J. Math. Kyoto Univ. (JMKYAZ)
16-1 (1976) 117-122

Note on integral closures of a noetherian integral domain

By

Jun-ichi NISHIMURA

(Received Dec. 9, 1974)

In this note we prove a theorem of Mori-Nagata ([4], (33.10)) using elementary facts that a complete local ring is a henselian ring ([4], (30.3)), and that if $\mathfrak{a}_n$ $(n=1,2,\cdots)$ are ideals of a complete semi-local ring with Jacobson radical $\mathfrak{m}$ such that $\mathfrak{a}_{n+1}\subseteq\mathfrak{a}_n$ for any n and such that $\bigcap\mathfrak{a}_n=(0)$, then for any given natural number n, there exists a natural number $m(n)$ such that $\mathfrak{a}_{m(n)}\subseteq\mathfrak{m}^n$ ([4], (30.1)). Using our proof, we can prove the finiteness property for integral extensions of a complete local integral domain ([4], (32.1), cf. [2], [5]), without using the structure theorem of complete local rings ([4], (31.1)). Finally we outline our proof of a theorem of Nagata ([4], (36.5)). In this article, we mean by a ring a commutative ring with identity and by a local ring a noetherian ring with only one maximal ideal. When $\mathfrak{p}$ is a prime ideal of a ring R, we denote by $k(\mathfrak{p})$ the field of quotients of $R/\mathfrak{p}$.

The writer expresses his hearty thanks to Professor M. Nagata for valuable suggestions.

Proposition 1. (Krull-Akizuki, [4], (33.2)) *Let R be a noetherian integral domain with field of quotients K, let L be a finite algebraic extension of K and let R' be a ring such that $R\subseteq R'\subseteq L$. If* altitude $R=1$, *then* i) *R' is a noetherian ring of altitude at most one,* ii) *for every prime ideal $\mathfrak{p}$ of R the number of the prime ideals $\mathfrak{p}'$ of R' such that $\mathfrak{p}=\mathfrak{p}'\cap R$ is finite, and* iii) *$[k(\mathfrak{p}'):k(\mathfrak{p})]$ is finite.*

This is well known and we omit the proof.

Proposition 2. *Let $(R, \mathfrak{m})$ be a henselian local integral domain with field of quotients K, let L be a finite algebraic extension of K and let $(\bar{R}, \bar{\mathfrak{m}})$ be the integral closure of R in L. Then $[\bar{R}/\bar{\mathfrak{m}}: R/\mathfrak{m}]$ is finite.*

Proof. We prove the assertion by induction on the altitude of R. If altitude $R=1$, the assertion is included in Proposition 1. Let altitude $R=n>1$, and assume that the assertion is valid for any henselian local integral domain of altitude at most $n-1$. Take a prime ideal $\mathfrak{p}$ of R of height one and a prime ideal $\bar{\mathfrak{p}}$ of $\bar{R}$ such that $\mathfrak{p}=\bar{\mathfrak{p}}\cap R$. Then $[k(\bar{\mathfrak{p}}):k(\mathfrak{p})]$ is finite by Proposition 1. Let $(\bar{\bar{R}}, \bar{\bar{\mathfrak{m}}})$ be the integral closure of $R/\mathfrak{p}$ in $k(\bar{\mathfrak{p}})$, then by the induction assumption, $[\bar{\bar{R}}/\bar{\bar{\mathfrak{m}}}: R/\mathfrak{m}]$ is finite, Therefore $[\bar{R}/\bar{\mathfrak{m}}: R/\mathfrak{m}]$ is finite. q.e.d.

Let $(R, \mathfrak{m})$ be a local integral domain with field of quotients K, let $(R^h, \mathfrak{m}^h)$ be the henselization of $(R, \mathfrak{m})$ and let $\bar{R}$ be the derived normal ring of R. Then we have three natural one-to-one correspondences between i) the maximal ideals $\{\mathfrak{m}_i\}$ of $\bar{R}$ and those $\{\bar{\mathfrak{m}}_i\}$ of $\bar{R}\otimes_R R^h$, ii) the maximal ideals $\{\bar{\mathfrak{m}}_i\}$ of $R\otimes_R R^h$ and the minimal prime ideals $\{\bar{\mathfrak{q}}_i\}$ of $\bar{R}\otimes_R R^h$, iii) the minimal prime ideals $\{\bar{\mathfrak{q}}_i\}$ of $\bar{R}\otimes_R R^h$ and those $\{\mathfrak{q}_i\}$ of R^h. Since R^h is noetherian, $\bar{R}$ is a quasi-semi-local ring. In these correspondences, $\bar{R}/\mathfrak{m}_i$ is isomorphic to $(\bar{R}\otimes_R R^h)/\bar{\mathfrak{m}}_i$, and $(\bar{R}\otimes_R R^h)_{\bar{\mathfrak{m}}_i}$ is the derived normal ring of $R^h/\mathfrak{q}_i$ ([4], [6]).

Thus we have

Proposition 3. *Let $(R, \mathfrak{m})$ be a local integral domain with field of quotients K, let L be a finite algebraic extension of K and let $\bar{R}$ be the integral closure of R in L. Then the number of maximal ideals $\mathfrak{m}_i$ of $\bar{R}$ is finite and, for each maximal ideal $\mathfrak{m}_i$ of $\bar{R}$, $[\bar{R}/\mathfrak{m}_i: R/\mathfrak{m}]$ is finite.*

Proposition 4. *Let $(R, \mathfrak{m})$ be a complete local integral domain with field of quotients K. If* altitude $R\geqq 2$, *then there exists a finite integral extension $(R', \mathfrak{m}')$ of R such that* i) $R\subseteq R'\subseteq K$, ii) depth $R'\geqq 2$.

Proof. Set $R_0=R$, $\mathfrak{m}_0=\mathfrak{m}$, and define $R_{i+1}=\{x\in K|\mathfrak{m}_i x\subseteq\mathfrak{m}_i\}$ for $i=0,1,2,\cdots$. Then, $(R,\mathfrak{m})=(R_0,\mathfrak{m}_0)\leq(R_1,\mathfrak{m}_1)\leq(R_2,\mathfrak{m}_2)\leq\cdots\leq(R_i,\mathfrak{m}_i)\leq\cdots$, and each $\mathfrak{m}_i$ is an $\mathfrak{m}_{i+1}$-primary ideal of R_{i+1}.

Suppose that depth $R_i=1$ for every i, then there exists a sequence of pairs (a_0,b_0), (a_1,b_1), $\cdots$, (a_i,b_i), $\cdots$ of elements of K such that $a_i,b_i\in R_i$, $(a_iR_i\colon b_i)_{R_i}=\mathfrak{m}_i$.

We claim that for sufficiently large n, $u_n=b_n/a_n$ belongs to R_n (which is a contradiction). $u_n\in R_n\Leftrightarrow\mathfrak{m}_{n-1}u_n\subseteq\mathfrak{m}_{n-1}\Leftrightarrow\mathfrak{m}_{n-1}u_n\subseteq R_{n-1}$ (for, by assumption, altitude $R_n\geqq2)\Leftrightarrow\mathfrak{m}_{n-2}\mathfrak{m}_{n-1}u\subseteq\mathfrak{m}_{n-2}\Leftrightarrow\mathfrak{m}_{n-2}\mathfrak{m}_{n-1}u_n\subseteq R_{n-2}\Leftrightarrow\cdots\Leftrightarrow\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{n-1}u_n\subseteq R_1\Leftrightarrow\mathfrak{m}_0\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{n-1}u_n\subseteq\mathfrak{m}_0$, and this last is implied by $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n\subseteq\mathfrak{m}_0$.

Therefore we are to prove the final inclusion for sufficiently large n.

Now consider the sequence of ideals in R_1: $\mathfrak{m}_1\supseteq\mathfrak{m}_1\mathfrak{m}_2\supseteq\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\supseteq\cdots\supseteq\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_i\supseteq\cdots$, if $\bigcap_{i=1}\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_i=\mathfrak{n}\neq(0)$, then for every j and for $y\in\mathfrak{n}(y\neq0)$, $R_j\subseteq(1/y)R_1$, which is a contradiction. Hence $\mathfrak{n}=(0)$. Then, by ([4], (30.1)), there exists a natural number n_0 such that for every $n\geqq n_0$, $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n\subseteq\mathfrak{m}_0$. q.e.d.

Proposition 5. *Let* $(R,\mathfrak{m})$ *be a complete local integral domain such that* altitude $R\geqq2$, *and let* $\bar{R}$ *be the derived normal ring of* R. *If* $\mathfrak{m}=(m_1,\cdots,m_r)$, *then* $\bigcap_{i=1}^r\bar{R}[1/m_i]=\bar{R}$.

Proof. If $x\in\bigcap_{i=1}^r\bar{R}[1/m_i]$, then there exists a natural number s such that $m_i^s x\in\bar{R}$ for every i. Let $R'=R[m_1^s x,\cdots,m_r^s x]$, then $(R'\colon x)_{R'}$ contains a power $\mathfrak{m}'^n$ of the maximal ideal $\mathfrak{m}'$ of R' for some n. By Proposition 4, we can take a finite R'-algebra R'' such that i) $R'\subseteq R''\subseteq\bar{R}$, ii) depth $R''\geqq2$. Since $R''=\bigcap_{\mathfrak{p}}R''_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through all prime ideals of R'' such that depth $R''_{\mathfrak{p}}=1$, x belongs to R''. q.e.d.

Proposition 6. *Let R be a noetherian integral domain with field of quotients K, let L be a finite algebraic extension of K and let* $\bar{R}$ *be the integral closure of R in L. Then* $\bar{R}$ *is a Krull domain.*

Proof. We prove Proposition 6 in several steps.

Step I. If the assertion of Proposition 6 is true for all noetherian integral domains of altitude at most $n-1$, then the assertion is true for all complete local integral domains of altitude at most n.

Indeed, this follows from Proposition 1 or Proposition 5 depending on whether $n=1$ or $n\geqq 2$.

Step II. If the assertion of Proposion 6 is true for all complete local integral domains of altitude at most n, then the assertion is true for all semi-local integral domains of altitude at most n.

Indeed, since $\bar{R}=\bigcap_{i=1}^{s}\bar{R}_{\mathfrak{m}_i}$, where $\mathfrak{m}_i$ runs through all maximal ideals of R, we may assume that $(R, \mathfrak{m})$ is local and that $L=K$. Let $\hat{R}$ be the completion of R, let $\mathfrak{n}$ be the nilradical of $\hat{R}$, let $R^*=\hat{R}/\mathfrak{n}$ and let $\bar{R}^*$ be the integral closure of R^* in the total quotient ring of R^*. Then, by our assumption, $\bar{R}^*$ is a direct product of Krull domains and $\bar{R}=\bar{R}^*\cap K$ (cf. [4], (33.10)). Hence $\bar{R}$ is a Krull domain.

Step III. If the assertion of Proposition 6 is true for all (semi-) local integral domains of altitude at most n, then the assertion is true for all noetherian integral domains of altitude at most n.

As for this step, since $\bar{R}=\bigcap\bar{R}_{\mathfrak{m}}$, where $\mathfrak{m}$ runs through all maximal ideals of R, it is sufficient to prove the following proposition.

Proposition 7. ([4], (33.11)) *Let R be a noetherian integral domain, let $\bar{R}$ be the derived normal ring of R, let $0\neq f\in R$ and let $\bar{\mathfrak{p}}$ be a prime ideal of $\bar{R}$ of height one. If $\bar{R}_{\bar{\mathfrak{p}}}$ is a discrete valuation ring and if $f\in\bar{\mathfrak{p}}$, then $\mathfrak{p}=\bar{\mathfrak{p}}\cap R$ is an associated prime ideal of fR.*

Proof. We may assume that $(R, \mathfrak{p})$ is a local ring. In the above natural correspondences, let $\bar{\mathfrak{p}}=\mathfrak{m}_1$. Then, since $\bar{R}_{\bar{\mathfrak{p}}}$ is a discrete valuation ring, $(\bar{R}\otimes_R R^h)_{\mathfrak{m}_1}$ is a discrete valuation ring. Hence altitude $R^h/\mathfrak{q}_1=1$. On the other hand, depth $R^h=$ depth $R+$ depth $R^h\otimes R/\mathfrak{p}$, and depth $R^h\leqq\min_{\mathfrak{q}\in\mathrm{Ass}(R^h)}$ {altitude $R^h/\mathfrak{q}$} ([3]). Therefore depth $R=1$.

Step IV. Repeating this process, we see that the assertion of Proposition 6 is true for all local integral domains.

Step V. If the assertion of Proposition 6 is true for all local integral domains, then the assertion is true for all noetherian integral domains.

Indeed, since $\bar{R}=\bigcap \bar{R}_{\mathfrak{m}}$, where $\mathfrak{m}$ runs through all maximal ideals of R, by Step IV and Proposition 7, we get the conclusion. q.e.d.

Proposition 8. (Nagata [4], (36.5)) *If R is a pseudo-geometric ring, then every R-algebra A of finite type is a pseudo-geometric ring.*

Outline of the proof. By ([4], (35.2)), we may assume that R is a (pseudo-geometric) normal domain with field of quotients K, $A=R[x]$, $x\in K$, $x\notin R$, and that for every non-zero ideal $\mathfrak{a}$ of A, $A/\mathfrak{a}$ is pseudo-geometric. Let $x=b/a$, $a, b\in R$ and let $S=1+aA$. Since $\bar{A}=\bigcap \bar{A}_{\mathfrak{m}}$, where $\mathfrak{m}$ runs through all maximal ideals of A and since $A[1/a]=R[1/a]$ is normal, it is sufficient to prove that the derived normal ring $\bar{A}_S$ of A_S is a finite A_S-algebra.

Let $A^*=\varprojlim_n A/a^nA$ (aA-adic completion of A). Then, by the theorem of Marot ([2]), A^* is a pseudo-geometric ring. Hence it is sufficient to prove that A^* is reduced (cf. [4], (32.2)).

Since Ass $(A/aA)=$ Ass (A/a^nA) for every natural number n, we can take a finite number of maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_r$ of A such that the induced homomorphism of A/a^nA to $\prod_{i=1}^r A_{\mathfrak{m}_i}/a^nA_{\mathfrak{m}_i}$ is injective for every n. Therefore we are to prove that $\widehat{A_{\mathfrak{m}_i}}=\varprojlim_n A/\mathfrak{m}_i^n$ is reduced for every i.

Now we may assume that R is a local ring with maximal ideal $\mathfrak{m}=\mathfrak{m}_i\cap R$. Let $f(X)$ be a monic polynomial over R such that $f(x)\equiv 0 \pmod{\mathfrak{m}_i}$ and let $y=f(x)$. Put $B=R[y]$ and $\mathfrak{n}_i=\mathfrak{m}_i\cap B$, then $y\in\mathfrak{n}_i$. By a lemma of Zariski ([4], (36.3)), $\widehat{B_{\mathfrak{n}_i}}$ is reduced, hence $B_{\mathfrak{n}_i}$ is psedo-geometric. Therefore $A_{\mathfrak{m}_i}$ is pseudo-geometric, hence, by ([4], (36.4)), $\widehat{A_{\mathfrak{m}_i}}$ is reduced.

Remark Proposition 4 is not true if we drop the assumption that R is complete (cf. [1]).

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] D. Ferrand-M. Raynaud: Fibres formelles d'un anneau local noethérien, Ann. Sc. E.N.S., t. 3, 1970, pp. 295-311.

[2] J. Marot: Sur les anneaux universellement japonais, C. R. Paris, t. 277, 1973, pp. 1029-1031.

[3] H. Matsumura: Commutative algebra, Benjamin, 1970.

[4] M. Nagata: Local rings, Interscience, 1962.

[5] J. Nishimura: Note on Krull domain, J. Math. Kyoto Univ. Vol. 15, 1975, pp. 397-400.

[6] M. Raynaud: Anneaux locaux henséliens, Lecture Note in Mathematics 169, Springer-Verlag.