_y(bv_x + cv_y)\} \\ &\quad + b'\{\bar{w}_x(av_x + bv_y) + \bar{w}_y(bv_x + cv_y)\} \\ &\quad + b'\{\bar{v}_x(aw_x + bw_y) + \bar{v}_y(bw_x + cw_y)\} \\ &\quad + c'\{\bar{w}_x(aw_x + bw_y) + \bar{w}_y(bw_x + cw_y)\}. \end{aligned}$$

Set $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, then there exists a real orthogonal (2, 2)-matrix T such that ${}^tTAT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} (\lambda_1, \lambda_2 > 0)$. Setting $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = {}^tT\begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = {}^tT\begin{bmatrix} w_x \\ w_y \end{bmatrix}$, we obtain

$$\begin{aligned} J_0 &= a'(\lambda_1|V_1|^2 + \lambda_2|V_2|^2) + b'(\lambda_1 V_1\bar{W}_1 + \lambda_2 V_2\bar{W}_2) + \\ &\quad + b'(\lambda_1\bar{V}_1 W_1 + \lambda_2\bar{V}_2 W_2) + c'(\lambda_1|W_1|^2 + \lambda_2|W_2|^2) \\ &\geqq c_3(|V_1|^2 + |V_2|^2 + |W_1|^2 + |W_2|^2) \\ &= c_3(|u_{xx}|^2 + 2|u_{xy}|^2 + |u_{yy}|^2), \end{aligned}$$

where we used the positivity of the matrix $A' = \begin{bmatrix} a' & b' \\ b' & c' \end{bmatrix}$.

(4.7)
$$\left|\iint_{\Omega} J_2 dxdy\right| \leqq c_4(\varepsilon\|u\|_{2,L^2}^2 + \varepsilon^{-1}\|u\|_{1,L^2}^2).$$

We will estimate

$$\iint_{\Omega} d\omega = \int_{\substack{y=0 \\ 0\leqq x\leqq d}} \omega + \int_{\substack{x=d \\ 0\leqq y\leqq d^\alpha}} \omega - \int_{\substack{y=x^\alpha \\ 0\leqq x\leqq d}} \omega \quad \equiv I_1 + I_2 + I_3$$

Because $u=v=dv=0$ on $y=0$, $I_1=0$. Similarly $I_2=0$. As $v=-\alpha x^{\alpha-1}w$ on $y=x^\alpha$, so

$$I_3 = -\int_0^d F\cdot\alpha(\alpha-1)x^{\alpha-2}\cdot|w|^2 dx$$

We pass to the coordinate system (ξ, η) defined by (3.1), then

$$dx = -(\beta-1)^{\beta}\cdot\xi^{-\beta}d\xi, \qquad x^{\alpha-2} = (\beta-1)^{-(\beta-2)}\xi^{\beta-2},$$

and

$$w = (\beta-1)^{-\beta}\xi^{\beta}\times\frac{\partial u}{\partial\eta}(\xi, 1)$$

Therefore

(4.8)
$$|I_3| \leqq c_5\int_A^\infty \xi^{2\beta-2}\left|\frac{\partial u}{\partial\eta}\right|^2 d\xi \leqq c_5 A^{-2}\int_A^\infty\left|\xi^\beta\frac{\partial u}{\partial\eta}\right|^2 d\xi.$$

If we take an infinitely differentiable function $\gamma(\eta)$ such that

$$\gamma(\eta) = \begin{cases} 0 & \text{if } \eta\leqq\frac{1}{3} \\ 1 & \text{if } \eta\geqq\frac{1}{2} \end{cases}$$

then

(4.9)
$$\int_A^\infty\left|\xi^\beta\frac{\partial u}{\partial\eta}\right|^2 d\xi = \int_A^\infty\int_0^1 \frac{\partial}{\partial\eta}\left\{\gamma(\eta)\xi^{2\beta}\left|\frac{\partial u}{\partial\eta}\right|^2\right\} d\eta d\xi$$
$$\leqq \int_A^\infty\int_0^1 |\gamma'(\eta)|\cdot\left|\xi^\beta\frac{\partial u}{\partial\eta}\right|^2 d\eta d\xi +$$

$$+2\int_A^\infty\int_0^1|\gamma(\eta)|\cdot\left|\xi^\beta\frac{\partial u}{\partial\eta}\right|\cdot\left|\xi^\beta\frac{\partial^2 u}{\partial\eta^2}\right|d\eta d\xi$$

$$\leqq c_6\|u\|^2_{2,L^2_{\xi,\eta}}\leqq c_7\|u\|^2_{2,L^2_x}$$

Thus

$$\left|\int_{\partial\Omega}\omega\right|\leqq c_8A^{-2}\|u\|^2_{2,L^2_x} \tag{4.10}$$

Taking sufficiently large A, we obtain (4.1) from (4.6), (4.7) and (4.10.)

(Q.E.D.)

Let us finish the proof of Proposition 3. Writing

$$(Lu,\ Mu)=(L_0u,\ M_0u)+\{(L'u,\ M_0u)+(L_0u,\ M'u)\}+\lambda\{(L_0u,\ u)$$
$$+(u,\ M_0u)\}+\lambda\{(L'u,\ u)+(u,\ M'u)\}+\lambda^2\|u\|^2_{L^2}+(L'u,\ M'u),$$

we estimate each term,

$$\mathrm{Re}\,(L_0u,\ M_0u)\geqq c_1\|u\|^2_{2,L^2}-c_2\|u\|^2_{1,L^2}$$

$$|(L'u,\ M_0u)+(L_0u,\ M'u)|\leqq c_9(\varepsilon\|u\|^2_{2,L^2}+\varepsilon^{-1}\|u\|^2_{1,L^2})$$

$$\lambda\,\mathrm{Re}\{(L_0u,\ u)+(u,\ M_0u)\}\geqq\lambda c_{10}(\|u\|^2_{1,L^2}-\|u\|^2_{L^2})$$

$$\lambda|(L'u,\ u)+(u,\ M'u)|\leqq\lambda c_{11}(\varepsilon\|u\|^2_{1,L^2}+\varepsilon^{-1}\|u\|^2_{L^2}).$$

$$|(L'u,\ M'u)|\leqq c_{12}\|u\|^2_{1,L^2}$$

Summing them up, we have

$$\mathrm{Re}\,(Lu,\ Mu)\geqq(c_1-\varepsilon c_9)\|u\|^2_{2,L^2}+(\lambda c_{10}-\lambda\varepsilon c_{11}-\varepsilon^{-1}c_9-c_2-c_{12})\|u\|^2_{1,L^2}$$
$$+(\lambda^2-\varepsilon^{-1}\lambda c_{11}-\lambda c_{10})\|u\|^2_{L^2}.$$

There exist a large λ and a small ε such that

$$c_1-\varepsilon c_9>0$$

$$\lambda c_{10}-\lambda\varepsilon c_{11}-\varepsilon^{-1}c_9-c_2-c_{12}\geqq 0$$

and

$$\lambda^2 - \varepsilon^{-1}\lambda c_{11} - \lambda c_{10} \geqq 0.$$

Proposition 3 is hence established.

5. Outline of the proof of Theorem 1.

Using a partition of unity in the same way as in [2], we have only to examine the case where the boundary has only one edge. And we have only to examine the case where the angle is equal to 0 and the case where the angle is equal to 2π, for in other cases this question is solved in [2].

In case of the angle 2π, we can apply the method of M. Š. Birman and G. E. Skvortsov after mapping a neighborhood of the edge onto a rectangle with a slit OE (fig. 1). But the two sides of OE must be distinguished.

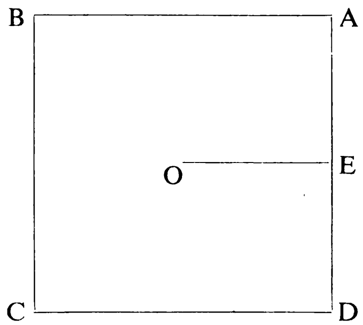

fig. 1

In turn, in case of the angle 0, this question is reduced to Theorem 2 after mapping a neighborhood of the edge onto the domain defined in Theorem 2.

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