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On Reeb components

By

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§1. Introduction

Let M be a compact orientable $n+1$ dimensional manifold and $\mathscr{F}$ a codimension one foliation on M tangent to ∂M of class C^r. $(M, \mathscr{F})$ is a *Reeb foliation* if all leaves in $\operatorname{Int} M$ are homeomorphic to $\mathbf{R}^n$. A *Reeb component* is a Reeb foliation whose leaves are proper. A Reeb foliation is always transversally orientable.

$(M, \mathscr{F})$ is *C^r conjugate* to $(M', \mathscr{F}')$ if there exists a foliation preserving C^r homeomorphism of M onto M'. $(M, \mathscr{F})$ is *C^r isotopic* to $(M, \mathscr{F}')$ if there exists a foliation preserving C^r homeomorphism of M which is C^r isotopic to the identity.

For $n=2$ Novikov ([8] for Reeb components), Rosenberg, Roussarie and Chatelet ([3], [10], [11] for Reeb foliations) has classified C^2 Reeb foliations by C^0 conjugacy and C^0 isotopy. In [6] it is shown that if $(M, \mathscr{F})$ is a Reeb foliation of class C^2 then M is homotopy equivalent to T^k (k dimensional torus) and $(M, \mathscr{F})$ is a Reeb component if and only if $k=1$.

The purpose of this note is to show that any Reeb component is an "ordinary Reeb component" if n is large. Here the *ordinary Reeb component* $(S^1 \times D^n, \mathscr{F}_R)$ (or $(S^1 \times D^n, \mathscr{F}'_R)$) is defined by $\omega = \Sigma x_i dx_i - \exp(1/(\Sigma x_i^2 - 1))dt$ where $D^n = \{(x_1, x_2, \dots, x_n) | \Sigma x_i^2 \leqq 1\}$ and t is the coordinate of $S^1 = \mathbf{R}/\mathbf{Z}$ (or $\omega' = \Sigma x_i dx_i + \exp(1/(\Sigma x_i^2 - 1))dt$ respectively). It is easy to see that ω and ω' are completely integrable non-singular one forms on $S^1 \times D^n$ and $\mathscr{F}_R$ and $\mathscr{F}'_R$ are Reeb components of class C^∞.

Theorem 1. *Let* $(M, \mathscr{F})$ *be a Reeb component of class* C^1. *If* $n \geqq 6$ *then* $(M, \mathscr{F})$ *is* C^0 *conjugate to* $(S^1 \times D^n, \mathscr{F}_R)$

Theorem 2. *Let* $(S^1 \times D^n, \mathscr{F})$ *be a Reeb component of class* C^1. *If* $n \geqq 6$ *then* $(S^1 \times D^n, \mathscr{F})$ *is* C^0 *isotopic to* $(S^1 \times D^n, \mathscr{F}_R)$ *or* $(S^1 \times D^n, \mathscr{F}'_R)$

Remark In Theorem 2 the isotopy can be taken to be identical on $S^1 \times \partial D^n$. In Theorems 1 and 2, if the holonomy groups of the boundaries are C^r conjugate then C^0 can be replaced by C^r.

To prove Theorem 1 it is convenient to introduce the notion of generalized Reeb component. $(M, \mathscr{F})$ is a *generalized Reeb component* if the holonomy groups of all leaves in $\operatorname{Int} M$ are trivial and if all leaves of $\mathscr{F}$ are proper. (A generalized Reeb component is transversely orientable. The proof is easy by using the double covering argument and Corollary 2.1.) This is a special case of the almost without holonomy foliations treated in [6] and the results in §2 (except Proposition 2.3.) are contained in [6] for C^2 case. But the assumption that all leaves of $\mathscr{F}$ are proper simplifies the arguments. So we give independent proofs for the results of §2.

The difficult point in the proof of Theorem 1 is to show that $\partial M = S^1 \times S^{n-1}$. And the proof of Theorem 2 is more easy than the proof of Theorem 1. So we leave it to the readers.

§2. Generalized Reeb components

In this section M is a compact manifold and $\mathscr{F}$ is a transversally orientable codimension one foliation on M of class C^1 tangent to ∂M. For $x \in M$, F_x is the leaf of $\mathscr{F}$ containing x and for a subset $S \subset M$, $Q_S = \{x \in M | F_x \cap S \neq \emptyset\}$.

Lemma 2.1. *Let* $g: [0, 1] \to M$ *be a map transverse to* $\mathscr{F}$ *and suppose that* $F_{g(t_1)} = F_{g(t_2)}$, $0 \leq t_1 < t_2 \leq 1$, *then there exists a closed curve* C *transverse to* $\mathscr{F}$ *such that* $Q_c = Q_{g([t_1, t_2])}$. *Moreover if* F' *is* *a leaf of* $\mathscr{F}$ *satisfying* $F' \cap g([t_1, t_1+\varepsilon])=\emptyset$ *for some* $\varepsilon>0$, *then we can take* C *so that* $F' \cap C=F' \cap g([t_1, t_2])$.

The proof is easy by a standard argument (see [4]).

Lemma 2.2. *Suppose no leaf of* $\mathscr{F}$ *is exceptional and there exists a non-compact proper leaf* F. *Then there exists a leaf* F_0 *and a closed curve* C_0 *transverse to* $\mathscr{F}$ *such that* $C_0 \cap F_0$ *consists of one point.*

Proof. There exists a closed curve C transverse to $\mathscr{F}$ passing throught F (see [4]). Let $g:[0, 1]\to M$ be a parametrization of C such that $g(0)=g(1)\in F$. If there exists t such that $g(t)\in F$ and $g((0, t))\cap F=\emptyset$, then for some small $\varepsilon_1, \varepsilon_2>0$, $g(\varepsilon_1)$ and $g(t+\varepsilon_2)$ belong to the same leaf and $g([\varepsilon_1, t+\varepsilon_2])\cap F=\{g(t)\}$. So by Lemma 2.1. there exists a closed transversal curve C_0 such that $C_0 \cap F=\{g(t)\}$. If there does not exists such t, putting $t_0=\inf\{t\in(0, 1)|g((0, t))\cap F\neq\emptyset\}$ we see $g(t_0)\notin F$ since F is proper. We assert that the leaf F_0 containing $g(t_0)$ is proper. Otherwise F_0 is locally dense, so there exists $\varepsilon>0$ such that $g(t_0-\varepsilon)\in F_0$. But from the definition of t_0, F and any neighborhood of $g(t_0-\varepsilon)$ intersect. This contradicts to the definition of t_0. By the same reason $g([0, t_0))\cap F_0=\emptyset$. Since F_0 is proper, for small $\varepsilon>0$, $g([0, t_0+\varepsilon])\cap F_0=\{g(t_0)\}$ and we can choose ε so that $g(t_0+\varepsilon)\in F$. Then, by Lemma 2.1., there exists a closed transversal curve C_0 such that $C_0 \cap F_0=\{g(t_0)\}$. q.e.d.

Problem. Under the assumption of Lemma 2.2. is there a closed transversal curve C such that $C\cap F=\{\text{one point}\}$?

Lemma 2.3. (*Sacksteder-Schwartz* [12]) *Let* C *be a closed curve transverse to* $\mathscr{F}$, *then for any* $x\in\partial Q_c$ *and for any neighborhood* U *of* x, *there exists a leaf with non-trivial holonomy group which intersects with* U.

For the proof see [12] or [5].

Lemma 2.4. *Let* $(M, \mathscr{F})$ *be a generalized Reeb component and*

C a closed curve transverse to $\mathscr{F}$ then we have $Q_c = \operatorname{Int} M$. Moreover for any leaf F in $\operatorname{Int} M$, the closure of F contains ∂M.

Proof. $Q_c = \operatorname{Int} M$ follows immediately from Lemma 2.3. If there exists a neighborhood U of $x \in \partial M$ such that $U \cap F = \emptyset$, since the holonomy group of each component of ∂M is non-trivial by Lemma 2.3., there exists $g: [0, 1] \to U$ transverse to $\mathscr{F}$ such that $g(0)$ and $g(1)$ belong to the same leaf and $g([0, 1]) \cap F = \emptyset$. Then by Lemma 2.1. there exists a closed transversal curve C' such that $Q_{c'} \cap F = \emptyset$. This is a contradiction. q.e.d.

Definition. A vector field X on M transverse to $\mathscr{F}$ is *nice* if X has a closed orbit C such that $C \cap F = \{\text{one point}\}$ for any leaf F in $\operatorname{Int} M$. We call such a orbit C a *nice orbit.*

Proposition 2.1. *Let $(M, \mathscr{F})$ be a generalized Reeb component, then there exists a nice vector field X on M.*

Proof. By Lemma 2.2 there exists a closed transversal curve C and a leaf F_0 such that $C \cap F_0 = \{\text{one point}\}$. Suppose that there exists a leaf F such that $F \cap C \ni x_1, x_2, x_1 \neq x_2$, then let $\widehat{x_1 x_2}$ be the arc of C such that $\widehat{x_1 x_2} \not\ni \{C \cap F_0\}$. By Lemma 2.1. there exists a closed transversal curve C' such that $Q_{c'} = Q_{\widehat{x_1 x_2}} \neq \operatorname{Int} M$. This contradicts to Lemma 2.4. It is easy to construct X so that C is an orbit of X. q.e.d.

Corollary 2.1. *Let $(M, \mathscr{F})$ be a generalized Reeb component. Then $\operatorname{Int} M$ is a fibration over S^1 with fibre F_{x_0}. Moreover there exists a foliation preserving flow ϕ_t on $\operatorname{Int} M$ whose orbits coincide with maximal solution curves of $X|\operatorname{Int} M$, and $\operatorname{Int} M = F_0 \times [0, 1]/(x, 0) \sim (\phi_1(x), 1)$. $\phi_1(F) = F$ for any leaf F in $\operatorname{Int} M$.*

Proof. We identify S^1 with C. Let $p: \operatorname{Int} M \to S^1$ be a map defined by $p(x) = C \cap F$, then clearly p is a fibration. Let dt be the natural one form on $S^1 = \mathbf{R}/\mathbf{Z}$, then there exists a positive function

f on Int M such that $\omega(fX)\equiv 1$ where $\omega=p^*dt$. ϕ_t is the flow associated to fX. q.e.d.

From now on $(M, \mathscr{F})$ is a generalized Reeb component. We fix a nice vector field X on M, a nice orbit C of X and a point $x_0\in C$. Then as in the proof of Corollary 2.1. we have the one form ω and the foliation preserving flow ϕ_t on Int M.

Lemma 2.5. *Let V be a component of ∂M and z a point of V. Let T be the maximal solution curve of X which contains z and y_0 be a point of $F_{x_0}\cap T$ (which is not empty by Lemma 2.4.). Then $F_{x_0}\cap T=\{y_n=\phi_n(y_0),\ n\in\mathbf{Z}\}$ and if X is outward normal at z, $\lim_{n\to\infty} y_n=z$.*

Proof. If there exists $y\in F_{x_0}\cap T$ such that $y=\phi_{n+t_0}(y_0)$, $0<t_0<1$, then by Lemma 2.1., there exists a closed transversal curve C' such that $Q_{C'}=Q_{\{\phi_t(y_0)|n\leqq t\leqq n+t_0\}}=Q_{\{\phi_t(x_0)|0\leqq t\leqq t_0\}}\neq$ Int M. This contradicts to Lemma 2.4. By the same argument we see that any segment $\{\phi_t(y_0)|n\leqq t\leqq n+1\}$ meets with all leaves of $\mathscr{F}$ in Int M. Since all leaves are proper, the set $\{y_n|n\in\mathbf{Z}\}$ does not accumulate to a point of Int $M\cap T$. So we have $\lim_{n\to\infty} y_n=z$ if X is outward normal at z. q.e.d.

The holonomy group of V at z is the image of a homomorphism Φ of $\pi_1(V, z)$ to G, where G is the group of germs at z of local diffeomorphisms of T. We denote by the same letter an element of $\pi_1(V, z)$ and a closed curve in V which represent the element. Also we use the same letter for an element of G and its representative local diffeomorphism.

We define a homomorphism $\theta: \pi_1(V, z)\to\mathbf{Z}$ as the composition of the following natural homomorphisms

$$\pi_1(V, z)\xrightarrow{H} H_1(V)\xrightarrow{i_*} H_1(M)\underset{\cong}{\xleftarrow{j_*}} H_1(\text{Int}\, M)\xrightarrow{p_*} H_1(S^1)\cong Z.$$

Then clearly $\theta(\alpha)=\int_{\alpha'}\omega$ where α' is a closed curve which represent

$j_*^{-1} \circ i_* \circ H(\alpha)$.

Proposition 2.2. *θ is not trivial. Let $k>0$ be the generator of* $\mathrm{Im}\,\theta$ *and suppose that X is outward at z, then there exists $\alpha \in \pi_1(V, z)$ such that $\Phi(\alpha)(y_n)=y_{n+k}$ for sufficiently large n and the holonomy group of V at z is free abelian of rank 1 generated by $\Phi(\alpha)$.*

Proof. By Lemma 2.3. there exists $\alpha \in \pi_1(V, z)$ such that $\Phi(\alpha)$ is not identity. Let l_1 be the lift of α to F_{x_0} with initial point y_n (n sufficiently large) then the end point of l_1 is y_m where $y_m=\Phi(\alpha)(y_n)$ and $m \neq n$. Let l_2 be the segment of T from y_m to y_n and l the composite of l_1 and l_2, then l represents $j_*^{-1} \circ i_* \circ H(\alpha)$. So $\theta(\alpha)=\int_l \omega = \int_{l_2} \omega = n-m \neq 0$ and θ is not trivial. By the same argument if $\theta(\alpha)=-k$ we have $\Phi(\alpha)(y_n)=y_{n+k}$ provided $\Phi(\alpha)$ is defined at y_n. To prove the final assertion it is sufficient to show that if $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ coincide on $\{y_n | n \text{ sufficiently large}\}$ then $\Phi(\alpha_1)=\Phi(\alpha_2)$. To prove this, it is sufficient to show that if $\Phi(\alpha_3)(y_n)=y_n$ for sufficiently large n then $\Phi(\alpha_3)=$identity. This is easy. q.e.d.

Remark. In the above proposition and proof, $\Phi(\alpha)$ is defined on T, so the phrase "sufficiently large n" is unnecessary. This is easily seen by using the foliation preserving flow ϕ_t.

To describe the structure of $\mathscr{F}$ near V we define a foliated manifold $V(N, h)$ as follows. Let N be a codimension one submanifold of a closed orientable manifold V such that $N-V$ is connected and the manifold V_N obtained from V by cutting along N has two boundary components N_1 and N_2 which are copies of N. Let h be a contracting diffeomorphism of $[0, \varepsilon)$, $\varepsilon>0$. $V(N, h)$ is obtained from $V_N \times [0, \varepsilon)$ by identifying $(x, t) \in N_1 \times [0, \varepsilon)$ with $(x, h(t)) \in N_2 \times [0, \varepsilon)$. Then $V(N, h)$ is a manifold with corners and there exists a dually foliated structure on $V(N, h)$ which is induced from the product structure of $V_N \times [0, \varepsilon)$. A result of Nishimori [7] is stated as follows.

Lemma 2.6. *There exists a submanifold N and a diffeomor-*

phism h satisfying above conditions. There exists a diffeomorphism g of $V(N, h)$ into M which preserves the dually foliated structures (where the dual structure of $\mathscr{F}$ is defined by X) such that $g(x, 0)=x$ for $x \in V \subset V_N \times [0, \varepsilon)/\sim$. Moreover if N' is a submanifold homologous to N then there exists $V(N', h')$ and g' which satisfy above conditions.

Proposition 2.3. *Suppose that $\partial M = V$ is connected and X is outward on V. Let $k>0$ be the generator of $\mathrm{Im}(\theta)$ and N as above then there exists a connected compact submanifold H of F_{x_0} whose boundary consists of k copies of N such that the following decomposition holds.*

$$F_{x_0} = H \cup \phi_k(H) \cup \phi_{2k}(H) \cup \cdots \cup \phi_{nk}(H) \cup \cdots$$

where $\mathrm{Int}\,\phi_{(n+1)k}(H) \supset \phi_{nk}(H)$ and $\phi_{(n+1)k}(H) - \mathrm{Int}\,\phi_{nk}(H)$ consists of k copies of V_N.

The proof is immediate from the following lemmas.

Lemma 2.7. *Let $g: V(N, h) \to M$ be as above. We identify $(x, \tau) \in (V_N - N_2) \times [0, \varepsilon)$ with a point of $V(N, h)$. For $t \geqq 0$ we define $\phi'_t(x, \tau) = g^{-1} \circ \phi_t \circ g(x, \tau)$ then ϕ'_t preserves the foliated structure on $V(N, h)$ and we have $\phi'_{nk}(x, \tau) = (x, h^n(\tau))$.*

The proof is immediate from the construction of $V(N, h)$ and Proposition 2.2.

Lemma 2.8. *Put $M' = M - \mathrm{Int}\, g(V_N \times [0, \varepsilon_0]/\sim)$, where $0 < \varepsilon_0 < \varepsilon$, then $H = M' \cap F_{x_0}$ is a compact connected manifold whose boundary consists of k copies of N.*

Proof. Without loss of generality we may assume that $z \in N$, $g(z, \varepsilon_0) \in F_{x_0}$, ε_0 is in the image of h^k and the α-limit set of the maximal solution curve of X containing z is C. Since M' is compact, $H = M' \cap F_{x_0}$ is a compact manifold whose boundary consists of k copies of N. Let N' be a component of ∂H, then $N' \cap \{y_n | n \in \mathbf{Z}\} = \{y_i\}$. *Let* $l': [0,1] \to V_N$ be a curve such that $l'(0) = z \in N_2$ and

$l'(1)=z\in N_1$. Define $l_0\colon[0,1]\to M$ by $l_0(t)=g((l'(t),\varepsilon_i))$ where ε_i is choosen so that $g((z,h(\varepsilon_i)))=y_i$ for $z\in N_1$, then l_0 is a curve in H from y_i to y_{i-k}. We define l_n inductively by $l_n=\phi^{-k}\circ l_{n-1}$, then l_n is a curve in H from y_{i-nk} to y_{i-nk-k}. By the assumption we have $\lim_{n\to\infty} y_{i-nk}=x_0$, so we can choose a curve l in H from y_{i-mk-k} to x_0 for some $m>0$. Then the composition of $l_0, l_1,\dots, l_m$ and l is a curve in H from y_i to x_0. Since H contains no closed component, this proves the connectedness of H. q.e.d.

§3. Reeb componenent

In this section $(M,\mathscr{F})$ is a Reeb component and we use the same notations as in §2.

Since $\operatorname{Int} M$ is a fibration over S^1 with fibre $\mathbf{R}^n$, we see that $\partial M=V$ is connected and $i^*\colon H_1(\partial M)\to H_1(M)$ is isomorphism for $n>2$ by the Poincaré duality, thus $\theta\colon\pi_1(V)\to\mathbf{Z}$ is surjective. So by Lemma 2.6. and Proposition 2.3. the following proposition holds.

Proposition 3.1. *There exists a foliation preserving diffeomorphism g of $V(N,h)$ into M, where N is a codimension one submanifold of $V=\partial M$ and h is a contracting diffeomorphism of $[0,\varepsilon)$. Moreover*

$$\mathbf{R}^n=F_{x_0}=H\cup\phi_1(H)\cup\phi_2(H)\cup\cdots\cup\phi_n(H)\cup\cdots$$

where H is a compact submanifold of $\mathbf{R}^n$, $\partial H\cong N$ and $\phi_{n+1}(H)-\operatorname{Int}\phi_n(H)=V_N$. We have $\lim_{n\to\infty}\phi_n(\partial H)=N$.

Let D_0 be an imbedded n disk in F_{x_0} such that $\operatorname{Int} D_0\supset H$. Fix $k>0$ such that $S_0=\partial D_0\subset\phi_k(H)-\operatorname{Int} H$ and $\pi\colon\tilde{M}\to M$ be the regular covering corresponding to $k\mathbf{Z}\subset\mathbf{Z}\cong\pi_1(M,x_0)$. Then $\tilde{F}$, $\tilde{X}$ and $\tilde{C}$ are defined naturally from F, X and C by π. It is easy to see $(\tilde{M},\tilde{F})$ is a Reeb component and $\tilde{X}$ is a nice vector field for $\tilde{F}$ and $\tilde{C}$ is a nice orbit of $\tilde{X}$. Let $\tilde{\phi}_t$ be the foliation preserving flow on $\operatorname{Int}\tilde{M}$ defined by $\tilde{X}$ and $\tilde{C}$, then we have $\pi\circ\tilde{\phi}_t(\tilde{x})=\phi_{tk}(\pi(\tilde{x}))$ for $\tilde{x}\in\operatorname{Int}\tilde{M}$ and $t\in\mathbf{R}$.

From now on we assume that $n\geqq 6$.

Lemma 3.1. *There exists a fibration $\tilde{p}: \partial\tilde{M} \to S^1$ with fibre S^{n-1}.*

Proof. Let l be a closed curve in N, since l is homotopic to a curve in F_{x_0}, l is homotopic to zero in M. So $\pi^{-1}(N)$ has k components. Fix a leaf $\tilde{F}_{x_0}$ in $\tilde{M}$ such that $\pi(\tilde{F}_{x_0}) = F_{x_0}$. Put $\tilde{H} = \pi^{-1}(H) \cap \tilde{F}_{x_0}$, then $\lim_{n\to\infty} \tilde{\phi}_n(\partial\tilde{H}) = \tilde{N}$ is a component of $\pi^{-1}(N)$. It is easy to see that we have the decomposition

$$\mathbf{R}^n \cong \tilde{F}_{x_0} = \tilde{H} \cup \tilde{\phi}_1(\tilde{H}) \cup \tilde{\phi}_2(\tilde{H}) \cup \cdots \cup \tilde{\phi}_n(\tilde{H}) \cup \cdots$$

such that $\tilde{\phi}_{n+1}(\tilde{H}) - \operatorname{Int} \tilde{\phi}_n(\tilde{H}) \cong \tilde{V}_{\tilde{N}}$ where $\tilde{V} = \partial\tilde{M}$. Since $\pi(\tilde{\phi}_1(\tilde{H}) - \operatorname{Int} \tilde{H}) = \phi_k(H) - \operatorname{Int} H \supset S_0$, $\tilde{\phi}_1(\tilde{H}) - \operatorname{Int} \tilde{H}$ contains $\tilde{S}_0$ where $\tilde{S}_0 \cong S^{n-1}$. Let $\tilde{S}_1 = \lim_{n\to\infty} \tilde{\phi}_n(\tilde{S}_0)$ then $\tilde{S}_1 \cong S^{n-1}$ and $\tilde{S}_1$ is homologous to $\tilde{N}$ in $\tilde{V}$. So by Lemma 2.6 and Proposition 2.3.,

$$\mathbf{R}^n \cong \tilde{F}_{x_0} = \tilde{D}_0 \cup \tilde{\phi}_1(\tilde{D}_0) \cup \tilde{\phi}_2(\tilde{D}_0) \cup \cdots$$

and $\tilde{V}_{\tilde{S}_1} \cong \tilde{\phi}_1(\tilde{D}_0) - \operatorname{Int} \tilde{D}_0$. Since $\tilde{\phi}_1(\tilde{D}_0)$ and $\tilde{D}_0$ are n-disks and $n \geqq 6$ we see that $\tilde{V}_{\tilde{S}_1} \cong S^{n-1} \times [0, 1]$ and $\tilde{V}$ is a fibration over S^1 with fibre S^{n-1}. q.e.d.

Corollary 3.1. *$\pi_1(\partial\tilde{M}) \cong \mathbf{Z}$ and $\pi_i(\partial M) \cong \pi_i(S^{n-1})$ for $i \geqq 2$.*

Proposition 3.2. *There exists a fibration $p: \partial M \to S^1$ with fibre S^{n-1}.*

Proof. Since $\pi_i(\partial M)$ is finitely generated, if $\pi_1(\partial M) \cong \mathbf{Z}$ then by the theorem of Browder-Levine [1] we see that ∂M is a fibration over S^1. Consider the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\partial\tilde{M}) & \xrightarrow{\pi_*} & \pi_1(\partial M) & \longrightarrow & \mathbf{Z}_k \\ & & {\scriptstyle H}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle H} & & \\ & & H_1(\partial\tilde{M}) & \xrightarrow{\pi_*} & H_1(\partial M). & & \end{array}$$

By the construction of $\tilde{M}$ we see that $\pi_*: H_1(\partial\tilde{M}) \to H_1(\partial M)$ is a monomorphism. So by a simple argument we see that $\ker H = 1$ and we have $\pi_1(\partial M) \cong H_1(\partial M) \cong \mathbf{Z}$. So ∂M is a fibration over S^1 with fibre G. By the homotopy exact sequence of fibrations we see that G is a homotopy sphere. But G is liftable to a leaf$\cong\mathbf{R}^n$ along ϕ, we have $G\cong S^{n-1}$. q.e.d.

By this proposition $\partial M\cong S^{n-1}\times[0, 1]/(x, 0)\sim(\psi(x), 1)$ for some diffeomorphism ψ of S^{n-1}. Let $W'=S^{n-1}\times[0, 1]\times[0, \varepsilon)\ni(x, s, t)$, on W' we define a foliation whose leaves are defined by $t=$constant. For some contracting diffeomorphism h of $[0, \varepsilon)$, let $W=W'/(x, 0, t)\sim(\psi(x), 1, h(t))$. On W there is a foliation induced from the foliation on W' and there exists a foliation preserving diffeomorphism g of W' into M.

Lemma 3.2. *There exists a diffeomorphism f of $\partial M\times[0, 1]$ into M such that $f|\partial M\times\{0\}$ is identity and for $0<t\leqq 1$, $f(\partial M\times\{t\})$ is transverse to $\mathscr{F}$.*

Proof. It is sufficient to show that there exists a diffeomorphism f' of $\partial W\times[0, \varepsilon_0]$ into W which satisfies obvious conditions. Define $h_s(t)=(1-s)t+sh(t)$ and $f'(x, s, t)=(x, s, h_s(t))$, then f' is a diffeomorphism and for $0<t\leqq\varepsilon_0<h(\varepsilon)$, $\{f'(x, s, t)|(x, s)\in S^{n-1}\times[0, 1]\}$ is transverse to $\mathscr{F}$ because $\frac{\partial h_s}{\partial s}(t)=h(t)-t\neq 0$. q.e.d.

Put $B=f(\partial M\times\{1\})$ then M is separated by B to M_1 and M_2 where $M_1=f(\partial M\times[0, 1])$. It is clear that the leaves of $\mathscr{F}/M_2$ are diffeomorphic to D^n.

Lemma 3.3. *$M_2\cong S^1\times D^n$ as a foliated manifold.*

Proof. M_2 is a D^n bundle over S^1 and the fibers are leaves of $\mathscr{F}|M_2$. But for $n\geqq 6$, $\pi_0(\mathrm{Diff}\,D^n)=0$ by Cerf [2] and the bundle is trivial. q.e.d.

Lemma 3.4. *There exists a vector field Y on M such that $Y|\mathrm{Int}\,M$ is transverse to $\mathscr{F}$ and all orbit of Y is periodic of period* 1.

Proof. By Lemma 3.3. there exists Y on M_2 which satisfies the conditions. It is easy to extend Y to M_1 by Lemma 3.2. q.e.d.

Now it is almost trivial to prove Theorem 1. To compare $(M, \mathscr{F})$ with $(S^1 \times D^n, \mathscr{F}_R)$, decompose $S^1 \times D^n$ into $S^1 \times D^n\left(\frac{1}{2}\right)$ and $S^1 \times S^{n-1} \times \left[\frac{1}{2}, 1\right]$ where $D^n\left(\frac{1}{2}\right) = \left\{(x_1,\dots, x_n)|\Sigma x_i^2 \leqq \frac{1}{2}\right\}$ and $S^{n-1} \times \left[\frac{1}{2}, 1\right] = \left\{(x_1,\dots, x_n)|\frac{1}{2} \leqq \Sigma x_i^2 \leqq 1\right\}$. Then $(M_2, \mathscr{F}|M_2) \cong \left(S^1 \times D^n\left(\frac{1}{2}\right), \mathscr{F}_R|S^1 \times D^n\left(\frac{1}{2}\right)\right)$ by Lemma 3.2. To examine the part M_1, we use Lemma 3.4. and the problem is reduced to the case $n=1$. This is easy but precise description is tedious.

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Added in proof: H. Rosenberg pointed out that, in the proof of Corollary 2.1, p is merely a submersion and to see that p is a fibration it is necessary to see that fX is complete. The completeness of fX follows from Lemma 2.5. by an easy argument.