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Perturbation theory for backward and forward random evolutions

By

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§1. Introduction

Research on the random evolution of a family of semigroups $\{T_i(t),\ t\geq 0,\ i=1,\dots,N\}$ with switching among semigroups controlled by a finite state, stationary Markov chain v was begun by Griego-Hersh [4] to study equations of the form

$$\frac{\partial \tilde{u}}{\partial t}=\tilde{A}\tilde{u}+Q\tilde{u} \tag{1.1}$$

where Q is the infinitesimal matrix of v, $\tilde{A}$ is the infinitesimal generator of a semigroup $\tilde{T}(t)$ on a Banach space $\tilde{B}$, and $\tilde{u}(t)=\tilde{T}(t)\tilde{f}$, where $\tilde{f}\in\tilde{B}$.

Quiring [11] gave a construction of random evolutions analogous to that of Griego and Hersh in which the Markov chain v is replaced by a diffusion process on the real line. In [10] Pinsky introduced discontinuous random evolutions as a representation for multiplicative operator functionals of a Markov chain. Using the theory of discontinuous random evolutions, Kertz [8] proved a type of singular perturbation theorem and gave new proofs of limit theorems for Markov processes on N lines. Limit theorems and applications for random evolutions have appeared in several other places [2], [4], and [5].

In this paper we give an alternative but equivalent formulation of random evolutions using a perturbation principle of Phillips [9].

This approach has the advantage of immediately furnishing the backward and forward equations for random evolutions. The backward equation is (1.1) above. The forward equation is

(1.2) $\frac{\partial \tilde{u}}{\partial t} = \tilde{A}\tilde{u} + Q^T\tilde{u}$, where Q^T is the matrix transpose of Q.

In section 2, we obtain a perturbation representation for $\tilde{T}(t)$ and solve (1.1) and (1.2) using an approach inspired by the work of Schoene [12]. § 3 contains an alternative perturbation formulation of random evolutions more in the spirit of the "renewal equation" approach to Markov chains in which the transition probabilities are shown to solve a pair of integral equations (renewal equations), thereby giving rise to the backward and forward Kolmogorov differential equations. The proofs in section 3 are analogous to those in [7] and are included for two main reasons — to show that perturbation representations for (backward and forward) random evolutions may be obtained independent of the famous Phillips Perturbation series, and in order that the proofs (which are analogous but longer due to the complicated notation) may be omitted in the extension to the nonstationary case.

In section 4, we generalize these results to the discontinuous case. § 5 contains the extension to the nonstationary case.

In [7] the author shows that the solution of the "transposed" equation (1.2) is not the "transpose" of the solution of (1.1), except in the special case that the semigroups commute with each other. He also studies the effect of "time-reversal" of the chain, in the case of a countable state space Markov chain with a finite "explosion time" and a relationship is established between time-reversal and the substitution of Q^T for Q in (1.1).

Surveys of the literature on random evolutions are given in the papers of Pinsky [10] and Cogburn-Hersh [2]. The reader is referred to Hille-Phillips [6] for the necessary facts about semigroups and to Chung [1] for information about Markov chains.

§2. Backward and forward random evolutions

Suppose $v=\{v(t), t\geq 0\}$ is a right-continuous Markov chain with state space $\{1,\dots, N\}$, stationary transition probabilities $p_{ik}(t)$, and infinitesimal matrix $Q=<q_{ik}>=<p'_{ik}(0)>$. P_i is the probability measure defined on sample paths $\omega(t)$ for v under the condition $\omega(0)=i$. E_i denotes integration with respect to P_i. For a sample path $\omega\in\Omega$ of v, $\tau_j(\omega)$ is the time of the jth jump, and $N(t, \omega)$ is the number of jumps up to time t.

Let $\{T_i(t), t\geq 0, i=1,\dots, N\}$ be a family of strongly continuous semigroups of bounded linear operations on a fixed Banach space B. A_i is the infinitesimal generator of T_i. Let $\mathscr{D}_i$ be the domain of A_i. $\tilde{B}$ is the N-fold cartesian product of B with itself. A generic element of $\tilde{B}$ is denoted by $\tilde{f}=<f_i>$ where $f_i\in B$, $i=1,\dots, N$. We equip $\tilde{B}$ with any appropriate norm so that $\|\tilde{f}\|\to 0$ as $\|f_i\|\to 0$ for each i.

Definition 2.1. A *backward random evolution* $\{R(t, \omega), t\geq 0\}$ is defined by the product

$$R(t)=T_{v(0)}(\tau_1)T_{v(\tau_1)}(\tau_2-\tau_1)\cdots T_{v(\tau_{N(t)})}(t-\tau_{N(t)})\,.$$

Definition 2.2. For $t\geq 0$ define the matrix operator $\tilde{T}(t)$ on $\tilde{B}$ specified componentwise by

$$(\tilde{T}(t)\tilde{f})_i=E_i[R(t)f_{v(t)}]\,.$$

The following results now follow from Griego and Hersh [4].

Theorem 2.3. *$\{\tilde{T}(t), t\geq 0\}$ is a strongly continuous semigroup of bounded linear operators on $\tilde{B}$.*

Theorem 2.4. *The infinitesimal generator $\tilde{A}$ of $\tilde{T}(t)$ is given by $\tilde{A}=\mathrm{diag}(A_1,\dots, A_N)+Q$ in matrix form, or considering $\tilde{A}$ as acting on column vectors we get*

$$(\tilde{A}\tilde{f})_i=A_if_i+\sum_j q_{ij}f_j$$

for $\tilde{f} \in \tilde{\mathscr{D}} = \mathscr{D}_1 \times \cdots \times \mathscr{D}_N$.

Below we shall construct a matrix $\tilde{R}(t)$ such that $\tilde{T}(t)\tilde{f} = \tilde{R}(t)\tilde{f}$ in usual matrix notation.

According to Theorem 3.5 of Phillips [9; p. 205], the matrix operator $\tilde{A} = \text{diag}(A_1, \ldots, A_N) + Q$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators, $\tilde{R}(t)$ on $\tilde{B}$. We proceed to obtain an explicit representation for $\tilde{R}(t)$.

In general we have from the above perturbation principle that if A is the infinitesimal generator of a semigroup of bounded linear operators $\tilde{L}(t)$ and P is a bounded linear operator, then $A+P$ is the infinitesimal generator of a semigroup of bounded linear operators $\tilde{R}(t)$ given by

$$\tilde{R}(t) = \sum_{n=0}^{\infty} \tilde{R}_n(t)$$

where

$$\tilde{R}_0(t) = \tilde{L}(t),$$

$$\tilde{R}_n(t) = \int_0^t \tilde{L}(s) P \tilde{R}_{n-1}(t-s) ds, \qquad n \geq 1.$$

In order to apply this we define $\tilde{L}(t) = \text{diag}(e^{-q_1 t} T_1(t), \ldots, e^{-q_N t} T_N(t))$, where $q_i = -q_{ii}$ for each i. Then $\tilde{L}(t)$ has generator $A = \text{diag}(A_1 - q_1, \ldots, A_N - q_N)$ so that we take P to be the matrix obtained by placing zeros along the diagonal in Q (leaving all other entries unchanged). Then it is easy to see that the operator $\tilde{R}(t)$ is given by (specified componentwise)

$$(\tilde{R}(t)\tilde{f})_i = \sum_k R_{ik}(t) f_k, \qquad 1 \leq i, \quad k \leq N,$$

where for f in B

$$R_{ik}(t) f = \sum_{n=0}^{\infty} R_{ik}^{(n)}(t) f$$

$$R_{ik}^{(0)}(t) f = \delta_{ik} e^{-q_i t} T_i(t) f \quad \text{(where } \delta_{ik} \text{ is the Kronecker delta)}$$

$$R_{ik}^{(n)}(t)f=\sum_{j\neq i}\int_0^t e^{-q_i s}T_i(s)q_{ij}R_{jk}^{(n-1)}(t-s)fds, \qquad n\geq 1$$

and $\tilde{R}_n(t)=<R_{ik}^{(n)}(t)>$. We can state these observations in the form of a theorem.

Theorem 2.5. $\tilde{R}(t)\tilde{f}=\tilde{T}(t)\tilde{f}$.

The following result now follows from Theorem 2.4.

Corollary 2.6. *The Cauchy problem for an unknown vector* $\tilde{u}(t)$, $t>0$

$$\frac{\partial u_i}{\partial t}=A_i u_i+\sum_j q_{ij}u_j,\ \tilde{u}(0+)=\tilde{f} \tag{2.1}$$

is solved by $\tilde{u}(t)=\tilde{R}(t)\tilde{f}$, *for* $\tilde{f}\in\tilde{\mathscr{D}}$.

The matrix operator $\tilde{C}=\text{diag}(A_1,\ldots,A_N)+Q^T$, where Q^T is the matrix transpose of Q, is the infinitesimal generator of a semigroup of bounded linear operators, $\tilde{S}(t)$ on $\tilde{B}$. We now proceed to obtain an explicit representation for $\tilde{S}(t)$.

Obviously, (from the above cited perturbation principle),

$$\tilde{S}(t)=\sum_{n=0}^{\infty}\tilde{S}_n(t)$$

where

$$\tilde{S}_0(t)=\tilde{L}(t),$$

$$\tilde{S}_n(t)=\int_0^t \tilde{L}(t-s)P^T\tilde{S}_{n-1}(s)ds, \qquad n\geq 1.$$

Let us define new operators as follows: for f in B, $1\leq i,\ k\leq N$, let

$$S_{ik}(t)f=\sum_{n=0}^{\infty}S_{ik}^{(n)}(t)f$$

where

$$S_{ik}^{(0)}(t)f=\delta_{ik}e^{-q_k t}T_k(t)f$$

and

$$S_{ik}^{(n)}(t)f = \sum_{j \neq k} \int_0^t e^{-q_k(t-s)} T_k(t-s) q_{jk} S_{ij}^{(n-1)}(s) f ds.$$

Then the operator $\tilde{S}(t)$ is given by (specified componentwise)

$$(\tilde{S}(t)\tilde{f})_k = \sum_i S_{ik}(t) f_k.$$

Definition 2.7. A *forward random evolution* $\{S(t, \omega), t \geq 0\}$ is defined by the product

$$S(t) = T_{v(\tau_n)}(t-\tau_n) T_{v(\tau_{n-1})}(\tau_n - \tau_{n-1}) \cdots T_{v(\tau_1)}(\tau_2 - \tau_1) T_{v(0)}(\tau_1)$$

where $N(t) = n$.

Hence, a forward random evolution is obtained by reversing the order of the operators in a backward random evolution.

Definition 2.8. For $t \geq 0$ define the *expectation semigroup* $\tilde{U}(t)$ on $\tilde{B}$ (specified componentwise) by

$$(\tilde{U}(t)\tilde{f})_k = \sum_i E_i[S(t) f_i;\ v(t) = k], \quad \text{where}$$

$$E_i[S(t) f_i;\ v(t) = k] = E_i[S(t) f_i I_{\{v(t)=k\}}].$$

We are now ready to prove the main result of this section.

Theorem 2.9. $\tilde{U}(t)\tilde{f} = \tilde{S}(t)\tilde{f}$.

Proof: We need to show that

$$\sum_i S_{ik}(t) f_i = \sum_i E_i[S(t) f_i;\ v(t) = k].$$

Now,

$$\sum_i S_{ik}(t) f_i = \sum_{n=0}^{\infty} \sum_i S_{ik}^{(n)}(t) f_i$$

and

$$\sum_i E_i[S(t) f_i;\ v(t) = k] = \sum_{n=0}^{\infty} \sum_i E_i[S(t) f_i;\ N(t) = n,\ v(\tau_n) = k].$$

We argue by induction on n that

$$E_i[S(t)f_i;\ N(t)=n,\ v(\tau_n)=k]=S_{ik}^{(n)}(t)f_i.$$

The case $n=0$ is true, so assume true for $n\geq 1$.

$$E_i[S(t)f_i;\ N(t)=n+1,\ v(\tau_{n+1})=k]$$

$$=\int_0^t E_i[S(t)f_i;\ N(t)=n+1,\ v(\tau_{n+1})=k|v(s),\ s\geq\tau']P_i\,(\tau\in d\tau')$$

(τ denotes the last discontinuity of $v(\cdot)$)

$$=\sum_{j\neq k}\frac{q_{jk}}{q_j}\int_0^t T_k(t-s)E_i[S(s)f_i;\ N(s)=n,\ \ v(\tau_n)=j]q_je^{-q_k(t-s)}ds$$

$$=\sum_{j\neq k}\int_0^t e^{-q_k(t-s)}T_k(t-s)q_{jk}S_{ij}^{(n)}(s)f_ids$$

(by the induction hypothesis)

$=S_{ik}^{(n+1)}(t)f_i$, giving the desired condition.

By standard semigroup theory we obtain the following theorem.

Theorem 2.10. *The Cauchy problem for an unknown vector* $\tilde{u}(t)$, $t>0$,

$$\frac{\partial u_k}{\partial t}=A_ku_k+\sum_i q_{ik}u_i,\quad \tilde{u}(0+)=\tilde{f} \tag{2.2}$$

is solved by $\tilde{u}(t)=\tilde{U}(t)\tilde{f}$, *for* $\tilde{f}\in\tilde{\mathscr{D}}$.

§3. Alternative perturbation representations for random evolutions

In this section we construct matrix representations for $\tilde{T}(t)$ and $\tilde{U}(t)$ that can not be directly obtained from Phillips perturbation principle. We include proofs for the backward case (the proofs for forward random evolutions are entirely parallel).

For $f\in B$, let

$$R_{ik}(t)f = \sum_{n=0}^{\infty} R_{ik}^{(n)}(t)f$$

where

$$R_{ik}^{(0)}(t)f = \delta_{ik} e^{-q_k t} T_k(t)f \tag{3.1}$$

$$R_{ik}^{(n+1)}(t)f = \sum_{j \neq k} \int_0^t R_{ij}^{(n)}(t-s) q_{jk} e^{-q_k s} T_k(s) f ds. \tag{3.2}$$

Let $\sum_{j \neq k} q_{jk} = r_k$ and $q = \mathrm{Max}(r_1, \dots, r_N)$.

Assume $\|T_i(t)\| \leq Ce^{\beta t}$ for each i. Since $T_i(t)$, $i = 1, \dots, N$, is strongly continuous, it is easily seen by induction on n that all the $R_{ik}^{(n)}(t)$ are defined and are strongly continuous in t. Furthermore, we have the estimates

$$\| R_{ik}^{(n)}(t) \| \leq Ce^{\beta t} \frac{(Cqt)^n}{n!}, \quad n = 0, 1, 2, \dots, \tag{3.3}$$

which can be proved by induction on n. In fact, (3.3) is true for $n = 0$; assuming it for $n \geq 1$, we have from (3.2)

$$\| R_{ik}^{(n+1)}(t) \| \leq Ce^{\beta t} \frac{(Cq)^{n+1}}{n!} \int_0^t (t-s)^n ds$$

$$= Ce^{\beta t} \frac{(Cqt)^{n+1}}{(n+1)!}$$

We see from (3.3) that the series is absolutely convergent, the sum $R_{ik}(t)$ satisfies the integral equation

$$R_{ik}(t) = \delta_{ik} e^{-q_k t} T_k(t) + \sum_{j \neq k} \int_0^t R_{ij}(t-s) q_{jk} e^{-q_k s} T_k(s) ds$$

and that

$$\| R_{ik}(t) \| \leq \sum_{n=0}^{\infty} \| R_{ik}^{(n)}(t) \| \leq Ce^{(Cq+\beta)t}. \tag{3.4}$$

The following simple lemma will play a significant role in what follows.

Lemma 3.1. $$R_{ik}(s+t)=\sum_l R_{il}(s)R_{lk}(t).$$

Proof. We first argue by induction on n that

$$R_{ik}^{(n)}(s+t)=\sum_{\nu=0}^{n}\sum_{l=1}^{N} R_{il}^{(\nu)}(s)R_{lk}^{(n-\nu)}(t).$$

For $n=0$ this is obvious. Assuming true for a given $n\geq 1$, we have

$$\sum_{\nu=0}^{n+1}\sum_l R_{il}^{(\nu)}(s)R_{lk}^{(n+1-\nu)}(t)=\sum_l R_{il}^{(n+1)}(s)R_{lk}^{(0)}(t)$$

$$+\sum_{\nu=0}^{n}\sum_l\sum_{j\neq k} R_{il}^{(\nu)}(s)\int_0^t R_{lj}^{(n-\nu)}(t-u)q_{jk}e^{-q_k u}T_k(u)du$$

$$=e^{-q_k t}R_{ik}^{(n+1)}(s)T_k(t)+\sum_{j\neq k}\int_0^t R_{ij}^{(n)}(s+t-u)q_{jk}e^{-q_k u}T_k(u)du$$

(by the induction hypothesis)

$$=e^{-q_k t}\sum_{j\neq k}\int_0^s R_{ij}^{(n)}(s+t-u)q_{jk}e^{-q_k u}T_k(u)T_k(t)du$$

$$+\sum_{j\neq k}\int_0^t R_{ij}^{(n)}(s+t-u)q_{jk}e^{-q_k u}T_k(u)du$$

$$=R_{ik}^{(n+1)}(s+t).$$

This completes the induction and summing over n we obtain

$$R_{ik}(s+t)=\sum_l\sum_{\nu=0}^{\infty} R_{il}^{(\nu)}(s)\sum_{n=\nu}^{\infty} R_{lk}^{(n-\nu)}(t)$$

$$=\sum_l R_{il}(s)R_{lk}(t).$$

Q. E. D.

For $t\geq 0$ define the matrix operator $\tilde{R}(t)$ on $\tilde{B}$ (specified componentwise) by

$$(\tilde{R}(t)\tilde{f})_i=\sum_k R_{ik}(t)f_k.$$

Theorem 3.2. $\{\tilde{R}(t),\ t\geq 0\}$ *is a strongly continuous semigroup of* *bounded linear operators on* $\tilde{B}$.

Proof. By the estimate (3.4),

$$\|(\tilde{R}(t)\tilde{f})_i\| \le \sum_k Ce^{(Cq+\beta)t}\|f_k\|, \text{ so } \tilde{R}(t)$$

is a bounded linear operator. Also, since the $R_{ik}^{(n)}(t)$ are strongly continuous the uniform convergence of the sum implies that $\tilde{R}(t)$ is strongly continuous in t. Thus, we need only check the semigroup property.

$$(\tilde{R}(s+t)\tilde{f})_i = (\tilde{R}(s)\tilde{R}(t)\tilde{f})_i.$$

By Lemma 3.1, we have

$$\begin{aligned}(\tilde{R}(s+t)\tilde{f})_i &= \sum_k R_{ik}(s+t)f_k \\ &= \sum_k [\sum_l R_{il}(s)R_{lk}(t)]f_k \\ &= \sum_l R_{il}(s)[\sum_k R_{lk}(t)f_k] \\ &= (\tilde{R}(s)\tilde{R}(t)\tilde{f})_i.\end{aligned}$$

This completes the proof of the theorem.

Theorem 3.3. $\tilde{R}(t)\tilde{f} = \tilde{T}(t)\tilde{f}$.

Proof. Let $f_i \in \mathscr{D}_i$. Then,

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}[(\tilde{R}(t)\tilde{f})_i - f_i] \\ &= \lim_{t\to 0}\frac{1}{t}[e^{-q_i t}T_i(t)f_i - f_i] \\ &\quad + \lim_{t\to 0}\frac{1}{t}[\sum_k \int_0^t e^{-q_k s}\sum_{j\neq k} R_{ij}(t-s)q_{jk}T_k(s)f_k ds] \\ &= A_i f_i - q_i f_i + \sum_{k\neq i} q_{ik}f_k \\ &= A_i f_i + \sum_k q_{ik}f_k.\end{aligned}$$

Therefore, the infinitesimal generator of $\tilde{R}(t)$ is $\tilde{A}$ with domain $\tilde{\mathscr{D}}$. Thus, (by standard semigroup theory) we obtain the result of the theorem.

The forward case can be handled in a similar manner. Assume $q_i \leq q'$ for each i.

For $f \in B$ define

$$S_{ik}(t)f = \sum_{n=0}^{\infty} S_{ik}^{(n)}(t)f$$

where

$$S_{ik}^{(0)}(t)f = \delta_{ik} e^{-q_i t} T_i(t)f$$

$$S_{ik}^{(n+1)}(t)f = \sum_{j \neq i} \int_0^t S_{jk}^{(n)}(t-s) q_{ij} e^{-q_i s} T_i(s) f ds.$$

Thus,

$$\|S_{ik}(t)\| \leq Ce^{(Cq'+\beta)t}$$

and

$$S_{ik}(t) = \delta_{ik} e^{-q_i t} T_i(t) + \sum_{j \neq i} \int_0^t S_{jk}(t-s) q_{ij} e^{-q_i s} T_i(s) ds.$$

Lemma 3.4. $S_{ik}(t+s) = \sum_l S_{lk}(t) S_{il}(s)$.

For $t \geq 0$ define the operator $\tilde{S}(t)$ on $\tilde{B}$ (specified componentwise) by

$$(\tilde{S}(t)\tilde{f})_k = \sum_i S_{ik}(t) f_i.$$

Theorem 3.5. *$\{\tilde{S}(t), t \geq 0\}$ is a strongly continuous semigroup of bounded linear operators on $\tilde{B}$.*

Theorem 3.6. $\tilde{S}(t)\tilde{f} = \tilde{U}(t)\tilde{f}$.

§4. Perturbation representations for discontinuous random evolutions

In this section we shall generalize the results of the previous sections. The proofs involved are entirely parallel to those of section 2 and thus are omitted.

Let $\{\Pi_{jk}\}$, $1 \leq j, k \leq N$, be uniformly bounded linear operators defined on B with $\Pi_{ii} = I$, the identity operator on B, for each i.

Definition 4.1. A *discontinuous backward random evolution* $\{R(t), t \geq 0\}$ is defined on v by the product

$$R(t) = T_{v(0)}(\tau_1)\Pi_{v(0)v(\tau_1)}T_{v(\tau_1)}(\tau_2 - \tau_1)\cdots T_{v(\tau_{N(t)})}(t - \tau_{N(t)}).$$

Definition 4.2. For $t \geq 0$ the *expectation semigroup* $\tilde{T}(t)$ corresponding to $R(t)$ is defined on $\tilde{B}$ by

$$(\tilde{T}(t)\tilde{f})_i = E_i[R(t)f_{v(t)}].$$

Analogous to the continuous case, we make the following definitions

$$R_{ik}(t) = \sum_{n=0}^{\infty} R_{ik}^{(n)}(t) \text{ where, for } f \text{ in } B,$$

$$R_{ik}^{(0)}(t)f = \delta_{ik}e^{-q_{i}t}T_i(t)f$$

$$R_{ik}^{(n+1)}(t)f = \sum_{j \neq i} \int_0^t e^{-q_{i}s}T_i(s)q_{ij}\Pi_{ij}R_{jk}^{(n)}(t-s)fds.$$

Dfinition 4.3. For $t \geq 0$ define the operator $\tilde{R}(t)$ on $\tilde{B}$ by $(\tilde{R}(t)\tilde{f})_i = \sum_k R_{ik}(t)f_k$.

Using the methods of section 2 (or §3) we get that $\tilde{R}(t)$ is a semigroup of bounded linear operators on $\tilde{B}$ and

Theorem 4.4.

i). $\tilde{R}(t)\tilde{f} = \tilde{T}(t)\tilde{f}$.

ii). $\tilde{u}(t) = \tilde{R}(t)\tilde{f}$, *solves the Cauchy problem*

$$\frac{\partial u_i}{\partial t} = A_i u_i + \sum_j q_{ij}\Pi_{ij}u_j, \quad \tilde{u}(0+) = \tilde{f}$$

for $\tilde{f} \in \tilde{\mathscr{D}}$.

Now, define a *discontinuous forward random evolution* $\{S(t), t\geq 0\}$ by reversing the order of the operators in Definition 4.1 above. Analogous to the continuous case, we define

$$(\tilde{U}(t)\tilde{f})_k = \sum_i E_i[S(t)f_i;\ v(t)=k].$$

Now, define

$$S_{ik}(t) = \sum_{n=0}^{\infty} S_{ik}^{(n)}(t) \quad \text{where}$$

$$S_{ik}^{(0)}(t)f = \delta_{ik} e^{-q_k t} T_k(t) f$$

$$S_{ik}^{(n+1)}(t)f = \sum_{j\neq k} \int_0^t e^{-q_k(t-s)} T_k(t-s) q_{jk} \Pi_{jk} S_{ij}^{(n)}(s) f ds.$$

For $t\geq 0$, define

$$(\tilde{S}(t)\tilde{f})_k = \sum_i S_{ik}(t) f_i.$$

Using the methods of section 2 (or § 3) we get that $\tilde{S}(t)$ is a semigroup of bounded linear operators on $\tilde{B}$ and

Theorem 4.5.

i). $\tilde{S}(t)\tilde{f} = \tilde{U}(t)\tilde{f}$.

ii). $\tilde{u}(t) = \tilde{S}(t)\tilde{f}$, *solves the Cauchy problem*

$$\frac{\partial u_k}{\partial t} = A_k u_k + \sum_i q_{ik} \Pi_{ik} u_i,\quad \tilde{u}(0+) = \tilde{f},\quad \tilde{f}\in\tilde{\mathscr{D}}.$$

Remark. Using the approach of § 3, alternative perturbation representations may be obtained for $\tilde{R}(t)$ and $\tilde{S}(t)$.

§ 5. Nonstationary random evolutions

The system of equations (2.1) taken with the system of equations (2.2) form a formally adjoint system. The relation between (2.1) and (2.2) shows up more clearly in the non-stationary case. Thus, in this section we shall extend the theory of backward and forward random evolutions to the nonstationary case.

Suppose $v(t, \omega)$ is a nonstationary Markov chain on $\{1, \dots, N\}$ with transition matrix

$$P(s, t) = < p_{ik}(s, t) > \quad \text{such that for} \quad 0 \le s \le t$$

$$p_{ik}(s, t) = \sum_{n=0}^{\infty} p_{ik}^{(n)}(s, t) \quad \text{where} \tag{5.1}$$

$$p_{ik}^{(0)}(s, t) = \delta_{ik} \exp\left\{-\int_s^t q_i(u)du\right\} \tag{5.2}$$

$$p_{ik}^{(n+1)}(s, t) = \sum_{j \neq i} \int_s^t \exp\left\{-\int_s^\tau q_i(u)du\right\} q_{ij}(\tau) p_{jk}^{(n)}(\tau, t) d\tau \tag{5.3}$$

or alternatively

$$p_{ik}^{(n+1)}(s, t) = \sum_{j \neq k} \int_s^t \exp\left\{-\int_\tau^t q_k(u)du\right\} q_{jk}(\tau) p_{ij}^{(n)}(s, \tau) d\tau, \tag{5.4}$$

so that $q_i(t) \ge 0$, $q_{ij}(t) \ge 0$, $q_i(t) = \sum_{j \neq i} q_{ij}(t)$. (Refer to Feller [3] for a discussion of such processes.)

Assume $q_i(t) \equiv -q_{ii}(t) \equiv \sum_{j \neq i} q_{ij}(t) \le q'$ for all i and t. Let $\sum_{j \neq k} q_{jk}(t) = r_k(t)$ and assume $r_k(t) \le q$ for all k and t.

Using ((5.1), (5.2), and (5.3)), let us now define new operators as follows: for $f \in B$ let

$$R_{ik}(s, t)f = \sum_{n=0}^{\infty} R_{ik}^{(n)}(s, t)f \quad \text{where for } 0 \le s \le t$$

$$R_{ik}^{(0)}(s, t)f = \delta_{ik} \exp\left\{-\int_s^t q_i(u)du\right\} T_i(t-s)f$$

$$R_{ik}^{(n+1)}(s, t)f = \sum_{j \neq i} \int_s^t \exp\left\{-\int_s^\tau q_i(u)du\right\} T_i(\tau - s) q_{ij}(\tau) R_{jk}^{(n)}(\tau, t) f d\tau.$$

Definition 5.1. For $0 \le s \le t$ define the operator $\tilde{R}(s, t)$ on $\tilde{B}$, specified componentwise, by

$$(\tilde{R}(s, t)\tilde{f})_i = \sum_k R_{ik}(s, t) f_k.$$

Modifying the techniques of § 3 we have the following results:

Theorem 5.2. i) *$\{\tilde{R}(s, t), 0 \leq s \leq t\}$ is a strongly continuous two parametric family of linear operators satisfying the convolution equation*

$$\tilde{R}(s, t)=\tilde{R}(s, u)\tilde{R}(u, t), \qquad 0\leq s\leq u\leq t.$$

ii) $\dfrac{\partial u_i(s, t)}{\partial s} = -A_i u_i(s, t) - \sum_j q_{ij}(s) u_j(s, t), \qquad 0\leq s\leq t,$
is solved by $\tilde{u}(s, t)=\tilde{R}(s, t)\tilde{f}$.

Using ((5.1), (5.2), and (5.4)), define operators as follows: for $f \in B$ let

$$S_{ik}(s, t)f=\sum_{n=0}^{\infty} S_{ik}^{(n)}(s, t)f \quad \text{where}$$

$$S_{ik}^{(0)}(s, t)f=\delta_{ik}\exp\left\{-\int_s^t q_i(u)du\right\}T_i(t-s)f$$

$$S_{ik}^{(n+1)}(s, t)f=\sum_{j\neq k}\int_s^t \exp\left\{-\int_\tau^t q_k(u)du\right\}T_k(t-\tau)q_{jk}(\tau)S_{ij}^{(n)}(s, \tau)fd\tau.$$

Definition 5.3. For $0\leq s\leq t$ define the operator $\tilde{S}(s, t)$ on $\tilde{B}$ (componentwise) by

$$(\tilde{S}(s, t)\tilde{f})_k=\sum_i S_{ik}(s, t)f_i.$$

Using arguments analogous to those in § 3 we have the following results:

Theorem 5.4. i) *$\{\tilde{S}(s, t), 0\leq s\leq t\}$ is a strongly continuous two parametric family of linear operators satisfying the equation*

$$\tilde{S}(s, t)=\tilde{S}(u, t)\tilde{S}(s, u), \qquad 0\leq s\leq u\leq t.$$

ii) $\dfrac{\partial u_k(s, t)}{\partial t} = A_k u_k(s, t) + \sum_i q_{ik}(t) u_i(s, t), \qquad 0\leq s\leq t,$
is solved by $\tilde{u}(s, t)=\tilde{S}(s, t)\tilde{f}$.

Let $P_{i,s}$ be the probability measure defined on sample paths for

v under the condition $v(s)=i$. $E_{i,s}$ denotes integration with respect to $P_{i,s}$.

Dfinition 5.5. A *backward random evolution* $\{R(s, t), 0\leq s\leq t\}$ is defined by the product

$$R(s, t)=T_{v(s)}(\tau_1-s)T_{v(\tau_1)}(\tau_2-\tau_1)\cdots T_{v(\tau_n)}(t-\tau_n)$$

where $\tau_1,\ldots, \tau_n$ are the jump times between s and t.

Definition 5.6. For $0\leq s\leq t$ define the operator $\tilde{T}(s, t)$ on $\tilde{B}$, specified componentwise, by

$$(\tilde{T}(s, t)\tilde{f})_i=E_{i,s}[R(s, t)f_{v(t)}].$$

Definition 5.7. A *forward random evolution* $\{S(s, t), 0\leq s\leq t\}$ is defined by the product

$$S(s, t)=T_{v(\tau_n)}(t-\tau_n)\cdots T_{v(\tau_1)}(\tau_2-\tau_1)T_{v(s)}(\tau_1-s)$$

where $\tau_1,\ldots, \tau_n$ are the jump times between s and t.

Definition 5.8. For $0\leq s\leq t$ define the operator $\tilde{U}(s, t)$ on $\tilde{B}$, specified componentwise, by

$$(\tilde{U}(s, t)\tilde{f})_k=\sum_i E_{i,s}[S(s, t)f_i;\ v(t)=k].$$

The proofs of the following results are analogous to that of Theorem 2.9 and are omitted for brevity.

Theorem 5.9.
i). $\tilde{R}(s, t)\tilde{f}=\tilde{T}(s, t)\tilde{f}$.
ii). $\tilde{S}(s, t)\tilde{f}=\tilde{U}(s, t)\tilde{f}$.

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