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# Asymptotic behaviours of two dimensional autonomous systems with small random perturbations

By

Kunio NISHIOKA

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## 0. Introduction.

Consider a following linear autonomuos system in $R^2$:

$$(0\cdot 1) \qquad \frac{dX(t)}{dt} = \mathrm{B}\cdot X(t),$$

where B is a $2\times 2$ constant matrix. If small linear "white noise type" perturbations act on the system $(0\cdot 1)$, we have a stochastic system:

$$(0\cdot 2) \qquad dX^\varepsilon(t) = \mathrm{B}\cdot X^\varepsilon(t)\,dt + \varepsilon\{\mathrm{C}\cdot X^\varepsilon(t)\,dB_1(t) + \mathrm{D}\cdot X^\varepsilon(t)\,dB_2(t)\},$$

where C and D are $2\times 2$ constant matrices and $B_i(t)$ $(i=1, 2)$ are independent one dimensional Brownian motions. Our interest is to study relations between properties[1] of the singular point $\{x=0\}$ of the system $(0\cdot 1)$ and of the system $(0\cdot 2)$ for sufficiently small $\varepsilon$.

With respect to radial parts, the relations are known, i.e., *if the origin is not a center for the system* $(0\cdot 1)$, *then*

$$(0\cdot 3) \qquad \lim_{\varepsilon\to 0}\lim_{t\to\infty}|X^\varepsilon(t)| = \lim_{t\to\infty}|X(t)| \qquad \text{a.s.},$$

*but if the origin is a center, then the equality* $(0\cdot 3)$ *is not necessarily valid.* Therefore, our purpose in this paper comes to establish such relations between an angular part $\theta(t)$ of $X(t)$ and the other one $\theta^\varepsilon(t)$ of $X^\varepsilon(t)$.

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[1] Many books, for example, Coddington and Levinson [1], discuss properties of the origin for the system (0. 1).

In case that $\Psi(\theta)$ (see the equality (0·8)) does not vanish, our results (Theorems 1 through 3) coincide with a slight modification of Nevel'son [7] However, in case that $\Psi(\theta)$ may vanish, the circumtances are different. In order to prove our results, we essentially need that the system (0·2) is linear and that the state space is two dimensional, because we know all asymptotic behaviours of $\theta^{\varepsilon}(t)$, which we studied in [8], only for that case It should be remarked that Friedman and Pinsky [2] also studied the asymptotic behav ours of $\theta^{\varepsilon}(t)$ and some of our results may be covered by theirs. But they are not interested in the limiting property of the system (0·2) as $\varepsilon \downarrow 0$.

For simplicity, we may assume that $D \equiv 0$ in the system (0·2):

$$(0\cdot 2') \qquad dX^{\varepsilon}(t) = B\cdot X^{\varepsilon}(t)\,dt + \varepsilon C\cdot X^{\varepsilon}(t)\,dB_1(t).$$

In fact, all cases which arise 'n the system (0·2) also arise in the system (0·2′). Making use of a simple calculation and Ito's formula, we have

$$(0\cdot 4) \qquad \frac{d\theta(t)}{dt} = \Phi_B(\theta(t)),$$

$$(0\cdot 5) \qquad d\theta^{\varepsilon}(t) = \Phi^{\varepsilon}(\theta^{\varepsilon}(t))\,dt + \varepsilon\Psi(\theta^{\varepsilon}(t))\,d\widetilde{B}(t),$$

where $\widetilde{B}(t)$ is a new one dimensional Brownian motion,

$$(0\cdot 6) \qquad \Phi^{\varepsilon}(\theta) = \Phi_B(\theta) + \varepsilon^2\Phi_C(\theta),$$

$$(0\cdot 7) \qquad \begin{cases} \Phi_B(\theta) = -(B\cdot e(\theta),\ e^*(\theta)) \\ \Phi_C(\theta) = (A(e(\theta)\cdot e(\theta),\ e^*(\theta)), \end{cases}$$

and

$$(0\cdot 8) \qquad \Psi^2(\theta) = (A(e(\theta))\cdot e^*(\theta),\ e^*(\theta)),$$

in which

$$(A(x))_{ij} = \sum_{m,n=1}^{2} c_{im}x_m c_{jn}x_n\ ,^{2)}$$

$e(\theta) = (\cos\theta, \sin\theta)$, and $e^*(\theta) = (\sin\theta, -\cos\theta)$. Note that $\Phi_{\varepsilon}(\theta+\pi) = \Phi^{\varepsilon}(\theta)$ and $\Psi^2(\theta+\pi) = \Psi^2(\theta)$.

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[2] $c_{ij}$ is an $(i,j)$ element of a matrix C, and so on.

Let H be a real constant regular matrix. If $Y=\mathrm{H}\cdot X$, then the system $(0\cdot 1)$ is transformed into

$$\frac{dY(t)}{dt}=(\mathrm{H}\cdot\mathrm{B}\cdot\mathrm{H}^{-1})\cdot Y(t),$$

where the transformed matrix $(\mathrm{H}\cdot\mathrm{B}\cdot\mathrm{H}^{-1})$ is one of the following canonical forms :

$$\text{(I)}\ \begin{pmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{pmatrix}\ b_2\neq 0, \qquad \text{(II)}\ \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\ b_1\neq b_2,$$

$$\text{(III)}\ \begin{pmatrix} b_1 & 0 \\ b_2 & b_1 \end{pmatrix}\ b_2>0, \qquad \text{(IV)}\ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

Thus, we may assume that the matrix B is one of the canonical forms (I) through (IV). For the system $(0\cdot 1)$, the origin is a *center* or a *spiral point*, if the matrix B is (I). It is an *improper node* or a *saddle point*, if B is (II). If B is (III), it is an *improper node*, and if B is (IV), it is a *proper node* (see Coddington and Levinson [1]).

## 1. A center and a spiarl point.

If the matrix B is (I), then it follows from the equality $(0\cdot 4)$ that $\theta(t)=\theta(0)-b_2t$. As for the behaviour of $\theta^{\varepsilon}(t)$, we have:

**Theorem 1.** *If the matrix* B *is (I), then it holds that, for any* $\delta>0$,

$$\lim_{\varepsilon\to 0} P_{\theta_0}\left\{\lim_{t\to\infty}\left|\frac{\theta^{\varepsilon}(t)}{t}+b_2\right|\leqq\delta\right\}=1,$$

*where* $\theta_0$ *is arbitrary.*

**Proof.** Note that there exists a constant $K$ such that $|\Phi^{\varepsilon}(\Theta)+b_2|\leqq\varepsilon^2K$ and $\Psi^2(\theta)\leqq K$. Then, integrating the equality (0.5), we have

$$\left|\frac{1}{t}(\theta^{\varepsilon}(t)-\theta^{\varepsilon}(0))+b_2\right|\leqq\varepsilon^2K+\frac{K}{t}|\tilde{B}(t)-\tilde{B}(0)|.$$

By virtue of the law of iterated logarithm, the theorem is obtained.

## 2. An improper node and a saddle point.

In case that the matrix B is (II), the system (0·4) has two stable equilibrium points (say $\alpha_1$ and $\alpha_2=\alpha_1+\pi$) and two unstable equilibrium points (say $\beta_1$ and $\beta_2=\beta_1+\pi$), i.e.,

$$\lim_{t\to\infty}\theta(t)=\begin{cases}\alpha_1 & \beta_2-\pi<\theta(0)<\beta_1\\ \beta_1 & \theta(0)=\beta_1\\ \alpha_2 & \beta_1<\theta(0)<\beta_2\\ \beta_2 & \theta(0)=\beta_2\end{cases}$$

Note that either $\alpha_1=0$ and $\beta_1=\pi/2$ or $\alpha_1=\pi/2$ and $\beta_1=\pi$.

**Theorem 2.** *If the matrix* B *is* (II), *then it holds that, for any* $\delta>0$, *and* $\theta_0\neq\beta_1$, $\beta_2$,

$$\lim_{\varepsilon\to0}\lim_{t\to\infty}P_{\theta_0}\{\theta^{\varepsilon}(t)\in U_{\delta}(\alpha_1)\quad or\quad U_{\delta}(\alpha_2)\}=1\,, \tag{2·1}$$

*where* $U_{\delta}(\ )$ *is* $\delta$*-neighbourhood of* $\alpha_1$.

In order to prove the theorem, we prepare the following lemma, which is a modification of Nevel'son [7].

**Lemma 1.** *Let* $f_{\varepsilon}(x)=f_0(x)+\varepsilon h(x)$. *For each* $\varepsilon>0$, *there exists a point* $a_{\varepsilon}\in(a,b)$ *such that* $\max_{a\leq x\leq b}f_{\varepsilon}(x)=f_{\varepsilon}(a_{\varepsilon})$, *and* $k+1$*-th derivative of* $f_{\varepsilon}(x)$ *exists in a neighbourhood of* $a_{\varepsilon}$ *for some* $k>0$ *independent of* $\varepsilon$. *Let* $g(x)$ *be continuous at* $a_0$ *and* $\int_a^b g(x)\exp\times\{(1/\varepsilon)f_{\varepsilon}(x)/\varepsilon\}ds$ *converge for some* $\varepsilon$. *Then as* $\varepsilon\to0$,

$$\int_a^b g(x)\exp\left\{\frac{1}{\varepsilon}f_{\varepsilon}(X)\right\}dx=\frac{\exp\{(1/\varepsilon)f_{\varepsilon}(a_{\varepsilon})\}\Gamma((1/k))g(a_{\varepsilon})}{k((1/\varepsilon))^{1/k}(-(f_{\varepsilon}^{(k)}(a_{\varepsilon})/k!)^{1/k}}$$

$$\times(2+o(\varepsilon^{1/k})),$$

*where* $\Gamma(p)$ *is the Gamma function.*

**Proof of theorem 2.** In the following proof, we assume that $\alpha_1=0$ and $\beta_1=\frac{1}{2}\pi$, without losing generality. As for the existence and a representation of an invariant measure density which appears in this and later proofs, see [8].

*Case* 1, $\Psi^2(\theta)>0$. There exists an invariant measure $\mu^\varepsilon(d\theta)$ such that for arbitrary $\theta_0$

$$(2\cdot 2)\qquad \lim_{t\to\infty} P_{\theta_0}\{\theta^\varepsilon(t)\in\cdot\}=\mu^\varepsilon(\cdot)$$

$$(2\cdot 3)\qquad \mu^\varepsilon(d\theta)=\frac{\nu_1^\varepsilon(\theta)+\nu_2^\varepsilon(\theta)}{\int_0^{2\pi}(\nu_1^\varepsilon(\psi)+\nu_2^\varepsilon(\psi))d\psi}d\theta$$

$$(2\cdot 4)\qquad \begin{cases} \nu_1^\varepsilon(\theta)=\dfrac{\int_\theta^{2\pi}W^\varepsilon(0,\psi)d\psi}{\varepsilon^2\Psi^2(\theta)W^\varepsilon(0,\theta)} \\ \nu_2^\varepsilon(\theta)=\dfrac{\int_0^\theta W^\varepsilon(0,\psi)d\psi}{\varepsilon^2\Psi^2(\theta)W^\varepsilon(2\pi,\theta)}, \end{cases}$$

in which (and later on) we set

$$W^\varepsilon(\theta_1,\theta_2)=\exp\left\{-\frac{1}{\varepsilon^2}\int_{\theta_2}^{\theta_1}\frac{2\Phi^\varepsilon(\psi)}{\Psi^2(\psi)}d\psi\right\}.$$

Let $\alpha_i^\varepsilon$ $(i=1,2)$ be stable equilibrium points and $\beta_i^\varepsilon$ be unstable equilibrium points of the dynamical system

$$(2\cdot 5)\qquad \frac{d\theta(t)}{dt}=\Phi^\varepsilon(\theta(t)).$$

It is clear that $\alpha_2^\varepsilon=\alpha_1^\varepsilon+\pi$ and $\beta_2^\varepsilon=\beta_1^\varepsilon+\pi$ and that $\lim_{\varepsilon\to 0}\alpha_i^\varepsilon=\alpha_i$ and $\lim_{\varepsilon\to 0}\beta_i^\varepsilon=\beta_i$.

If we apply Lemma 1 to $\nu_i^\varepsilon(\theta)$ in the same way as Nevel'son [7] did, then we have

$$\int_{[0,2\pi)\setminus\Sigma_i U_\delta(\alpha_i^\varepsilon)}(\nu_1^\varepsilon(\theta)+\nu_2^\varepsilon(\theta))\cdot d\theta=o\left(\int_{\Sigma_i U_\delta(\alpha_i)}(\nu_1^\varepsilon(\theta)+\nu_2^\varepsilon(\theta))d\theta\right),$$

from which it follows that

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(U_\delta(0)+U_\delta(\pi))=1\,.$$

If $\Psi(\theta)$ vanishes, then it does, at most, at four points in $[0,2\pi)$, say $0\leqq\gamma_1\leqq\gamma_2\leqq\gamma_3(=\gamma_1+\pi)\leqq\gamma_4(=\gamma_2+\pi)<2\pi$. Note that $\gamma_i$'s are independent of $\varepsilon$.

*Case* 2, $\gamma_i\neq 0$ $(i=1,2)$. There exists an invariant measure density $\nu^{\varepsilon}(\theta)$, which includes a neighbourhood of 0 and one of $\pi$ in its support. Suppose that $0<\gamma_1<\gamma_2<\frac{1}{2}\pi$, then

$$(2\cdot 6)\qquad \nu^{\varepsilon}(\theta)=\begin{cases}\dfrac{\int_{\gamma_1}^{\theta}W^{\varepsilon}(\gamma_1,\psi)\,d\psi}{\varepsilon^2\Psi^2(\theta)\,W^{\varepsilon}(\gamma_1,\theta)} & \gamma_1\leqq\theta<\gamma_2\\ \dfrac{\int_{\gamma_2}^{\theta}W^{\varepsilon}(\gamma_2,\psi)\,d\psi}{\varepsilon^2\Psi^2(\theta)\,W^{\varepsilon}(\gamma_2,\theta)} & \gamma_2\leqq\theta<\gamma_1+\pi\\ \nu^{\varepsilon}(\theta-\pi) & \gamma_1+\pi\leqq\theta<\gamma_1+2\pi\,.\end{cases}$$

We estimate $\int_0^{2\pi}\nu^{\varepsilon}(\theta)\,d\theta$. For any $\delta>0$,

$$\int_0^{2\pi}\nu^{\varepsilon}(\theta)\,d\theta=\int_{\Sigma_i U_\delta(\alpha_i^{\varepsilon})}\nu^{\varepsilon}(\theta)\,d\theta+\int_{\Sigma_i U_\delta(\gamma_i)}\nu^{\varepsilon}(\theta)\,d\theta$$
$$+\int_{[0,2\pi)\setminus(\Sigma_i U_\delta(\alpha_i^{\varepsilon})+\Sigma_i U_\delta(\gamma_i))}\nu^{\varepsilon}(\theta)\,d\theta\,.$$

Since it holds that $\Phi^{\varepsilon}(\gamma_i)<0$ uniformly with respect to $\varepsilon$, it follows from the equality $(2\cdot 6)$ that

$$\int_{\Sigma_i U_\delta(\gamma_i)}\nu^{\varepsilon}(\theta)\,d\theta\leqq M\,,$$

where $M$ is a constant independent of $\varepsilon$. By Lemma 1, we have

$$\int_{\Sigma_i U_\delta 2\alpha_i^{\varepsilon})}\nu^{\varepsilon}(\theta)\,d\theta=\frac{2A_1^{\varepsilon}A_2^{\varepsilon}}{\Psi^2(\alpha_1^{\varepsilon})\,W^{\varepsilon}(\beta_2^{\varepsilon},\alpha_1^{\varepsilon}+2\pi)}(2+o(\varepsilon))$$

and

$$\int_{[0,2\pi)\setminus(\Sigma_i U_\delta(\alpha_i^{\varepsilon})+\Sigma_i U_\delta(\gamma_i))}\nu^{\varepsilon}(\theta)\,d\theta=o\Big(\int_{\Sigma_i U_\delta(\alpha_i^{\varepsilon})}\nu^{\varepsilon}(\theta)\,d\theta\Big),$$

where

$$A_1^{\varepsilon}=\frac{1}{2}\Gamma\Big(\frac{1}{2}\Big)\Big[-\frac{1}{2}\Big(-\frac{2\Phi^{\varepsilon}(\theta)}{\Psi^2(\theta)}\Big)'_{\theta=\beta_2^{\varepsilon}}\Big]^{-1/2}$$
$$A_2^{\varepsilon}=\frac{1}{2}\Gamma\Big(\frac{1}{2}\Big)\Big[\frac{1}{2}\Big(-\frac{2\Phi^{\varepsilon}(\theta)}{\Psi^2(\theta)}\Big)'_{\theta=\alpha_1^{\varepsilon}}\Big]^{-1/2}.$$

Thus, as $\varepsilon\to 0$,

$$(2\cdot 7)\qquad \frac{\int_{\Sigma U_\delta(\alpha_i^{\varepsilon})}\nu^{\varepsilon}(\theta)\,d\theta}{\int_{[0,2\pi)}\nu(\theta)\,d\theta}\to 1\,,$$

which proves the theorem, because

$$\lim_{t\to\infty} P_{\theta_0}\{\theta^\varepsilon(t)\in\cdot\} = \frac{\int \nu^\varepsilon(\theta)d\theta}{\int_{[0,2\pi)}\nu^\varepsilon(\theta)d\theta}. \tag{2·8}$$

For the other $\gamma_i$, we can prove the theorem in the same manner as the above.

*Case* 3. $\gamma_1=0$. In this case, 0 and $\pi$ are natural boundary points, because it follows, from the assumption that $\gamma_1=0$, that $c_{21}=0$, which proves that $\Phi^\varepsilon(0)=0$. If $\gamma_1\neq\gamma_2$, then it is easy to see that

$$\begin{aligned} &\frac{k_1}{\theta}\leqq -\frac{2\Phi^\varepsilon(\theta)}{\Psi^2(\theta)}\leqq\frac{k_2}{\theta} && \theta\in[0,\delta] \\ &\frac{k_3}{\theta}\leqq -\frac{2\Phi^\varepsilon(\theta)}{\Psi^2(\theta)}\leqq\frac{k_4}{\theta} && \theta\in[-\delta,0] \end{aligned} \tag{2·9}$$

where $\delta$ and $k_i$ are positive constants independent of $\varepsilon$. From the inequality (2·9), we see that

$$\begin{aligned} &\left(\frac{\theta_2}{\theta_1}\right)^{k_1/\varepsilon^2}\leqq W^\varepsilon(\theta_1,\theta_2)\leqq\left(\frac{\theta_2}{\theta_1}\right)^{k_2/\varepsilon^2} && \theta_1,\theta_2\in(0,\delta] \\ &\left(\frac{\theta_4}{\theta_3}\right)^{k_3/\varepsilon^2}\leqq W^\varepsilon(\theta_3,\theta_4)\leqq\left(\frac{\theta_4}{\theta_3}\right)^{k_4/\varepsilon^2} && \theta_3,\theta_4\in[-\delta,0), \end{aligned}$$

which proves that 0 and $\pi$ are *attracting* (see [8]). Hence, we obtain that

$$P_{\theta_0}\{\lim_{t\to\infty}\theta^\varepsilon(t)=0 \text{ or } \pi\}=1 \qquad \theta_0\neq\tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi.$$

If $\gamma_1=\gamma_2$, then we can prove in a similar way.

**Remark.** If $\beta_i$ $(i=1,2)$ are not natural boundary points, then the equality (2·1) is valid for $\theta_0=\beta_1, \beta_2$. But, if they are natural boundary points, then

$$P_{\beta_i}\{\theta_\varepsilon(t)=\beta_i\}=1.$$

## 3. An improper node.

Since $\Phi_B(\theta)=b_2\cos^2\theta$ in case that the matrix B is (III), the system (0·4) has only two stab e equilibrium points $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, i.e.,

$$\lim_{t\to\infty}\theta(t)=\begin{cases}\frac{1}{2}\pi & -\frac{1}{2}\pi<\theta(0)\leqq\frac{1}{2}\pi\\ \frac{3}{2}\pi & \frac{1}{2}\pi<\theta(0)\leqq\frac{3}{2}\pi\end{cases}$$

**Theorem 3.** *If the matrix* B *is* (III), *then it holds that, for any* $\delta>0$ *and any* $\theta_0$,

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}P_{\theta_0}\{\theta^{\varepsilon}(t)\epsilon U_{\delta}(\tfrac{1}{2}\pi)\ \ or\ \ U_{\delta}(\tfrac{3}{2}\pi)\}=1\,. \tag{3·1}$$

In order to prove the theorem, we need the lemma due to Nevel'son [7]:

**Lemma 2.** (*Nevel'son*) *Let* $f(x)$ *be a non-negative increasing function in some neighbourhood of* $x=a$ *such that the order of the first non-vanishing derivative of* $f(x)$ *at* $a$ *is* $k>1$ (*with* $k$ *odd*). *Moreover,* $f^{(k+1)}(x)$ *exists in the neighbourhood of* $x=a$, *and* $g(u,x)$ *be continuous at* $(a,a)$. *Then, for sufficiently small* $\delta>0$, *it holds that*

$$\int_{a-\delta}^{a+\delta}dx\int_{x}^{a+\delta}du\, g(u,x)\exp\left\{-\frac{1}{\varepsilon}(f(u)-f(x))\right\}$$

$$=g(a,a)\left(\frac{f^{(k)}(a)}{\varepsilon k!}\right)^{-2/k}A_k(1+o(\varepsilon^{1/k}))$$

*as* $\varepsilon\to 0$, *where*

$$A_k=\int_{-\infty}^{\infty}dp\int_{0}^{\infty}dq\exp\{p^k-(p+q)^k\}.$$

**Proof of Theorem 3.** We discuss the proof for each type of the matrix C.

*Case* 1. $\Phi_C(\frac{1}{2}\pi)>0$. Note that $\Phi^{\varepsilon}(\theta)>0$ for any $\theta$. If $\Psi(\theta)$ does not vanish, then there exists an invariant measure $\mu^{\varepsilon}(d\theta)$, written by the equalities (2·3) and (2·4). Applying Lemma 1 to the equality (2·4), we have

$$\nu_1^{\varepsilon}(\theta)+\nu_2^{\varepsilon}(\theta)=\frac{1}{\Phi^{\varepsilon}(\theta)}(1+o(\varepsilon^2)),$$

from which we obtain the equality (3·1), using the euality (2·3) and that

(3·2) $$\Phi^{\varepsilon}(\tfrac{1}{2}\pi)\to 0 \quad \text{as} \quad \varepsilon\to 0\,.$$

If $\Psi(\theta)$ vanishes, then $\gamma_i\neq\frac{1}{2}\pi$ $(i=1, 2)$. Actually, if $\gamma_i=\frac{1}{2}\pi$ $(i=1$, or $2)$, then it follows that $c_{12}=0$, which is equivalent that $\Phi_C(\frac{1}{2}\pi)=0$. Thus in case that $\Psi(\theta)$ vanishes, $\theta^{\varepsilon}(t)$ has an invariant measure density $\nu^{\varepsilon}(\theta)$ such that

(3·3) $$\nu^{\varepsilon}(\theta)=\begin{cases}\dfrac{\int_0^{\gamma_2}W^{\varepsilon}(\eta_1,\psi)\,d\psi}{\varepsilon^2\Psi^2(\theta)\,W^{\varepsilon}(\eta_1,\theta)} & \gamma_1<\theta\leqq\gamma_2\\[2ex] \dfrac{\int_0^{\gamma_2}W^{\varepsilon}(\eta_2,\psi)\,d\psi}{\varepsilon^2\Psi^2(\theta)\,W^{\varepsilon}(\eta_2,\theta)} & \gamma_2<\theta\leqq\gamma_1+\pi\\[2ex] \nu^{\varepsilon}(\theta-\pi) & \gamma_1+\pi<\theta\leqq\gamma_1+2\pi\end{cases}$$

with some $\eta_i$'s. Applying Lemma 1 to the equality (3·3), we see

$$\begin{aligned}\nu^{\varepsilon}(\theta)&=\frac{1}{\Phi^{\varepsilon}(\theta)}(1+o(\varepsilon^2)) && \theta\notin\sum_i U_\delta(\gamma_i)\\ \nu^{\varepsilon}(\theta)&\leqq M && \theta\in\sum_i U_\delta(\gamma_i),\end{aligned}$$

which proves the equality (3·1).

*Case 2.* $\Phi_C(\frac{1}{2}\pi)=0$ *and* $\Phi_C'(\frac{1}{2}\pi)>0$. In this case, there are two stable equilibrium points $\alpha_i^{\varepsilon}$ and two unstable equilibrium points $(2i-1/2)\pi$ $(i=1, 2)$ for the dynamical system (2·5). It is easy to see that

(3·4) $$\alpha_i^{\varepsilon}\uparrow\frac{2i-1}{2}\pi \quad \text{as} \quad \varepsilon\to 0\,.$$

If $\Psi(\theta)$ does not vanish, then $\theta^{\varepsilon}(t)$ has an inveraiant measure density, written by the equations (2·3) and (2·4). We estimate $\int_0^{2\pi}\nu^{\varepsilon}(\theta)\,d\theta$ $(i=1, 2)$. For any $\delta>0$, there exists some $\varepsilon$ such that $\alpha_i^{\varepsilon}\in U_\delta$ $\times((2i-1/2)\pi)$, and

$$\int_0^{2\pi}\nu_1^{\varepsilon}(\theta)\,d\theta=\int_{I_1}\nu_1^{\varepsilon}(\theta)\,d\theta+\int_{I_2}\nu_1^{\varepsilon}(\theta)\,d\theta,$$

where $I_1=[0, 2\pi)\backslash\sum_i U_\delta((2i-1/2)\pi)$ and $I_2=\sum_i U_\delta((2i-1/2)\pi)$. Applying Lemma 1 to the equality (2·4), we have

$$\left\{\int_{I_1}\nu_1^{\varepsilon}(\theta)\,d\theta=\int_{I_1}\frac{1}{2\Phi^{\varepsilon}(\theta)}(1+o(\varepsilon^2))\,d\theta\right.$$

$$(3\cdot5)\qquad \begin{cases} \displaystyle\int_{I_2} \nu_1^{\varepsilon}(\theta)\,d\theta = \frac{2\varepsilon^{-1/3} W(\alpha_1^{\varepsilon}, \frac{1}{2}\pi)}{\Psi^2(\alpha_1^{\varepsilon})} A_1^{\varepsilon} A_2^{\varepsilon} (4 + o(\varepsilon^{2/3})) \\ \displaystyle\int_0^{2\pi} \nu_2^{\varepsilon}(\theta)\,d\theta = o(\varepsilon^2), \end{cases}$$

in which

$$(3\cdot6)\qquad \begin{cases} A_1^{\varepsilon} = \dfrac{\Gamma(\frac{1}{3})}{\frac{1}{2}((2\Phi^{\varepsilon}(\theta)/\Psi^2(\theta)))''_{\theta=(1/2)\pi}} \\ A_2^{\varepsilon} = \dfrac{\Gamma(\frac{1}{2})}{((2\Phi^{\varepsilon}(\theta)/\Psi^2(\theta)))'_{\theta=\alpha_1^{\varepsilon}}}. \end{cases}$$

This and the equality (2·4) prove the equality (3·1).

If $\Psi(\theta)$ vanishes and if $\gamma_i \neq \frac{1}{2}\pi$, then it is not difficult to obtain the equality (3·1) in the same way as in Case 2 of the proof of Theorem 2. However, if $\gamma_i = \frac{1}{2}\pi$ for some $i$ (it does not arise that $\gamma_1 = \gamma_2 = \frac{1}{2}\pi$ by virtue of the assumption that $\Phi'_G(\frac{1}{2}\pi) > 0$), then the circumstance is different. We cannot state if a natural boundary point $\frac{1}{2}\pi$ is *repelling*.[3] If it is repelling, then there exists an invariant measure density $\nu^{\varepsilon}(\theta)$, given by

$$\nu^{\varepsilon}(\theta) = \begin{cases} \dfrac{1}{\Psi^2(\theta)\, W^{\varepsilon}(\xi, \theta)} & \gamma_1 < \theta < \dfrac{1}{2}\pi \\ \nu^{\varepsilon}(\theta - \pi) & \gamma_3 < \theta < \dfrac{3}{2}\pi \\ 0 & \text{otherwise,} \end{cases}$$

where we assume that $\gamma_2 = \frac{1}{2}\pi$, without losing generality, and $\xi$ is some point in $(\gamma_1, \frac{1}{2}\pi)$. Estimating $\int_0^{2\pi} \nu^{\varepsilon}(\theta)\,d\theta$ in the same way as in the equality (3·5), we obtain

$$(3\cdot7)\qquad \begin{cases} \displaystyle\int_{I_1} \nu^{\varepsilon}(\theta)\,d\theta = \frac{2}{\Phi^{\varepsilon}(\frac{1}{2}\pi - \delta)\, W^{\varepsilon}(\xi, \frac{1}{2}\pi - \delta)} (1 + o(\varepsilon^2)) \\ \displaystyle\int_{I_2} \nu^{\varepsilon}(\theta)\,d\theta \geq \sum_i \int_{(2i-1/2)\pi-\delta}^{\alpha^{\varepsilon}} \nu^{\varepsilon}(\theta)\,d\theta = \frac{2(1+o(\varepsilon))}{\varepsilon \Psi^2(\alpha_1^{\varepsilon})\, W^{\varepsilon}(\xi, \alpha_1^{\varepsilon})} A_2^{\varepsilon}, \end{cases}$$

where $A_2^{\varepsilon}$ is given by the equality (3·6). It follows from the equality (3·7) that

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[3] See [8].

$$\int_{I_1} \nu^\varepsilon(\theta)d\theta = o\Big(\int_{I_2} \nu^\varepsilon(\theta)d\theta\Big)$$

which proves the equality (3·1) by virtue of the equation (2·8). If $\frac{1}{2}\pi$ is attracting, then the equation (3·1) is clear.

*Case* 3. $\Phi_C(\frac{1}{2}\pi)=0$ and $\Phi_C'(\frac{1}{2}\pi)=0$. It holds that

$$\begin{cases} \Phi^\varepsilon(\theta)>0 & \theta\neq\frac{1}{2}\pi,\ \frac{3}{2}\pi \\ \Phi^\varepsilon(\theta)=0 & \theta=\frac{1}{2}\pi,\ \frac{3}{2}\pi, \end{cases}$$

for sufficiently small $\varepsilon$. Thus, it is not difficult to obtain the equality (3·1) making use of Lemma 2 in case that $\Psi(\theta)$ does not vanish, or that $\Psi(\theta)$ vanishes at $\theta\neq(2i-1/2)\pi$ $(i=1,2)$. But, if $\Psi(\theta)$ vanishes at $\theta=(2i-1/2)\pi$, then we see, by calculating $W^\varepsilon$, that $\frac{1}{2}\pi+0$ or $\frac{1}{2}\pi-0$ is attracting. The equality (3·1) is obtained.

*Case* 4. $\Phi_C(\frac{1}{2}\pi)=0$ and $\Phi_C'(\frac{1}{2}\pi)<0$. For the dynamical system (2·5), there are two stable equilibrium points $(2i-1/2)\pi$ and two unstable equilibrium points $\beta_i^\varepsilon$ $(i=1,2)$ such that

$$\beta_i^\varepsilon\uparrow\frac{2i-1}{2}\pi \qquad \text{as} \quad \varepsilon\to0\,.$$

Thus, there is little different in proving the equality (3·1) between Case 2 and Case 4.

*Case* 5. $\Phi_C(\frac{1}{2}\pi)<0$. In this case, the dynamical system (2·5) has two stable equilibrium points $\alpha_i^\varepsilon$ and two unstable equilibrium points $\beta_i^\varepsilon$ $(i=1,2)$ such that

$$\begin{cases} \alpha_i^\varepsilon\uparrow\dfrac{2i-1}{2}\pi & \text{as} \quad \varepsilon\to0 \\ \beta_i^\varepsilon\downarrow\dfrac{2i-1}{2}\pi & \text{as} \quad \varepsilon\to0\,. \end{cases} \tag{3·8}$$

If $\Psi(\theta)$ does not vanish, then there exists an invariant measure $\mu^\varepsilon(d\theta)$, written by the equalities (2·3) and (2·4). Estimating $\int_0^{2\pi}\nu^\varepsilon(\theta)d\theta$ according to the same procedure as in Case 2, we obtain the equality (3·1). If $\Psi(\theta)$ vanishes, then $\gamma_1\neq\frac{1}{2}\pi$ $(i=1,2)$ by virtue of the assumption that $\Phi_C(\frac{1}{2}\pi)<0$. Thus, an invariant measure density, given by the equality (3·3), exists. For any $\delta>0$, there exists some $\varepsilon>0$ such that $\alpha_i^\varepsilon\in U_\delta((2i-1/2)\pi)$ and $\beta_i^\varepsilon\in U_\delta((2i-1/2)\pi)$. Let $J_1=\sum_i U_\delta(\gamma_i)$, $J_2=\sum_i U_\delta((2i-1/2)\pi)$, and $J_3=[0, 2\pi]\backslash J_1\backslash J_2$. Estimating $\int_0^{2\pi}\nu^\varepsilon(\theta)d\theta$ in the same manner as in Case 2 of the proof of Theorem 2, we see

$$(3\cdot 9)\quad \begin{cases} \int_{J_1}\nu^\varepsilon(\theta)d\theta\leqq M \\ \int_{J_2}\nu^\varepsilon(d)d\theta\geqq \sum_i\int_{\alpha_i^\varepsilon}^{\beta_i^\varepsilon}\nu^\varepsilon(\theta)d\theta=\dfrac{2B_1^\varepsilon B_2^\varepsilon}{\Psi^2(\alpha_1^\varepsilon)W^\varepsilon(\alpha_1^\varepsilon,\beta_1^\varepsilon)}(1+o(\varepsilon)) \\ \int_{J_3}\nu^\varepsilon(\theta)d\theta=\int_{J_3}\dfrac{1}{2\Phi^\varepsilon(\theta)}(1+o(\varepsilon^2))d\theta\,, \end{cases}$$

where $M$ is a constant independent of $\varepsilon$, and

$$B_1^\varepsilon=\Gamma\left(\frac{1}{2}\right)\left[\left(\frac{2\Phi^\varepsilon(\theta)}{\Psi^2(\theta)}\right)'_{\theta=\beta_1^\varepsilon}\right]^{-1}$$

$$B_2^\varepsilon=\Gamma\left(\frac{1}{2}\right)\left[\left(\frac{2\Phi^\varepsilon(\theta)}{\Psi^2(\theta)}\right)'_{\theta=\alpha_1^\varepsilon}\right]^{-1}.$$

The equality (3·9) proves the equality (3·1) by verture of the equalities (2·8) and (3·8).

## 4. A proper node.

If the matrix B is (IV), then it is clear that $\theta(t)=\theta(0)$ for the system (0·4). However, *there is a counter example such that for some $\delta>0$ and some $\theta_0$*

$$(4\cdot 1)\qquad \lim_{\varepsilon\to 0}P_{\theta_0}\{\lim_{t\to\infty}\theta^\varepsilon(t)\epsilon U_\delta(\lim_{t\to\infty}\theta(t))\}=0\,.$$

**Example.** Let the matrix C be such that

$$\begin{pmatrix} c_1 & 0 \\ 0 & c_2\end{pmatrix}\quad c_1<c_2\,.$$

Then, we can solve the stochastic differential equation (0·2′):

$$(4\cdot 2)\quad x_i^\varepsilon(t)=x_i^\varepsilon(0)\exp\{(b-\tfrac{1}{2}\varepsilon^2c_i)t+c_i(B_1(t)-B_1(0))\}\qquad (i=1,2).$$

Applying the law of iterated logarithm to the solution (4·2), we see that for $x_1^\varepsilon(0)\neq 0$

$$\lim_{t\to\infty}\frac{x_2^\varepsilon(t)}{x_1^\varepsilon(t)}=0 \qquad \text{a.s.}.$$

Thus, for any $\varepsilon>0$

$$p_{\theta_0}\{\lim_{t\to\infty}\theta^\varepsilon(t)=0 \quad \text{or} \quad \pi\}=1 \qquad \theta_0\neq 0, \pi,$$

from which the equality (4·1) holds.

From the above-obtained relations between the systems (0·1) and (0·2), we have the following remark:

**Remark.** *If the orgin is a spiral point, an improper node, or a paddle point in the system* (0·1), *then the system* (0.2), *preserves the property of the origin in the system* (0.1) *with probability arbitrarily close to one, for sufficiently small* $\varepsilon$. *But, if the origin is a center or a proper node in the system* (0·1), *then it is not necessarity true in the system* (0·2).

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