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On some behavior spaces and Riemann-Roch theorem on open Riemann surfaces[1)]

By

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Introduction.

The purpose of this paper is to obtain a formulation of the Riemann-Roch theorem on open Riemann surfaces by using the real Hilbert space of square integrable complex differentials and introducing a special Λ_p-behavior space, as has been done by Shiba [9]. In our case, only A-periods are normalized, and B-periods are completely arbitrary and this character of our behavior space is in contrast with Λ_0-behavior in [9]. Besides this, the period normalization in this paper gives much hope to obtain some relations between these behavior spaces and the classical works. Also it seems that in a similar way, we can get the Riemann-Roch theorem by treating the complex Hilbert space. But in this case the ideal boundary becomes small [7], [8]. To get the Riemann-Roch theorem for general open Riemann surfaces, Kusunoki [2, 3] made restrictions only on the real part of differentials. In Kusunoki's line, some works have been done. As in [9], our formulation of the Riemann-Roch theorem is valid for general surfaces with large boundaries.

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1. Preliminaries.

The totality of square integrable complex differentials on a Riemann surface W forms a Hilbert space over the complex field $\boldsymbol{C}$, if we introduce the usual inner product defined by

$$(\lambda_1, \lambda_2) = \iint_W \lambda_1 \wedge \bar{\lambda}_2{}^* = \iint_W (a_1\bar{a}_2 + b_1\bar{b}_2)\,dxdy$$

where $\lambda_j = a_j(z)dx + b_j(z)dy$ with local parameter $z = x+iy$. We denote it by $\tilde{\Lambda} = \tilde{\Lambda}(W)$. As usual $\bar{\lambda} = \bar{a}dx + \bar{b}dy$ and $\lambda^* = -bdx + ady$ stand for the complex conjugate and conjgate of λ respectively. The norm in $\tilde{\Lambda}$ is denoted by $\|\lambda\| = (\lambda, \lambda)^{1/2}$. Square integrable real differentials on W also form a Hilbert space $\Gamma = \Gamma(W)$ over the real field $\boldsymbol{R}$ with the same inner product as above. It can be easily checked that $\tilde{\Lambda}$ forms a linear space over $\boldsymbol{R}$, and in this meaning we denote it by $\Lambda = \Lambda(W)$. Λ forms a real Hilbert space with respect to the new inner product defined by

$$\langle \lambda_1, \lambda_2 \rangle = \mathrm{Re}(\lambda_1, \lambda_2).$$

The norm in Λ will be denoted by $|\!|\!|\cdot|\!|\!|$. It is trivial that $|\!|\!|\cdot|\!|\!| = \|\cdot\|$ and so $\tilde{\Lambda}$ and Λ have the same topological structure.

It should be noticed that, through this paper, the notations Γ and Λ are different from those in Ahlfors-Sario [1]. With only these exceptions we follow Ahlfors-Sario [1] for notations and terminology. For instance $\Gamma_c, \Gamma_e, \Gamma_{co}, \Gamma_{eo}, \Gamma_h, \cdots$ will be used to denote the subspaces of the real Hilbert space Γ, and also $\Lambda_c, \Lambda_e, \Lambda_{co}, \Lambda_{eo}, \Lambda_h, \cdots$ will stand for corresponding subspaces of Λ. The orthogonality relation between these last subspaces certainly is taken with respect to the inner product $\langle\ ,\ \rangle$. The following orthogonal decompositions are valid (cf. [9]):

$$\Lambda_c = \Gamma_c \dotplus i\Gamma_c, \quad \Lambda_{co} = \Gamma_{co} \dotplus i\Gamma_{co}, \quad \Lambda_h = \Lambda_c \cap \Lambda_c{}^*$$

$$\Lambda_c = \Lambda_h \dotplus \Lambda_{eo}, \quad \Lambda_h = \Lambda_{hse}^* \dotplus \Lambda_{hm}, \quad \Lambda = \Lambda_h \dotplus \Lambda_{eo} \dotplus \Lambda_{eo}^*.$$

The following lemma is frequently used in the sequel.

Lemma 1.1. *Let Ω be a canonical regular region on W, and $\Xi(W) = \{A_j, B_j\}^g{}_{j=1}$ a canonical homology basis on W modulo di-*

viding cycles, such that $\Xi \cap \bar{\Omega}$ forms a canonical homology basis on $\bar{\Omega}$ modulo $\partial\Omega$. If φ_1, φ_2 are C^1-differentials which are semiexact and closed respectively, then

$$(\varphi_1, \varphi_2^*)_\Omega = \int_{\partial\Omega}\left(\int \varphi_1\right)\bar{\varphi}_2 + \sum_\Omega \left(\int_{A_j}\varphi_1 \int_{B_j}\bar{\varphi}_2 - \int_{B_j}\varphi_1 \int_{A_j}\bar{\varphi}_2\right).$$

This can be proved by cutting Ω along A_j, B_j, and applying Green's formula.

Note that because of closedness of φ_2 the integral $\int_{\partial\Omega}(\int \varphi_1)\bar{\varphi}_2$ is independent of the additive constant of $\int \varphi_1$.

2. Λ_p-behavior space.

Definition 2.1. A linear subspace Λ_p of Λ_{hse} will be called a *behavior space* if

(1) There exists a closed subspace Λ_1 of Λ_{hse} such that

$$\Lambda_p \supset \Lambda_1 + i\Lambda_1^{\perp *}$$

where $\Lambda_1^{\perp}$ is the orthogonal complement of Λ_1 in Λ_h

(2) $\langle \lambda_p, i\lambda_p^* \rangle = 0$ for each $\lambda_p \in \Lambda_p$

(3) $\int_{A_j} \lambda_p = 0$, $j = 1, 2, \cdots$ for every $\lambda_p \in \Lambda_p$.

From this definition it is easy to verify that if Λ_p is a behavior space, so is $\bar{\Lambda}_p$, where $\bar{\Lambda}_p = \{\bar{\lambda}_p : \lambda_p \in \Lambda_p\}$.

Definition 2.2. A meromophic differential defined on a neighborhood U of the ideal boundary β of W is said to have Λ_p-behavior if there exist $\lambda_p \in \Lambda_p$, $\lambda_{e0} \in \Lambda_{eo} \cap \Lambda^1$ such that on a neighborhood U of β

$$\varphi = \lambda_p + \lambda_{eo}.$$

Definition 2.3. A meromorphic function f (not necessarily single-valued) defined near β is said to have Λ_p-behavior if differential df has Λ_p-behavior in the above sense.

3. The existence and uniqueness theorems.

Theorem 3.1. (*uniqueness*). *Let φ be a first kind differential* *which has Λ_p-behavior. Then it is identically zero if*

$$\int_{A_j}\varphi=0 \quad (j=1,2,\cdots,g)$$

where $g(\leq\infty)$ is the genus of W.

Proof. It should be observed that the condition in the theorem is only for finite number of A_j. Since φ has Λ_p-behavior, there exist $\lambda_p\in\Lambda_p$, $\lambda_{eo}\in\Lambda_{eo}\cap\Lambda^1$ such that on a neighborhood U of β φ can be written as

$$\varphi=\lambda_p+\lambda_{eo}.$$

Now let Ω be a canonical regular region on W such that its relative boundary $\partial\Omega$ is contained in U. We may assume that $\Xi\cap\bar{\Omega}$ forms a canonical homology basis of $\bar{\Omega}$ modulo the border. Then, by Lemma 1.1 and $\int_{A_j}\varphi=0$ $(j=1,2,\cdots,)$ we can write

$$\begin{aligned}\|\!|\varphi|\!\|_\Omega^2&=\|\varphi\|_\Omega^2=(\varphi,\varphi)_\Omega=-i(\varphi,\varphi^*)_\Omega\\&=i\int_{\partial\Omega}\Big(\int\varphi\Big)\bar{\varphi}-i\sum_{\Omega}\Big(\int_{A_j}\varphi\int_{B_j}\bar{\varphi}-\int_{B_j}\varphi\int_{A_j}\bar{\varphi}\Big)\\&=i\int_{\partial\Omega}\Big(\int\varphi\Big)\bar{\varphi}=i\int_{\partial\Omega}\Big(\int(\lambda_p+\lambda_{eo})\Big)(\overline{\lambda_p+\lambda_{eo}})\\&=-i(\lambda_p+\lambda_{eo},\lambda_p^*+\lambda_{eo}^*)_\Omega+i\sum\Big(\int_{A_j}\lambda_p\int_{B_j}\bar{\lambda}_p-\int_{B_j}\lambda_p\int_{A_j}\bar{\lambda}_p\Big).\end{aligned}$$

From the condition (3) in the definition of Λ_p-behavior we obtain

$$\|\!|\varphi|\!\|_\Omega^2=\|\varphi\|_\Omega^2=(\lambda_p,i\lambda_p^*)_\Omega-i\varepsilon_\Omega$$

where $\varepsilon_\Omega=(\lambda_{eo},\lambda_p^*)_\Omega+(\lambda_p,\lambda_{eo}^*)_\Omega+(\lambda_{eo},\lambda_{eo}^*)_\Omega$. By making use of the orthogonal decompositions in section 1, it follows that $\lim_{\Omega\to W}\varepsilon_\Omega=0$ Then, we get the equality

$$\|\!|\varphi|\!\|^2=(\lambda_p,i\lambda_p^*)=\langle\lambda_p,i\lambda_p^*\rangle.$$

The right side is also zero because of the condition (2) in Definition 2.1, and so we get $\varphi\equiv0$.

Now we will prove the existence of certain first kind differentials which have Λ_p-behavior.

Theorem 3.2. *Let $\alpha_j \neq 0$ be given complex numbers. Then there exist square integrable first kind differentials $\phi_{\alpha_j}(B_j)$ which have the following properties:*

(i) *$\phi_{\alpha_j}(B_j)$ have Λ_p-behavior*

(ii) $$\int_{A_k} \phi_{\alpha_j}(B_j) = \begin{cases} -\alpha_j & (k=j) \\ 0 & (k \neq j) \end{cases}$$

(iii) *The $\phi_{\alpha_j}(B_j)$ are uniquely determined for each j.*

Proof. The cycles B_j can be regarded as oriented analytic Jordan curves. Let R be a relatively compact ring domain containing a B_j and v be a C^2-function on $R-B_j$ defined as follows:

$$v = \begin{cases} \alpha_j \text{ on the left side of } B_j \\ 0 \text{ on the right side of } B_j. \end{cases}$$

Then v can be extended to $W-B_j$ such that it becomes a C^2-function with relatively compact support in W. Denote the extension by $\hat{v}$. Then $d\hat{v} \in \Lambda_c^1(W)$ and so it can bewritten as:

$$d\hat{v} = \lambda_1 + \lambda_1^{\perp} + \lambda_{eo}$$

where

$$\lambda_1 \in \Lambda_1, \quad \lambda_1^{\perp} \in \Lambda_1^{\perp}, \quad \Lambda_1 + i\Lambda_1^{*\perp} \subset \Lambda_p.$$

Now we set

$$\phi_{\alpha_j}(B_j) = \lambda_1^{\perp} + i(\lambda_1^{\perp})^* = d\hat{v} - (\lambda_1 - i\lambda_1^{*\perp}) - \lambda_{eo} = d\hat{v} - \lambda_p - \lambda_{eo}.$$

It can be seen from this equation that $\phi_{\alpha_j}(B_j)$ is a first kind differential and has Λ_p-behavior, since $d\hat{v}$ has compact support. Also for any cycle γ

$$\int_\gamma \phi_{\alpha_j}(B_j) = \alpha_j (B_j \times \gamma) - \int_\gamma \lambda_p.$$

Now if we take A_k instead of γ, then (ii) is satisfied. The uniqueness follows easily from Theorem 3.1.

To prove the existence of second and third kind differentials we need the following lemma [10].

Lemma 3.1. *Let Ω be a regularly imbedded connected subregion of W whose relative boundary $\partial\Omega$ is compact, and V be the complement of $\bar{\Omega}$. For any closed C^1-differential σ defined on a neighborhood of $\bar{V}$, the following two statements are equivalent:*

(i) *$\sigma|V$, the restriction of σ onto V, can be extended as a closed C^1-differential $\hat{\sigma}$ on W such that the support of $\hat{\sigma}$ has a compact intersection with $\bar{\Omega}$.*

(ii) $\int_{\partial\Omega}\sigma=0.$

Theorem 3.3. *Let θ_j be an analytic singularity given at each point p_j on W $(j=1,2,\cdots,n)$. Consider a differential θ which is equal to θ_j near p_j and the sum of residues of θ is zero. Then there exists a differential $\varphi=\varphi_\theta$ such that*

(i) *φ has Λ_p-behavior*

(ii) *φ is regular analytic except at p_j $(j=1,2,\cdots,n)$*

(iii) *φ has singularity θ, that is, $\|\theta-\varphi\|_{U_j}<\infty$ for a punctured neighborhood U_j of p_j $(j=1,2,\cdots,n)$.*

The proof can be carried out in the same manner as Ahlfors-Sario [1], Shiba [9], if we use our orthogonal decomposition

$$\Lambda=\Lambda_1+\Lambda_1^{\perp}+\Lambda_{eo}+\Lambda_{eo}^*.$$

Namely define

$$\tau=\hat{\theta}-\lambda_1-\lambda_{eo}'=\lambda_1^{\perp}+\lambda_{eo}''^*+i\hat{\theta}^*.$$

Then τ is a complex harmonic differential with singularity θ. Consequently $\lambda_{eo}', \lambda_{eo}''\in\Lambda_{eo}\cap\Lambda^1$, since $\tau\in\Lambda^1$, $\hat{\theta}\in C^1$, $\lambda_1\in\Lambda_1$. If we set $\varphi=\frac{1}{2}(\tau+i\tau^*)$ it is easily seen that φ has the desired properties.

Remark. We can see that the differentials constructed above are uniquely determined, if we require that φ should satisfy

$$\int_{A_j}\varphi=0 \quad (j=1,2,\cdots,g).$$

Now we show that this normalization is always possible.

Indeed, let x_j be A_j-periods of φ such that only a finite number of x_j are different from zero. We set

$$\varphi_p=\varphi+\sum_j \phi_{x_j}(B_j).$$

It is clear that φ_p preserves the singularity, and satisfies the normalization:

$$\int_{A_j}\varphi_p=\int_{A_j}\varphi+\sum_j\int_{A_j}\phi_{x_j}(B_j)=\int_{A_j}\varphi-x_j=0.$$

As for uniqueness we need only Theorem 3. 1.

The following normalized differentials whose existence is guaranteed by Theorem 3. 3, and the holomorphic (first kind) differentials $\phi_{\alpha_j}(B_j)$ obtained by Theorem 3. 2, will play an important role in the proof of the Riemann-Roch theorem.

(I) $\varphi_{p_j,n}$, (resp. $\tilde{\varphi}_{p_j,n}$) : differential with Λ_p-behavior, regular analytic except at p_j where it has singularity $dz/z_j{}^n$ (resp. $idz/z_j{}^n$) $(n=2,3,\cdots)$

(II) ψ_{p_j,q_j}, (resp. $\tilde{\psi}_{p_j,q_j}$) : meromorphic differential with Λ_p-behavior, which has residues 1 at p_j, -1 at q_j (resp. i at p_j, $-i$ at q_j) and regular elsewhere.

4. Dual boundary behaviors.

Definition 4. 1. Let $\Lambda_p{}^{(k)}=\Lambda_p(\Lambda_1{}^{(k)}, 0, \boldsymbol{C})$ $(k=1,2)$ be two behavior spaces corresponding to the subspaces $\Lambda_1{}^{(1)}, \Lambda_1{}^{(2)}\subset\Lambda_{hse}$. We say that $\Lambda_p{}^{(1)}$-behavior and $\Lambda_p{}^{(2)}$ behavior are *dual to each other* if for all $\lambda_p{}^{(1)}\in\Lambda_p{}^{(1)}$, $\lambda_p{}^{(2)}\in\Lambda_p{}^{(2)}$

$$(\lambda_p{}^{(1)}, \overline{\lambda_p{}^{(2)*}})=0 \quad (\Leftrightarrow \langle\lambda_p{}^{(1)}, \overline{\lambda_p{}^{(2)*}}\rangle=\langle\lambda_p{}^{(1)}, i\overline{\lambda_p{}^{(2)*}}\rangle=0).$$

The following lemma is a nice consequence of this definition.

Lemma 4. 1. *Suppose that* $\Lambda_p=\Lambda_p(\Lambda_1, 0, \boldsymbol{C})$ *is a behavior which satisfies the condition:*

(i) $(\lambda_p, i\lambda_p{}^{1*})=0 \quad (\Leftrightarrow \langle\lambda_p, i\lambda_p{}^{1*}\rangle=\langle\lambda_p, \lambda_p{}^{1*}\rangle=0)$

for all $\lambda_p, \lambda_p{}^1\in\Lambda_p$.

Then Λ_p-behavior and $\overline{\Lambda}_p$-behavior are dual to each other.

Proof. Since $\overline{\Lambda}_p$ is a behavior space, we need only check the condition in definition 4. 1. For this purpose take $\lambda_p, \lambda_p^1 \in \Lambda_p$ then by (i) we get

$$(\lambda_p, (\overline{\lambda_p^1})^*) = \langle \lambda_p, \lambda_p^{1*} \rangle + i \langle \lambda_p, i\lambda_p^{1*} \rangle$$
$$= 0 \qquad \text{q.e.d.}$$

The following lemma [2, 3], will be used in the proof of Riemann-Roch theorem. Therefore we prove it in our terminology.

Lemma 4. 2. *Let $\Lambda_p^{(1)}$ and $\Lambda_p^{(2)}$ be dual boundary behaviors to each other. Let φ be an abelian differential (of first or second kind) with $\Lambda_p^{(1)}$-behavior and ψ any abelian differential with $\Lambda_p^{(2)}$-behavior. Let W_0 be the planar surface obtained from W by cutting along A_j and B_j cycles. Then,*

(i) *there exists a single valued meromorphic function f on W_0 such that $df = \varphi$,*

(ii) $$2\pi i \sum \operatorname{Res} f\psi = -\sum_{j=1}^{g} \left(\int_{A_j} \varphi \int_{B_j} \psi - \int_{B_j} \varphi \int_{A_j} \psi \right).$$

Proof. (i) is obvious by assumptions. To prove (ii) we apply Lemma 1. 1 to the region Ω_0 obtained from a sufficiciently large canonical region Ω by taking off mutually disjoint parametric disks about the singularities of φ and ψ. We may suppose that $\Xi \cap \overline{\Omega}$ forms a canonical homology basis of $\overline{\Omega}$ modulo $\partial\Omega$ then

$$2\pi i \sum \operatorname{Res} f\psi = -\sum_{\Omega} \left(\int_{A_j} \varphi \int_{B_j} \psi - \int_{B_j} \varphi \int_{A_j} \psi \right) + \int_{\partial\Omega} f\psi.$$

By assumption we know that $\varphi = \lambda_p^{(1)} + \lambda_{eo}$, $\psi = \lambda_p^{(2)} + \lambda_{eo}^1$ near the ideal boundary, in particular near $\partial\Omega$. By use of Lemma 1. 1, and from the definitions of Λ_p and its dual behavior we can write

$$\int_{\partial\Omega} f\psi = -\left(\lambda_p^{(1)}, \overline{\lambda_p^{(2)*}}\right)_{\Omega} + \sum_{\Omega} \left(\int_{A_j} \lambda_p^{(1)} \int_{B_j} \lambda_p^{(2)} - \int_{B_j} \lambda_p^{(1)} \int_{A_j} \lambda_p^{(2)} \right) + \varepsilon_{\Omega}$$
$$= -(\lambda_p^{(1)}, \overline{\lambda_p^{(2)*}})_{\Omega} + \varepsilon_{\Omega} \to 0 \quad (\Omega \to W).$$

Thus we get the desired result.

5. The Riemann-Roch Theorem.

Let $\delta=\delta_p/\delta_q$ be a finite divisor on W, where $\delta_p=P_1^{m_1}P_2^{m_2}\cdots P_r^{m_r}$ and $\delta_q=q_1^{n_1}q_2^{n_2}\cdots q_s^{n_s}$ are disjoint integral divisors. Let $\Lambda_p^{(1)}$ and $\Lambda_p^{(2)}$ be dual boundary behaviors. We consider the following sets which evidently form linear spaces over $\boldsymbol{R}$:

$S(\Lambda_p^{(1)}; 1/\delta)=\{f$: (i) single valued meromorphic function on W, (ii) has $\Lambda_p^{(1)}$-behavior, (iii) is multiple of of $1/\delta\}$

$M(\Lambda_p^{(1)}; 1/\delta_p)=\{f$: (i) is a multi-valued meromorphic function on W. (ii) has $\Lambda_p^{(1)}$-behavior, (iii) is a multiple of $1/\delta_p$ (iv) periods of df are normalized, i.e., $\int_{A_j}df=0\}$

$D(\Lambda_p^{(2)}; \delta)=\{\alpha$: (i) a meromorphic differential on W, (ii) has $\Lambda_p^{(2)}$-behavior, (iii) is a multiple of $\delta\}$

$E(\Lambda_p^{(2)}; 1/\delta_q)=\{\alpha$: (i) a meromorphic differential on W, (ii) has $\Lambda_p^{(2)}$-behavior, (iii) is a multiple of $1/\delta_q\}$

In the case that $\delta_q\neq 1$ we identify the elements f_1, f_2 of M if and only if $f_1-f_2=$ constant.

The following well-known algebraic lemma should be provided.

Lemma 5.1. *Let X and Y be two linear spaces over a field K, and consider a biliner form (x, y) defined over $X\times Y$. Denote the left kernel by X_0 and the right kernel by Y_0. If the quotient space X/X_0 is finite dimensional, then there is an isomorphism $X/X_0\simeq Y/Y_0$.*

Theorem 5.1. (*Riemann-Roch*). *Suppose that $\Lambda_p^{(1)}$-and $\Lambda_p^{(2)}$-behaviors are dual to each other. Let $\delta=\delta_p/\delta_q$ be a finite divisor on W, where δ_p and δ_q are disjoint integral divisors. Then*

$$\dim S(\Lambda_p^{(1)}; 1/\delta)=2[\deg\delta_p+1-\min(\deg\delta_q, 1)]$$
$$-[\dim E(\Lambda_p^{(2)}; 1/\delta_q)/D(\Lambda_p^{(2)}; \delta)].$$

Proof. We follow essentially the proof of Kusunoki [2]. We define a function $h_p(f, \alpha)$ on $M\times E$ by

$$h_p(f, \alpha)=\operatorname{Re}(\sum_j \operatorname{Res}_{p_j} f\alpha) \quad \text{for} \quad f\in M,\ \alpha\in E.$$

Since α is regular at each p_j, additive constants of f have no effect on the residues of $f\alpha$ at each p_j. Therefore $h_p(f,\alpha)$ is well-defined. Then by Lemma 4. 2 we can write

$$h_p(f,\alpha)=\frac{1}{2\pi}\operatorname{Im}\left[\sum_{j=1}^{g}\int_{B_j}df\int_{A_j}\alpha\right]-\operatorname{Re}\left[\sum_k \operatorname*{Res.}_{q_k} f\alpha\right]$$

since df is normalized, i.e., $\int_{A_j}df=0$. Thus, if f belongs to the left-kernel of $h_p(f,\alpha)$, i.e., $0=h_p(f,\alpha)$ for every $\alpha\in E$, then we get $\operatorname{Im}\int_{B_k}df=0$, $\operatorname{Re}\int_{B_k}df=0$ by taking $\alpha\equiv\phi_1(B_k)$ and $\alpha\equiv\phi_i(B_k)$ respectively. Thus $\int_{B_k}df=0$. Therefore f is single-valued on the whole W, since by assumption we already know that $\int_{A_k}df=0$. If δ is an integral divisor, then $\delta=\delta_p$ and so $f\in S$. If δ is non-integral, then we take $\alpha\equiv\psi^{(2)}_{q_1,q_k}$. It can be seen that $\operatorname{Im}f(q_1)=\operatorname{Im}f(q_k)$ and $\operatorname{Re}f(q_1)=\operatorname{Re}f(q_k)$ $(k=1,2,\cdots,s)$. Thus $f-f(q_1)$ has zeros at $q_k(2\leq k\leq s)$. Moreover, if we take $\varphi^{(2)}_{q_k,\nu}$ and $\tilde{\varphi}^{(2)}_{q_k,\nu}$ as α $(1\leq k\leq s,\ 2\leq\nu\leq n_k)$ it follows that $f-f(q_1)$ has at least n_k zeros at q_k. By the equivalent relation in M we get $f\in S$. Conversely, it is obvious that the left-kernel of h_p contains S. In a similar way we can see that D is the right-kernel of h_p. Indeed, since $f\alpha$ is regular analytic at each p_j for $f\in M$, $\alpha\in D$, then D is contained in the right-kernel. The converse is proved by taking the integrals $\int\varphi^{(1)}_{p_j,\mu}$ and $\int\tilde{\varphi}^{(1)}_{p_j,\mu}$ as f $(1\leq j\leq r,\ 2\leq\mu\leq m_j+1)$. To get the final result we must see that M is a finite-dimensional space. For $\delta_q\neq1$ the following integrals span M;

$$\int\varphi^{(1)}_{p_j,\mu}\quad\text{and}\quad\int\tilde{\varphi}^{(1)}_{p_j,\mu}\qquad\begin{matrix}1\leq j\leq r\\ 2\leq\mu\leq m_j+1\end{matrix}$$

If $\delta_q=1$, the above integrals and 1, i make a basis of M. So we find that

$$\dim M=\begin{cases}2\sum_{j=1}^{r}m_j+2=2\deg\delta_p+2 & (\delta_q=1)\\[2ex] 2\sum_{j=1}^{r}m_j=2\deg\delta_p & (\delta_q\neq1)\end{cases}$$

So in any case we have $\dim M=2[\deg\delta_p+1-\min(\deg\delta_q,1)]$. Then we can apply Lemma 5. 1. q.e.d.

If the genus of W is finite, Theorem 5. 1 reduces to the following rather classical form:

Corollary 5. 1. *If $\Lambda_p^{(1)}$-and $\Lambda_p^{(2)}$-behaviors are dual to each other, then for any finite divisor δ on W*

$$\dim S - \dim D = 2(\deg \delta - g + 1).$$

Proof. We can find a basis for E:

(a) if $\delta_q = 1$ $\{\phi_{a_j}^{(2)}(B_j), \phi_{ib_j}^{(2)}(B_j)\}_{j=1}^{g}$ span E, where $a_j, b_j \in \boldsymbol{R}$.

(b) if $\delta_q \neq 1$ $\{\phi_{a_j}^{(2)}(B_j), \phi_{ib_j}^{(2)}(B_j), \varphi_{q_k,\nu}^{(2)}, \tilde{\varphi}_{q_k,\nu}^{(2)}, \psi_{q_1,q_l}^{(2)}, \tilde{\psi}_{q_1,q_l}^{(2)}\}_{2\leq l\leq s,\, 2\leq\nu\leq n_k}^{1\leq j\leq g,\, 1\leq k\leq s}$

span E provided that in both cases we choose a_j and b_j as in Theorem 2. 1, then

$$\dim E = \begin{cases} 2g & (\delta_q = 1) \\ 2[g + \sum_{k=1}^{s} (n_k - 1) + s - 1] & (\delta_q \neq 1). \end{cases}$$

So, $\dim E = 2[g - \min(\deg \delta_q, 1) + \deg \delta_q]$ and the result easily follows from Theorem 5. 1.

6. Generalization.

Divide the set of positive integers $\{1, 2, \cdots, g\}$ into two disjoint sets J_1, J_2, and let $\{L_j\}$ be a set of straight lines L_j $(j \in J_1)$ passing through the origin $z = 0$.

Definition 6. 1. A linear subspace $\Lambda_p = \Lambda_p(J_1, J_2)$ of Λ_{hse} is called a behavior space if

(1) there exists a closed subspace Λ_1 of Λ_{hse} such that

$$\Lambda_p \supset \Lambda_1 + i\Lambda_1^{\perp *}$$

where $\Lambda_1^{\perp}$ is the orthogonal complement of Λ_1 in Λ_h

(2) $\langle \lambda_p, i\lambda_p^* \rangle = 0$ for each $\lambda_p \in \Lambda_p$

(3) $\int_{A_j, B_j} \lambda_p \in L_j$ if $j \in J_1$, and $\int_{A_j} \lambda_p = 0$ if $j \in J_2$.

We can similarly formulate the Riemann-Roch theorem in terms of such behavior spaces. As a special case where $J_2 = \phi$, we have Shiba's result [9], and our result is the case $J_1 = \phi$. Given L_j we can prove that a behavior space Λ_p actually exists.

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