J. Math. Kyoto Univ. (JMKYAZ)
16-1 (1976) 209-222

Perturbation of random processes and ergodicity of some simple infinite system

By

Toshio NIWA

(Received April 26, 1975)

0. Introduction

In spite of their great interests in statistical mechanics, very little are knowh about the ergodic properties of infinite systems of particles except the system of hard rods moving in one-dimention [2]. Recently Hardy et al. [1] have studied some interesting two-dimensional system. As is simple its dynamics, it is possible to obtain some ergodic properties, however, only for "linearized" time evolution.

In this paper we propose some simple model systems which are generalizations of the system of Hardy et al. in part, but the domain where collisions do occur is bounded. These models can be seen, in some sense, as the finite systems surround by ideal gasses. We investigate some ergodic properties of these models. We show that these systems are Bernoulli systems (Theorem 1 of section 1), therefore, they have mixing properties, and that the time correlation functions are decreasing exponentially (Theorem 2 of section 1).

Unfortunately, our systems have no interactions between particles except those of which are in some bounded domain. So the systems are to be considered as "perturbed ideal gasses". However it seems to me that the dissipative character of the interactions together with the statistical nature of the systems, that is, the infinite many of degrees of freedom of the systems will play some important role for the ergodicity even for the unrestricted systems.

In section 1, we describe the models in detail and state the main results. In section 2, we prove them by reformutating them in different ways. The concepts raising in them may be interesting in themselves.

Constant discussions with Mr. Y. Tsujii were useful for me. I also thank to Professor H. Totoki to whom the proof of the Theorem 3 of section 2 is partialy due.

1. Descript.ons of the models and results

1.1 Let $\boldsymbol{Z}^\nu$ be ν-dimensional integral lattice. On each lattice site there are at most 2ν particles. The velocity of a particle is one of the 2ν unit vectors $(1,0,\cdots,0)$, $(0,1,\cdots,0)$, $\cdots$, $(0,0,\cdots,-1)$. The configurations where there are at least two particles with the same velocity on the same lattice site are excluded.

More precisely, the phase space $\mathcal{X}$ of allowed configulations of particles is

$$\mathcal{X}=\{X;\, X\colon \boldsymbol{Z}^\nu\times\boldsymbol{P}\to\{0,1\}\}$$

where

$$\boldsymbol{P}=\{v=(v_1,v_2,\cdots,v_\nu)\in\boldsymbol{Z}^\nu;\ |v|=|v_1|+|v_2|+\cdots+|v_\nu|=1\}.$$

Then, naturally we have

$$\mathcal{X}=\{0,1\}^{\boldsymbol{Z}^\nu\times\boldsymbol{P}}$$
$$\cong\prod_{a\in\boldsymbol{Z}^\nu}\mathcal{X}_a\cong\prod_{v\in\boldsymbol{P}}\mathcal{X}_v,$$

where

$$\mathcal{X}_a=\{X_a;\, X_a=X|_{\{a\}\times\boldsymbol{P}}\colon \{a\}\times\boldsymbol{P}\cong\boldsymbol{P}\to\{0,1\}\},$$

and

$$\mathcal{X}_v=\{X_v;\, X_v=X|_{\boldsymbol{Z}^\nu\times\{v\}}\colon \boldsymbol{Z}^\nu\times\{v\}\cong\boldsymbol{Z}^\nu\to\{0,1\}\}.$$

These spaces are compact with product topology.

1. 2 Now let us define the time evolution T.

T is made up of the free motion T_0 and the collision C.
T_0 is merely a translation:

$$T_0X(a, v) = X(a-v, v).$$

C is defined by the interaction $\varphi = \prod_{a\in \mathbf{Z}^\nu} \varphi_a$;

$$\varphi_a : \mathcal{X}_a \times \prod_{b\in R(a)\setminus\{a\}} \mathcal{X}_b \to \mathcal{X}_a$$

$$(CX)_a = \varphi_a(X_a, X_b; b\in R(a)\setminus\{a\}), \quad \text{for } \forall X\in\mathcal{X}$$

where $R(a) = \{a+b; b\in R_\varphi\}$, and the interaction range R_φ of φ is a bounded set of $\mathbf{Z}^\nu$.

It is natural to assume that the interaction φ preserves the number of particles on each site

$$\sum_{v\in \mathbf{P}} X_a(v) = \sum_{v\in \mathbf{P}} (CX)_a(v),$$

and for each $a\in \mathbf{Z}^\nu$ $\varphi_a(\cdot, X_b; b\in R(a)\setminus\{a\})$ is a bijection of $\mathcal{X}_a$ for every fixed X_b, $b\in R(a)\setminus\{a\}$.

As an example, let $\nu=2$.

Assume the interaction φ has zero range, that is, $R_\varphi = \{(0, 0, \cdots, 0)\}$, and preserves the number of particles and momentum on each latties site; that is,

$$\sum_{v\in \mathbf{P}} X_a(v) = \sum_{v\in \mathbf{P}} (CX)_a(v),$$

and

$$\sum_{v: X_a(v)=1} v = \sum_{v: (CX)_a(v)=1} v \qquad \text{for } \forall a\in \mathbf{Z}^\nu,\ \forall X\in\mathcal{X}$$

then φ_a is trivial, that is, $(CX)_a = X_a$, or the one considered in [1].

Now we define the time evolution map T by $T=CT_0$. (We can also define the T by $T=T_0CT_0$. We can handle this case similarly.)

As is mentioned in the introduction we consider only the case where φ_a are trivial except these a's which belong to some fixed finite set V of Z^ν:

$$(CX)_a = X_a \quad \text{if} \quad a\notin V.$$

1. 3 We denote by $(\mathcal{X}, \mathfrak{A}, \mu)$ the measure spase $\mathcal{X}$, where $\mathfrak{A}$ is the algebra generated by the cylinder sets of $\mathcal{X}$, and μ is a measure on $\mathfrak{A}$.

It is an interesting problem to define the equilibrium states μ, that is T-invariant states μ on $\mathcal{X}$ [1].

In this paper we consider only the case where μ has no correlations between sites and velocities:

$\mu=\prod_{a\in Z^\nu}\otimes\mu_a$, where $\mu_a=\mu_0(\forall a\in \boldsymbol{Z}^\nu)$ is a probability measure on $\mathcal{X}_a$, and $\mu=\prod_{v=\boldsymbol{P}}\otimes\mu_\nu$, where μ_ν is a probability measure on $\mathcal{X}_v$. Then T-invariance of μ is equivelent to the φ_a-invariance of μ_a, that is

$$\mu_a(X_a)=\mu_a((CX)_a) \quad \text{for} \quad \forall X\in\mathcal{X}.$$

Now we give the definition which plays the essential role in our paper.

Definition 1. An interaction $\varphi=\prod_{a\in Z^\nu}\varphi_a$ is said to be *dissipative* if the system defined by the interaction has following property:

For any bounded subset K of $\boldsymbol{Z}^\nu$, let $X\in\mathcal{X}$ be such configulation that

$$X(a,v)=0 \qquad \text{unless} \quad a\in K.$$

Then for some number $n>0$ depending on X,

$$T^nX(a,v)=0 \qquad \text{for all } a\in K \text{ and } v\in\boldsymbol{P}.$$

The interaction given in the above example is dissipative. More generally it is not hard to see that if the interaction φ has zero range and preserves the number of particles and total momentum on each lattice site then φ is dissipative.

1. 4 We are now in the place that we can state our results.

Theorem 1. *Assume that the system* $(\mathcal{X},\mu,T)$ *satisfies the following conditions*:

(1) *The interaction*, $\varphi=\prod_{a\in Z^\nu}\varphi_a$ *which defiines the time evolution* $T=C\cdot T_0$ *is dessipative.*

(2) φ_a *is trivial if* a *does not belong to some fixed bounded set* V *of* $\boldsymbol{Z}$.

(3) *The state* μ *has no correlatsons between sites and velocities, that is,* μ *is of the form*, $\mu=\prod_{a\in Z^\nu}\otimes\mu_a=\prod_{v\in p}\otimes\mu_v$, $\mu_a=\mu_0$.

$(\forall a\in\boldsymbol{Z}^\nu)$, *where* μ_a *are* φ_a*-invariant. Then the dynamical sys-*

tem $(\mathfrak{X}, \mu, T)$ *is a Bernoulli system* [6].

Theorem 2. *Let* $(\mathfrak{X}, \mu, T)$ *be as in Theorem* 1. *Then for any cylinder sets* A *and* B *of* $\mathfrak{X}$, *we have*

$$|\mu(T^n A \cap B) - \mu(A)\mu(B)| \leq \text{const.}\, r^n$$

for all $n \geqq 0$, *where const. and* r *depend only on the supports of* A *and* B, *and* $0 < r < 1$.

2. Processes with interactions and the proofs of theorems.

2.1 Let us consicer the physical systems $\mathscr{S}_i = \mathscr{S}_i(M_i, H_i)$ $(i=1, 2, \cdots, N)$ with Hamiltonians H_i and phare spaces M_i respectively. Let $\{\varphi_t^{(i)}\}$ be the time evolutions of $\mathscr{S}_i$ induced from H_i. If they are in equilibrium states, they are represented by invariant probability measures μ_i on M_i respectively.

If these systems $\mathscr{S}_i$ are coupled together and the mutual interactions are negligible, then the coupled system $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2 \times \cdots \times \mathscr{S}_N$ has the Hamiltonian $H = H_1 + H_2 + \cdots + H_N$ and the phase space $M = M_1 \times M_2 \times \cdots \times M_N$ (product space of $M_1, M_2, \cdots$, and M_N).

As mutual interactions are negligible, the obtained equilibrium state of the coupled system $\mathscr{S}$ is represented by the direct propudt measure $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_N$ of the measures $\mu_1, \mu_2, \cdots$, and μ_N on $M = M_1 \times M_2 \times \cdots \times M_N$ [4].

Now let $(M, \mu, \{\varphi^n\})$ be a dynamical system with discrete time. We can represent it by a symbolic dynamics (Ω, ρ, T); Ω is the set of doubly infinite sequences $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots)$ of elements of S:

$$\Omega = \prod_{n \in \mathbf{Z}} S_n \to \omega = \{\omega_n\},$$

$$\omega_n \in S_n = S = \{a_0, a_1, \cdots, a_{s-1}\}.$$

T is the shift of Ω:

$$(T\omega)_n = \Omega_{n-1}.$$

ρ is a T-invariant probability measure on Ω.

Hence-forth we call the (Ω, ρ, T) s-shift. Then the representation of the dynamical system $(M, \mu, \{\varphi^n\})$ by the s-shift (Ω, ρ, T) means the mapping π of M to Ω such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M \\ \downarrow{\scriptstyle\pi} & T & \downarrow{\scriptstyle\pi} \\ \Omega & \longrightarrow & \Omega \end{array}$$

commutes and $\pi(\mu)=\rho$ (see [3], [5]).

Let (Ω_i, ρ_i, T_i) be s_i-shift representing the dynamical system $(M_i, \mu_i, \{\varphi_i{}^n\})$ obtained from the system $\mathscr{S}_i(M_i, H_i)$. Then the product system $\mathscr{S}=\mathscr{S}_1\times\mathscr{S}_2\times\cdots\times\mathscr{S}_N$ is represented by the product s-shift (Ω, ρ, T_0), where $s=s_1\cdot s_2\cdot\cdots\cdot s_N$ and $(\Omega, \rho, T)=\prod_{i=0}^{N}(\Omega_i, \rho_i, T_i)$, if the Hamiltonian H of the coupled system $\mathscr{S}$ is exactly the sum of H_1, $H_2, \cdots$, and H_N, that is, there are no mutual interactions. If we take mutual interactions into considerations, they are represented by a map C of Ω, and the time evolution of the coupled system $\mathscr{S}$ will be represented by the composed map $T=C\cdot T_0$. The fact that the mutual interactions are "negligible" is represented by the C-invariance of $\rho=\rho_i\otimes\rho_2\otimes\cdots\otimes\rho_N$.

In this context it is interesting that in some cases ρ is completely determined by the map C([1], [4]).

We do not dwell on this problem.

2.2 In this way we arrive at the following notation. Let $(\boldsymbol{X}, \rho, T_0)$ be s-shift:

$$\boldsymbol{X}=\prod_{n\in\boldsymbol{Z}} S_n, \quad S_n=S=\{0, 1, 2, \cdots, s-1\}.$$

For any $\omega=\{\omega_n\}\in\boldsymbol{X}$

$$(T_0\omega)_n=\omega_{n-1},$$

and ρ is a T_0-invariant probability measure on $\boldsymbol{X}$. We denote by π_K the projection of $\boldsymbol{X}$ onto

$$S_K=\prod_{n\in\boldsymbol{K}} S_n$$

for any subset K of $\boldsymbol{Z}$:

$$\pi_K: \boldsymbol{X}\to S_K: \pi_K(\omega)=\omega_K=\{\omega_n\}_{n\in K} \quad \text{for } \forall\omega=\{\omega_n\}_{n\in\boldsymbol{Z}}.$$

Now we give the following

Definition 2. A mapping C of $(\boldsymbol{X}, \rho)$ is called an *interaction* or a *collision* of $(\boldsymbol{X}, \rho, T_0)$ if it satisfies the following conditions.

(1) C is an automorphism of $(\boldsymbol{X}, \rho)$, that is, invertible ρ-preserving measurable transformation of $(\boldsymbol{X}, \rho)$.

(2) For any finite subset K of $\boldsymbol{Z}$, there exists a finite subset K' of $\boldsymbol{Z}$ such that for any two elements ω, ω' of $\boldsymbol{X}$, $\pi_{K'}(\omega)=\pi_{K'}(\omega')$ implies $\pi_K(C\omega)=\pi_K(C\omega')$. In particular if C satisfies in addition the following conditisn, the C is called a *local collision on* K.

(3) C is the identity on$\boldsymbol{Z}\backslash K$, that is

$$\pi_{Z\backslash K}(C\omega)=\pi_{Z\backslash K}(\omega) \qquad \text{for all } \omega\in \boldsymbol{X}.$$

The obtained dynamical system $(\boldsymbol{X}, \rho, T)$ is called a *process with interactson* C, where T is defined by $T=C\cdot T_0$.

As mentioned above it is an interesting problm to investigate the relation between interactions C and shift-invariant measures ρ.

We do not dwell on this problem here. We give another notion.

Definition 3. An interaction C of $(\boldsymbol{X}, \rho, T_0)$ is called to be *dissipative*, if it has following properties:

(1) $C\theta=\theta$, where θ denotes the "vacuum" element of X, that is, $\theta_n=0$ for all $n\in \boldsymbol{Z}$.

(2) Take any finite set K of $\boldsymbol{Z}$, if $\omega\in \boldsymbol{X}$ satisfies $\pi_{Z\backslash K}(\omega)=\pi_{Z\backslash K}(\theta)$ then there exists a number $n>0$ depending on ω such that

$$\pi_K(T^n\omega)=\pi_K(\theta).$$

Remark. Let C be a dissipative local collision on $K=(0, k]$, then there exists a number $n_K>0$ independent of ω such that for any $\omega\in \boldsymbol{X}$ and $n\geq n_K$, if

$$\pi_{(-n,0]}(\omega)=\pi_{(-n,0]}(\theta) \quad \text{then} \quad \pi_K(T^n\omega)=\pi_K(\theta)$$

Here we use the notation

$$(a, b]=\{n\in \boldsymbol{Z}; a<n\leq b\}$$

for any a, $b\in \boldsymbol{Z}$.

2.3 We will now on deal with only the dissipative local collisions on $K=(0, k]$. Then we get the following

Theorem 3. *Let* $(\boldsymbol{X}, \rho, T)$ *be a process with interaction* C. *If* C *is a dissipative local collision on* $K=(0, k]$ *and* ρ *is a direct product probabillity measupe on* $\boldsymbol{X}$ *such that*

$$1>\rho_0=\rho\{\omega;\ \omega_0=0\}>0.$$

then $(\boldsymbol{X}, \rho, T)$ *is a Bernoulli system.*

Theorem 4. *Under the same assmptions as theorem 3, we have*

$$|\rho(T^n A\cap B)-\rho(A)\rho(B)|\leq \text{const.}\ r^n$$

for all $n\geq 0$ *and for any cylinder sets* A *and* B *of* $\boldsymbol{X}$. *Here const. and* r *are constant numbers depending on the supports of* A *and* B, *and* $0<r<1$.

2.4 *Proof of theorem 3:*

Let $\xi=\{C_0, C_1, \cdots, C_{s-1}\}$ be a partition of $\boldsymbol{X}$ such that

$$C_i=\{\omega\in\boldsymbol{X};\ \omega_0=i\}\qquad \text{for}\quad i\in S=\{0, 1, \cdots, s-1\}.$$

As $T^{-n}C_i=\{\omega\in\boldsymbol{X};\ \omega_{-n}=i\}$ for all $n\geq 0$ and $i\in S$, it is clear that ξ, $T^{-1}\xi$, $\cdots$, $T^{-n}\xi$, $\cdots$ are mutually independent. Therefore ξ is a Bernoulli-partion for T, that is, $\{T^n\xi;\ n\in\boldsymbol{Z}\}$ are also mutually independent.

We have to show that

$$\bigvee_{n=-\infty}^{\infty} T^n\xi=\varepsilon\quad (=\text{partition into individual points}).$$

Let $\quad J_i=\{\omega\in\boldsymbol{X};\ \pi_{(-n_K(i+1), -n_K i]}(\omega)=\pi_{(-n_K(i+1), -n_K i]}(0)\}$

where n_K is the number defined in the above remark.

Let $$\zeta_0=\bigvee_{n=0}^{\infty} T^{-n}\xi.$$

We note that $J_i\,(i\geq 0)$ are contained in $\mathscr{B}(\zeta_0)$, σ-algebra of $\boldsymbol{X}$ generated by ζ_0. As ρ is a direct product probability measure with $\rho_0>0$, so it is clear that

$$\rho(\boldsymbol{X}\setminus\bigcup_{i=1}^{\infty} J_i)=0.$$

Now let ω and ω' of $\bigcup_{i=0}^{l} J_i$ be ζ_0-equivalent, that is, belong to the same element of ζ_0. If $\omega\in J_l(0\leq l\leq j)$, then $\omega'\in J_l$, and by the dissipativeness and the locality of the interaction C we have

$$(T^{n_K(l+1)}\omega)_i=(T^{n_K(l+1)}\omega')_i \qquad \text{for} \quad \forall i\geq k.$$

Hence

$$(T^{n_K(j+1)}\omega)_i=(T^{n_K(j+1)}\omega')_i \qquad \text{for} \quad \forall i\leq k.$$

Therefore we get

$$(T^{n_K(j+1)+p}\omega)_i=(T^{(n_K(j+1)+p}\omega')_i \qquad \text{for} \quad \forall i\leq k+p.$$

This means

$$T^{n_K(j+1)+p}\zeta_0\geq T_0{}^{k+p}\zeta_0 \qquad \text{on} \quad \bigcup_{i=0}^{j} J_i.$$

Hence

$$\bigvee_{n=0}^{\infty} T^n\zeta_0\geq\bigvee_{n=0}^{\infty} T_0{}^n\zeta_0=\varepsilon \qquad \text{on} \quad \bigcup_{i=0}^{j} J_i.$$

As j is arbitrary, so

$$\bigvee_{n=0}^{\infty} T^n\zeta_0=\varepsilon \pmod 0,$$

and

$$\bigvee_{n=-\infty}^{\infty} T^n\xi=\bigvee_{n=0}^{\infty} T^n\zeta_0=\varepsilon. \qquad \textbf{q.e.d.}$$

2. 5 Now let us prove the theorem 4.

We use the following notation: Let L be a subset of $\boldsymbol{Z}$ and $\alpha_L\in\prod_{n\in L} S_n$.

$$[\alpha_L]=\{\omega\in\boldsymbol{X};\ \pi_L(\omega)=\alpha_L\}.$$

First we assume that B is a cylinder set on $(0,b]$ $(b\geq k)$, i.e.

$$B=\{\omega\in\boldsymbol{X};\ \pi_{(0,b]}(\omega)\in B_{(0,b]}\}$$

for some $B_{(0,b]}\subset S_{(0,b]}$.

Let

$$J_i=[\pi_{(-m(i+1),-mi]}(\theta)] \qquad \text{for} \quad i\geq 0,$$

where $m=b-k+n_K$.

We denote from now on that

$$\pi_{(-m(i+1),-mi]}=\pi_i$$

for the brevity.

Let

$$I_j=J_j-\bigcup_{i=1}^{j=1}J_i,$$

and $\Omega_j=\{\omega;\pi_\ell(\omega)\neq\pi_\ell(\theta)\text{ for }0\leq\forall\ell\leq j-1\}$.
Assume that $A=[\alpha_K]$ fore some $\alpha_K\in S_K$, and denote

$$A_j=A\cap I_j.$$

Then

$$A_j=\bigcup_{\omega\in\Omega_j}[\alpha_K]\cap[\pi_{(-mj,0]}(\omega)]\cap[\pi_j(\theta)]. \tag{1}$$

By the locality and the dissipativeness of C we can get

$$T^{m(j+1)}([\alpha_K]\cap[\pi_{(-mj,0]}(\omega)]\cap[\pi_j(\theta)])$$
$$=[\pi_{(0,b]}(\theta)]\cap[\varphi_j(\alpha_K,\pi_{(-mj,0]}(\omega)))_{(b,m(j+1)+k]}]$$

Here $\varphi_j(\alpha_K,\pi_{(-mj,0]}(\omega))$ is some element of $S_{(b,m(j+1)+k]}$ depending on $\alpha_k\in S_K$ and $\pi_{(-mj,0]}(\omega)\in S_{(-mj,0]}$, whose explicit from is not necessary to know.

Hence

$$T^{m(j+1)+p}([\alpha_K]\cap[\pi_{(-mj,0]}(\omega)]\cap[\pi_j(\theta)])$$
$$=T^p(O)\cap T^p[(\varphi_j(\alpha_K,\pi_{(-mj,0]}(\omega)))_{(b,m(j+1)+k]}]. \tag{2}$$

where $O=[\pi_{(0,b]}(\theta)]$.

Taking the sum for ω over Ω_j we get

$$T^{m(j+1)+p}(A_j)\cap B$$
$$=(T^p(O)\cap B)\cap(\bigcup_{\omega\in\Omega_j}T^p[(\varphi_j(\alpha_K,\pi_{(-mj,0]}(\omega)))_{(b,m(j+1)+k]}]).$$

Hence

$$\rho(T^{m(j+1)+p}(A_j)\cap B)=\rho(T^p(O)\cap B)$$
$$\times\sum_{\pi_{(-mj,0]}(\omega);\,\omega\in\Omega_j}{}'\rho([(\varphi_j(\alpha_K,\pi_{(-mj,0]}(\omega)))_{(b,m(j+1)+k]}]). \tag{3}$$

On the hther hand, taking the sum over Ω_j in (2) we have

$$\rho(A_j)=\rho(O)\times\sum_{\pi_{(-mj,0]}(\omega);\,\omega\in\Omega_j}([\varphi_j(\alpha_K,\pi_{(-mj,0]}(\omega)))_{(b,m(j+1)+K]}]). \quad (4)$$

Note also that

$$\rho(A_j)=\rho(A)\rho(I_j). \quad (5)$$

Combining these (3), (4) and (5) we get

$$\rho(T^{m(j+1)+p}(A_j)\cap B)$$

$$=\frac{\rho(A)}{\rho(O)}\rho(T^p(O)\cap B)\rho(I_j).$$

Finary we can get

$$\rho(T^n(A)\cap B)$$

$$=\rho(T^n(\bigcup_{j=0}^{q}A_j)\cap B)+\rho(T^n(A\backslash\bigcup_{j=0}^{q}A_j)\cap B)$$

$$=\frac{\rho(A)}{\rho(O)}\sum_{j=0}^{q}\rho(I_j)\cdot\rho(T^{n-m(j+1)}(O)\cap B)+\rho(T^n(A\backslash\bigcup_{j=0}^{q}A_j)\cap B). \quad (4)$$

for $n\geq m(q+1)$.

Summing up over S_K, the left hand side becomes

$$\sum_{A=[\alpha_K];\,\alpha_K\in S_K}(T^n(A)\cap B)=\rho(T^n(\boldsymbol{X})\cap B)=\rho(B),$$

and the right hand side becomes

$$\frac{1}{\rho(O)}\sum_{j=0}^{q}\rho(I_j)\rho(T^{n-m(j+1)}(O)\cap B)$$

$$+\sum_A\rho(T^n(A\backslash\bigcup_{j=0}^{q}A_j)\cap B)$$

$$=\frac{1}{\rho(O)}\sum_{j=0}^{q}\rho(J_j)\rho(T^{n-m(j+1)}(O)\cap B)$$

$$+\rho(T^n(\boldsymbol{X}\backslash\bigcup_{j=0}^{q}J_j)\cup B).$$

Hence

$$\rho(B)=\frac{1}{\rho(O)}\sum_{j=0}^{q}\rho(I_j)\rho(T^{n-m(j+1)}(O)\cap B)$$

$$+\rho(T^n(\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j)\cap B). \tag{7}$$

Consequently from (6) and (7) we get

$$\rho(T^n(A)\cap B)=\rho(A)\{\rho(B)-\rho(T^n(\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j)\cap B)\}$$

$$+\rho(T^n(A\setminus\bigcup_{j=0}^{q}A_j)\cap B).$$

Hence we have for any cylinder set A on $(0,k]$,

$$|\rho(T^n(A)\cap B)-\rho(A)\rho(B)|$$

$$=|\rho(T^n(A\cap(\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j))\cap B)$$

$$-\rho(A)\rho(T^n[\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j)\cap B)|$$

$$\leq\rho(T^n(\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j)\cup B)$$

$$\leq\rho(\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j),\quad \text{for } n\geq(b-k+n_K)(q+1).$$

Note that

$$\rho(\boldsymbol{X}\setminus\bigcup_{j=0}^{q}J_j)=(1-\rho_0{}^m)^q.$$

From this we can easily get

$$|\rho(T^n(A)\cap B)-\rho(A)\rho(B)|\leq\text{const.}\,r^n$$

for all $n\geq0$, where $r=(1-\rho_0{}^m)<1$ and $\text{const.}=r^{(b-k+n_K)}$.

Thus we have proved theorem for the cylinder set B on $(0,b]$ and the cylinder set A on K.

In general, let A and B be the cylinder sets on $[-a,a]$, $a\geq k$.

Then, $T^{a+1}(A)$ and $T^{a+1}(B)$ are the cylinder sets on $(0,2a+1]$. We can assume that $T^{a+1}(A)$ is a thin cylinder set on $(0,2a+1]$, that is, we can set that

$$T^{a+1}(A)=A_1\cap A_2,$$

where $A_1=[\alpha_{(0,k]}]$ for some $\alpha_{(0,k]}\in S_{(0,k]}$ and $A_2=[\alpha'_{(k,2a+1]}]$ for some $\alpha'_{(k,2a+1]}\in S_{(k,2a+1]}$.

Then

$$T^{n+a+1}(A)=T^{n}(A_1)\cap T^{n}(A_2).$$

Note that for $n>2a+1-k$, $T^{n}(A_2)$ is a cylinder set on $(\bar{a}, 2a+1]$, $\bar{a}=n+k>2a+1$, and $T^{n}(A_1)\cap T^{a+1}(B)$ is acylinder set on $(0, \bar{a}]$. Hence $T^{n}(A_1)\cap T^{a+1}(B)$ and $T^{n}(A_2)$ are mutually independent. Therefore

$$\begin{aligned}
&\rho(T^{n}(A)\cap B)-\rho(A)\rho(B)\\
&\quad=\rho(T^{a+1}(T^{n}(A)\cap B))-\rho(T^{a+1}(A))\rho(T^{a+1}(B))\\
&\quad=\rho(T^{n}(T^{a+1}(A))\cap T^{a+1}(B))-\rho(T^{a+1}(A))\rho(T^{a+1}(B))\\
&\quad=\rho(T^{n}(A_1)\cap T^{n}(A_2)\cap T^{a+1}(B))-\rho(A_1\cap A_2)\rho(T^{a+1}(B))\\
&\quad=\rho(T^{n}(A_2))\rho(T^{n}(A_1)\cap T^{a+1}(B))-\rho(A_1)\rho(A_2)\rho(T^{a+1}(B))\\
&\quad=\rho(A_2)\{\rho(T^{n}(A_1)\cap T^{a+1}(B))-\rho(A_1)\rho(T^{a+1}(B))\}.
\end{aligned}$$

As A_1 is a cylinder set on $(0, k]$ and $T^{a+1}(B)$ on $(0, 2a+1]$, we have

$$\begin{aligned}
&|\rho(T^{n}(A_1)\cap T^{a+1}(B))-\rho(A_1)\rho(T^{a+1}(B))|\\
&\qquad\leq \text{const.}\, r^{n} \qquad \text{for all } n\geq 0.\\
&|\rho(T^{n}(A)\cap B)-\rho(A)\rho(B)|\\
&\qquad\leq \text{const.}\, r^{n} \qquad \text{for all } n\geq 0.
\end{aligned}$$

q.e.d.

2.6 Appling these theorems we can verify the theorems of section 1.

We show that the system $(\mathscr{X}, \mu, T)$ which satisfies the conditions (1), (2) and (3) of the theorem 1 of the section 1 is a special case of a process $(\boldsymbol{X}, \rho, T)$ with interaction C where C is a dissipative local collision.

Let $V_k=\{(x^1, \cdots, x^\nu)\in \boldsymbol{Z}^\nu;\ |x^i|\leq k \text{ for } i=1, \cdots, \nu\}$ be such a bounded set of $\boldsymbol{Z}^\nu$ that

$$\text{if}\quad a\notin V_k \quad\text{then}\quad \varphi_a=\text{trivial}.$$

It is easy to see that we con concentrate our consideration on such sites $x=(x^1, x^2, \cdots, x^\nu)\in \boldsymbol{Z}^\nu$ that for some $i=1, \cdots, \nu\, |x^i|\leq k$. Because there is no interaction outside the V_k, so the particles on the sites

$x=(x^1, x^2, \cdots, x^\nu)$ where $|x^i|>k$ for all $i=1, 2, \cdots, \nu$ move like ideal gas.

For the simplicity we consider only the case when $\nu=2$ and $k=1$. It is not hard to see in general case.

Now we constract a mapping f of $\mathcal{X}_{\tilde{V}}=\coprod_{a\in\tilde{V}}\mathcal{X}_a$ to $\boldsymbol{X}=S^{\boldsymbol{Z}}$ where $\tilde{V}=\{a=(x^1, \cdots, x^\nu)\in \boldsymbol{Z}^\nu;$ for some $i=1, \cdots, \nu, |x^i|\leq k\}$.

For a configuration $X(a; v)$, $a\in\tilde{V}$, the image $\{\omega_n\}\in S^Z$ of it under f is given by

$$\omega_n=(\varepsilon_n^{\ 1}, \varepsilon_n^{\ 2}, \cdots, \varepsilon_n^{\ 12}).\ \{0, 1\}^{12}=S$$

where

$$\varepsilon_n^{\ 1}=X((n-2, 1); (1, 0))$$

$$\varepsilon_n^{\ 2}=X((n-2, 0); (1, 0))$$

$$\varepsilon_n^{\ 3}=X((n-2, -1); (1, 0))$$

$$\varepsilon_n^{\ 4}=X((-1, n-2); (0, 1))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\varepsilon_n^{\ 12}=X((-1, -n+2); (0, -1))$$

The interactions $\{\varphi_a;\ a\in V\}$ induce the local collision C on $K=(0, 2k+1]$.

The dissipativeness of C follows from the dissipativeness of $\varphi=\{\varphi_a\}$.

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY

References

[1] J. Hardy et al: Time evolution of a two-dimensional model system I, Jour. Math. Phys. Vol. 14, No. 12 (1973) 1746–1759.

[2] Ya. G. Sinai: Ergodic properties of a gas of one-dimensional hard rods with an infinite number of degrees of freedom, Funct. Anal. Appli. Vol. 6, No. 1 (1972) 41–50.

[3] ______: Ergodicity of Boltzmann gas model, Proc. I. U. P. A. P. Meeting, Copenhagen, 1966, Benjamin (1967) 559–573.

[4] A. I. Khinchin: Mathematical foundations of statistical mechaics, Chap. II §8, Dover, New York, (1945).

[5] W. Krieger: On entropy and generators of measurepreserving transformations, Trans. Amer. Math. Soc. 149 (1970) 453–464.

[6] P. Billingsley: Ergodic Theory and Information, J. Wiley & Sons, Inc., New York-London-Sydney, (1965).