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On nonapproximability of $K(\pi, n)$-spaces by homogeneous spaces

By

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§0. Introduction. The following question has been proposed by Prof. W. Browder: *Given a finitely generated abelian group* π *and an integer* $n>2$, *do there exist, for arbitrary positive integers* m, *homogeneous spaces* X_m, *such that* $\pi_i(X_m)\approx\pi_i(\mathbf{K}(\pi, n))$ *for* $i\leq m$? Prof. W. Browder has told us, orally that the question can be settled if either $n>3$ or π is finite by cohomological arguments. In this paper we prove that *if* $n=3$ *and* $m\geq 63$ *then* π *must be finite*, thus completing the answer to the above question. Note that, for m small, it is possible to find X_m such that $\pi_i(X_m)\approx\pi_i(\mathbf{K}(\mathbf{Z}, 3))$ for $i\leq m$; for example, for $m=14$, we can take X_m simply connected, compact, simple Lie group of type $\mathbf{E}_8$. The method of the proof is purely homotopical and involves a counting argument. Our proofs depend heavily upon the computation of the lower homotopy groups of simple Lie groups, by Kachi [4], Kervaire [5] and Mimura and Toda ([6], [7] and [8]).

My sincere thanks are due to Prof. W. Browder for suggesting this problem, to Prof. Mimura for supplying some informations about the homotopy groups of exceptional Lie groups, and also to Gopal Prasad without whose help and encouragement this paper would not have materialized.

§1. The main result

Let X and Y be any topological spaces and $m\geq 1$ be an integer. We say X is an m-approximation to Y if there is a map $f: X\rightarrow Y$

such that $f_{\#}: \pi_i(X) \to \pi_i(Y)$ is an isomorphism, for $0 \leq i \leq m$; if X and Y are CW-complexes and if Y is a $\mathbf{K}(\pi, n)$ space, then this is equivalent to saying $\pi_i(X) \approx \pi_i(Y)$, $0 \leq i \leq m$.

Theorem. *Let π be a finitely generated abelian group and X be a homogeneous space such that X is an m-approximation to a $\mathbf{K}(\pi, 3)$ for some $m \geq 63$. Then π is finite.*

It follows from the standard results that a 2-connected homogeneous space is homotopy type of $\mathbf{G}/\mathbf{H}$ where $\mathbf{G}$ is a compact 1-connected Lie group and $\mathbf{H}$ is a closed 1-connected subgroup of $\mathbf{G}$. Therefore, throughout the paper, all the groups are assumed to be compact and 1-connected. So, let $\mathbf{G}/\mathbf{H}$ be an m-approximation to a $\mathbf{K}(\pi, 3)$-space, $m \geq 4$. Then by the homotopy exact sequence of the fibration $\mathbf{H} \hookrightarrow \mathbf{G} \to \mathbf{G}/\mathbf{H}$, we have

(1.1) $0 \longrightarrow \pi_3(\mathbf{H}) \longrightarrow \pi_3(\mathbf{G}) \longrightarrow \pi \longrightarrow 0$ is exact and

(1.2) $\iota_{\#}: \pi_i(\mathbf{H}) \longrightarrow \pi_i(\mathbf{G})$ is an isomorphism for $4 \leq i \leq m-1$, where $\iota: \mathbf{H} \longrightarrow \mathbf{G}$ is the inclusion map.

Since $\mathbf{G}$ and $\mathbf{H}$ are 1-connected, there are direct product decompositions,

$$\mathbf{G} = \prod_{i \in I} \mathbf{G}_i \quad \text{and} \quad \mathbf{H} = \prod_{j \in J} \mathbf{H}_j \tag{1.3}$$

where $\mathbf{G}_i$ and $\mathbf{H}_j$ are 1-connected, simple groups, I and J are finite sets.

For a simple group $\mathbf{S}$, let $l_{\mathbf{G}}(\mathbf{S})$ (resp. $l_{\mathbf{H}}(\mathbf{S})$) denote the number of simple factors of $\mathbf{G}$ (resp. $\mathbf{H}$) isomorphic to $\mathbf{S}$ and let $l(\mathbf{S}) = l_{\mathbf{G}}(\mathbf{S}) - l_{\mathbf{H}}(\mathbf{S})$. For any finitely generated abelian group A, let $r_n(A)$ denote the number of factors in A isomorphic to the cyclic group $\mathbf{Z}_n$ of order n, where $n = p^k$, for some prime p and an integer $k \geq 1$; let $r_\infty(A)$ denote the rank of A. Then from (1.2) and (1.3) it follows that

$$\Sigma_{i,n} = \sum_{\substack{\mathbf{S} \\ \mathbf{S}, \text{simple}}} r_n(\pi_i(\mathbf{S})) \cdot l(\mathbf{S}) = 0 \tag{1.4}$$

for $4 \leq i \leq m-1$ and for every $n = p^k$. Since $\pi_3(\mathbf{S}) \approx \mathbf{Z}$ for any simple group $\mathbf{S}$ (See Bott [1]), it follows from (1.1) and (1.3) that

(1.5) $$\sum_{\substack{\mathbf{S} \\ \mathbf{S},\text{simple}}} l(\mathbf{S}) = r_\infty(\pi).$$

In what follows the symbols $\mathbf{A}_n$ $(n \geq 1)$, $\mathbf{B}_n$ $(n \geq 3)$, $\mathbf{C}_n$ $(n \geq 2)$, $\mathbf{D}_n$ $(n \geq 4)$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ denote the compact 1-connected Lie groups of the corresponding type, as usual.

(1.6) **Proposition:** *Let $p \geq 31$, be a prime. Then for each simple group* $\mathbf{S}$, *the p-primary components of* $\pi_{2p}(\mathbf{S})$ *and* $\pi_{2p+1}(\mathbf{S})$ *are given as below*:

$\pi_i(\mathbf{S}; p)$:

i \ $\mathbf{S}$	$\mathbf{A}_n(n<p)$, $\mathbf{B}_n(n<p/2)$, $\mathbf{C}_n(n<p/2)$ $\mathbf{D}_n\left(n \leq \frac{p+1}{2}\right)$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$	$\mathbf{D}_{p+1}$	Any other
$2p$	$\mathbf{Z}_p$	0	0
$2p+1$	0	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$

Proof. Suppose $\mathbf{S}$ is isomorphic to one of the groups shown in column number 2 of the above table. Then by Serre [9], $\mathbf{S}$ is p-regular and hence $\pi_i(\mathbf{S}; p)$ is given by $\pi_i(X_{\mathbf{S}}; p)$ where $X_{\mathbf{S}}$ is a product of spheres having a single S^3 factor. Since $\pi_{2p}(S^k; p) \approx \mathbf{Z}_p$ if $k=3$ and ≈ 0 if $k \neq 3$ (see [10]), it follows that $\pi_{2p}(\mathbf{S}; p) \approx \mathbf{Z}_p$. If $\mathbf{S}$ is not isomorphic to a group shown in column number 2, then $\mathbf{S}$ is a classical group in the stable range for $\pi_{2p}(\mathbf{S})$ and hence by Bott's periodicity theorem and his computations of the stable homotopy groups of classical groups it follows that $\pi_{2p}(\mathbf{S}; p) = 0$ (note that $2p \equiv 2$ or 6 mod 8). Similar arguments give $\pi_{2p+1}(\mathbf{S}; p)$ also for $\mathbf{S} \not\approx \mathbf{B}_n$ and $\not\approx \mathbf{D}_n$. If $\mathbf{S} \approx \mathbf{B}_n$ or $\mathbf{D}_n$, we can use the computations of Kervaire [5].

(1.7) **Lemma.** $l(\mathbf{S}) = 0$ *if* $\mathbf{S} \approx \mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, $(n \leq 7)$, $\mathbf{C}_n$ $(n \leq 3)$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$ *or* $\mathbf{E}_7$. *Also* $l(\mathbf{E}_8) = r_\infty(\pi)$, $l(\mathbf{A}_8) + l(\mathbf{C}_4) = 0$ *and* $l(\mathbf{B}_8) + l(\mathbf{D}_8) = 0$.

We shall prove this in the next section. Assuming the validity of it for a while, the proof of the theorem is completed as follows:

Let $M_1=\{\mathbf{S}|\mathbf{S}$ is simple, and $\pi_{62}(\mathbf{S}; 31)\approx \mathbf{Z}_{31}\}$ and $M_2=\{\mathbf{S}|\mathbf{S}\approx \mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, $(n\leq 8)$, $\mathbf{C}_n$ $(n\leq 4)$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8\}$. We need,

(1.8) **Lemma.** *If* $\mathbf{S}\in M_2$ *and* $\mathbf{S}'$ *is a simple group imbeddable in* $\mathbf{S}$, *then* $\mathbf{S}'\in M_2$, *where "imbeddable" means existence of homomorphism of* $\mathbf{S}'$ *into* $\mathbf{S}$ *with finite kernel.*

Proof. $\mathbf{S}\in M_2$ implies the rank, $r(\mathbf{S})\leq 8$ and $\mathbf{S}'$ is imbeddable in $\mathbf{S}$ implies $r(\mathbf{S}')\leq r(\mathbf{S})$. The only simple groups of rank≤ 8 which are not in M_2 are $\mathbf{C}_n$, $5\leq n\leq 8$. Hence it is enough to prove that $\mathbf{C}_5$ is not imbeddable in $\mathbf{S}$, for $\mathbf{S}\in M_2$. From Dynkin [3] (theorem 11.2), all elements of M_2 except $\mathbf{B}_8$ are imbeddable in $\mathbf{E}_8$ and $\mathbf{C}_5$ is not imbeddable in $\mathbf{E}_8$. Thus it remains to see that $\mathbf{C}_5$ is not imbeddable in $\mathbf{B}_8$. If possible assume that $\mathbf{C}_5$ is imbeddable in $\mathbf{B}_8$. Then the natural real representation of $\mathbf{B}_8$ will give a 17-dimensional real representation ρ of $\mathbf{C}_5$. We claim that ρ is not irreducible. In fact, if ρ is irreducible, then, being odd dimensional, its complexification is an irreducible 17-dimensional complex representation. On the other hand, it can be shown, using H. Weyl's dimension formula, that $\mathbf{C}_5$ has no irreducible complex representation of dimension 17. Hence ρ is not irreducible. Therefore, it follows that an imbedding of $\mathbf{C}_5$ in $\mathbf{B}_8$ gives an imbedding of $\mathbf{C}_5$ in $\mathbf{D}_8$. Since $\mathbf{D}_8$ is imbeddable in $\mathbf{E}_8$ and $\mathbf{C}_5$ is not imbeddable in $\mathbf{E}_8$, we arrive at a contradiction. This complete the proof of the lemma.

(1.9) Proof of the theorem: By (1.4), (1.6) and the definition of M_1, it follows that

$$\sum_{62,31}=\sum_{\mathbf{S}\in M_1} l(\mathbf{S})=0,$$

and by (1.7) and the definition of M_2, it follows that

$$\sum_{\mathbf{S}\in M_1-M_2} l(\mathbf{S})=-l(\mathbf{E}_8)=-r_\infty(\pi). \tag{1.10}$$

Let $\quad \Gamma_1=\bigoplus_{\mathbf{H}_j\in M_2}\pi_{62}(\mathbf{H}_j; 31); \qquad \Lambda_1=\bigoplus_{\mathbf{G}_i\in M_2}\pi_{62}(\mathbf{G}_i; 31),$

$$\Gamma_2 = \bigoplus_{\mathbf{H}_j \notin M_2} \pi_{62}(\mathbf{H}_j;\ 31); \qquad \Lambda_2 = \bigoplus_{\mathbf{G}_i \notin M_2} \pi_{62}(\mathbf{G}_i;\ 31).$$

Then clearly $\pi_{62}(\mathbf{H};\ 31) \approx \Gamma_1 \oplus \Gamma_2$ and $\pi_{62}(\mathbf{G};\ 31) \approx \Lambda_1 \oplus \Lambda_2$. By lemma (1.8), it follow that $\iota_\#(\Gamma_2) \subset \Lambda_2$. Since by assumption, $\iota_\#$ is a monomorphism, we have $r_{31}(\Lambda_2) \leq r_{31}(\Gamma_2)$. By (1.10), we have

$$0 \leq r_{31}(\Lambda_2) - r_{31}(\Gamma_2) = \sum_{\mathbf{S} \in M_1 - M_2} l(\mathbf{S}) = -r_\infty(\pi)$$

and hence $r_\infty(\pi) = 0$. This complete the proof of the theorem modulo the lemma (1.7).

§2. The lemma (1.7)

The table below gives $\pi_i(\mathbf{S})$, where $\mathbf{S}$ is a simple group and $5 \leq i \leq 18$. Only the information relevent to the proof of the lemma have been tabulated. The sources of these informations are Kachi [4], Kervaire [5], Mimura [6] and Mimura and Toda [8]. For the odd primary components of lower homotopy groups of exceptional Lie groups, we refer to unpublished work of Mimura [7]. The notations are as in Mimura and Toda [8]: the symbols ∞, $+$ and an integer r indicate an infinite cyclic group, direct sum and a cyclic group $\mathbf{Z}_r$ of order r respectively. We put various values for i and n in the equation (1.4) and prove lemma (1.7).

(2.1) $i=6$, $n=4, 3, 2$: $\underline{l(\mathbf{A}_1) = l(\mathbf{A}_2) = l(\mathbf{G}_2) = 0}$.

(2.2) $i=5$, $n=2, \infty$, together with (2.1): $\sum_{n\geq 2} l(\mathbf{C}_n) = 0 = \sum_{n \geq 1} l(\mathbf{A}_n)$.

(2.3) $i=7$, $n=\infty$, together with (2.2): $\sum_{n\geq 3} l(\mathbf{B}_n) + \sum_{n\geq 4} l(\mathbf{D}_n) + l(\mathbf{D}_4) = 0$.

(2.4) $i=8$, $n=3, 2$, together with (2.3): $\underline{l(\mathbf{A}_3) = 0;}$

$$l(\mathbf{B}_3) + l(\mathbf{D}_4) + l(\mathbf{B}_4) + l(\mathbf{F}_4) = 0.$$

(2.5) $i=9$, $n=\infty$, $l(\mathbf{D}_5) + l(\mathbf{E}_6) = 0$.

(2.6) $i=10$, $n=5, 3, 8, 4, 2$:

Table of $\pi_i(\mathbf{S})$

S \ i	5	6	7	8	9	10	11	12	13	14	15	16	18
$\mathbf{A}_1$	2	12											
$\mathbf{A}_2$	∞	6											
$\mathbf{A}_3$	∞	0	∞	24									
$\mathbf{A}_4$	∞	0	∞	0	∞	120	0						
$\mathbf{A}_5$	∞	0	∞	0	∞	0	∞	720					
$\mathbf{A}_6$	∞	0	∞	0	∞	0	∞	0	∞	7!			
$\mathbf{A}_7$	∞	0	∞	0	∞	0	∞	0	∞	0	∞	8!	
$\mathbf{A}_8$	∞	0	∞	0	∞	0	∞	0	∞	0	∞	0	9!
$\mathbf{A}_9$	∞	0	∞	0	∞	0	∞	0	∞	0	∞	0	0
$\mathbf{C}_2$	2	0	∞	0	0	120	2						
$\mathbf{C}_3$	2	0	∞	0	0	0	∞	2	2	2·7!			
$\mathbf{C}_4$	2	0	∞	0	0	0	∞	2	2	0	∞	0	9!
$\mathbf{C}_5$	2	0	∞	0	0	0	∞	2	2	0	∞	0	0
$\mathbf{B}_3$	0	0	∞	2+2	2+2	8	∞+2	0	2	7!/2 +8+2			
$\mathbf{D}_4$	0	0	∞+∞	2+2+2	2+2+2	8+24							
$\mathbf{B}_4$	0	0	∞	2+2	2+2	8	∞+2	0	2	8+2			
$\mathbf{D}_5$	0	0	∞	2	∞+2	4	∞						
$\mathbf{B}_5$	0	0	∞	2	2	2	∞						
$\mathbf{D}_6$	0	0	∞	2	2	0	∞+∞	2+2	2+2	4+48	∞+2		
$\mathbf{B}_6$	0	0	∞	2	2	0	∞	2	2	8	∞+2		
$\mathbf{D}_7$	0	0	∞	2	2	0	∞	0	∞	4	∞		
$\mathbf{B}_7$	0	0	∞	2	2	0	∞	0	0	2	∞		
$\mathbf{D}_8$	0	0	∞	2	2	0	∞	0	0	0	∞+∞	2+2+2	24+8
$\mathbf{B}_8$	0	0	∞	2	2	0	∞	0	0	0	∞	2+2	8
$\mathbf{D}_9$	0	0	∞	2	2	0	∞	0	0	0	∞	2	4
$\mathbf{B}_9$	0	0	∞	2	2	0	∞	0	0	0	∞	2	2
$\mathbf{D}_{10}$	0	0	∞	2	2	0	∞	0	0	0	∞	2	0
$\mathbf{G}_2$	0	3	0										
$\mathbf{F}_4$	0	0	0	2	2	0							
$\mathbf{E}_6$	0	0	0	0	∞	0							
$\mathbf{E}_7$	0	0	0	0	0	0	∞	2	2	0			
$\mathbf{E}_8$	0	0	0	0	0	0	0	0	0	0	∞	2	24

$l(\mathbf{A}_4)+l(\mathbf{C}_2)=0$

$l(\mathbf{A}_4)+l(\mathbf{C}_2)+l(\mathbf{D}_4)=0$ and hence $\underline{l(\mathbf{D}_4)=0}$

$l(\mathbf{A}_4)+l(\mathbf{C}_2)+l(\mathbf{B}_3)+2l(\mathbf{D}_4)+l(\mathbf{B}_4)=0$ and hence

$l(\mathbf{B}_3)+l(\mathbf{B}_4)=0$

$\underline{l(\mathbf{D}_5)=0=l(\mathbf{B}_5)}$, and by (2.4) $\underline{l(\mathbf{F}_4)=0}$ and finally by (2.5), $\underline{l(\mathbf{E}_6)=0}$.

(2.7) $i=11$, $n=2, \infty$, together with (2.6) and (2.3):

$$\underline{l(\mathbf{C}_2)=0}; \text{ hence from (2.6) } \underline{l(\mathbf{A}_4)=0} \text{ and } l(\mathbf{D}_6)+l(\mathbf{E}_7)=0.$$

(2.8) $i=12$, $n=5$: $\underline{l(\mathbf{A}_5)=0}$.

(2.9) $i=14$, $n=3$, 32, 16, 7, 8, 4 and 2:

$$\underline{l(\mathbf{D}_6)=l(\mathbf{C}_3)=l(\mathbf{A}_6)=l(\mathbf{B}_3)=0} \text{ and hence } \underline{l(\mathbf{E}_7)=l(\mathbf{B}_4)=0}$$

(by (2.7) and (2.6)). Further $\underline{l(\mathbf{B}_6)=l(\mathbf{D}_7)=l(\mathbf{B}_7)=0}$. Also by (2.2), (2.3), (2.6) and (1.5), $\underline{l(\mathbf{E}_8)=r_\infty(\pi)}$.

(2.10) $i=16$, $n=5$: $l(\mathbf{A}_7)=0$.

(2.11) $i=18$, $n=5$, 3, 8: $\underline{l(\mathbf{A}_8)+l(\mathbf{C}_4)=0}$; $l(\mathbf{D}_8)+l(\mathbf{E}_8)=0$;

$$2l(\mathbf{D}_8)+l(\mathbf{B}_8)+l(\mathbf{E}_8)=0$$

and hence $\underline{l(\mathbf{D}_8)+l(\mathbf{B}_8)=0}$.

Now the under-lined equations in (2.1) to (2.11) together prove lemma (1.7).

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