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A remark to the ordering theorem of L. de Branges

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0. Introduction

In [2], L. de Branges gave a remarkable theorem on the order relation between Hilbert spaces of entire functions contained isometrically in some space $L^2(\sigma)$. However, he put an assumption which, from the point of view of applications, is an undesirable restriction. In fact, in order to prove the uniqueness of the correspondence between a generalized second order differential operator and its spectral function (cf. [1]), it is necessary to prove the ordering theorem for the spaces consisting of entire functions of minimal exponential type. The purpose of this note is to give a complete proof of this ordering theorem which we have used in [1].

1. Statement and proof of the theorem

Following our paper [1], we introduce definitions and notations which will be used later.

Definition 1.1. *A Hilbert space of entire functions H satisfying the following properties will be called a $K-B$ space.*

$(H.1)$ *If $f\in H$, then its conjugate also belongs to H and has the same norm.*

$(H.2)$ *Put $\mathscr{D}(A)=\{\varphi\in H: \lambda\varphi(\lambda)\in H\}$ and $A\varphi(\lambda)=\lambda\varphi(\lambda)$ for $\varphi\in\mathscr{D}(A)$. Then A becomes a closed symmetric operator.*

(*H*.3) *If* $f\in H$ *and* $f(z)=0$ *for some* $z\in \boldsymbol{C}$, *then* $f(\lambda)/(\lambda-z)\in H$.

(*H*.4) *Put* $\Delta(\lambda)=\sup\{|f(\lambda)|^2 : f\in H, (f, f)\leqq 1\}$. *Then* Δ *is locally bounded in* $\boldsymbol{C}$.

From the property (*H*.4), the Hilbert space H has a reproducing kernel $J_\lambda(\mu)$, i.e., $f(\lambda)=(f, J_\lambda)$ for every $f\in H$. de Branges proved that there exist real entire functions (i.e., entire functions with real values on the real line.) P, Q such that

$$J_\lambda(\mu)=\frac{1}{\mu-\bar{\lambda}}\{P(\mu)Q(\bar{\lambda})-P(\bar{\lambda})Q(\mu)\}. \tag{1.1}$$

We note here that, for any two pairs $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ satisfying the relation (1.1), there exists a matrix S of $\boldsymbol{SL}(2, \boldsymbol{R})$ such that

$$(P_1(\lambda), Q_1(\lambda))=(P_2(\lambda), Q_2(\lambda))S.$$

By one of the pairs $\{P, Q\}$, we define the characteristic function E of H;

$$E(\lambda)=P(\lambda)+iQ(\lambda). \tag{1.2}$$

Then it is easy to see that for any $\lambda\in \boldsymbol{C}_+$

$$|E(\lambda)|>|E(\bar{\lambda})|,$$

hence $E(\lambda)$ has no zeros in $\boldsymbol{C}_+$.

The ordering theorem of de Branges may be stated as follows.

Ordering theorem. (*L. de Branges* [2].) *Let* H_1 *and* H_2 *be* $K-B$ *spaces included isometrically in the same space* $L^2(\sigma)$ *for some Radon measure* σ *on* $\boldsymbol{R}^1$. *Let* E_1 *and* E_2 *be the characteristic functions for* H_1 *and* H_2 *respectively. Suppose that* $\log^+|E_1/E_2|$ *is dominated by a harmonic function on* $\boldsymbol{C}_+$. *Then either* H_1 *contains* H_2 *or* H_2 *contains* H_1.

This note is devoted to prove the above theorem, without assuming that $\log^+|E_1/E_2|$ is dominated by a harmonic function on $\boldsymbol{C}_+$ but under the condition that both E_1 and E_2 are entire functions of minimal exponential type, i.e., $\log|f(z)|=o(|z|)$ as $|z|\to\infty$. Fortunately, the key lemma of de Branges in proving the theorem is available for any minimal exponential type entire functions.

Lemma 1.2. (*L. de Branges* [2], *Lemma* 8, p. 107.) *Let $f_1(z)$ and $f_2(z)$ be entire functions of minimal exponential type satisfying*

$$\min\{|f_1(z)|, |f_2(z)|\}\leqq\frac{1}{|\mathrm{Im}\, z|}$$

for all complex z. Then either f_1 or f_2 vanishes identically.
To make use of this lemma, we have to prove several lemmas.

Lemma 1.3. *Let σ_1 and σ_2 be complex Radon measures with finite total variations on $\boldsymbol{R}^1$. Let f_1 and f_2 be entire functions such that*

$$\log|f_k(z)|\leqq a|z|, \qquad k=1, 2, \tag{1.3}$$

for every sufficiently large $|z|$. Suppose that

$$f(z)=f_1(z)\int_{-\infty}^{\infty}\frac{\sigma_1(dt)}{t-z}+f_2(z)\int_{-\infty}^{\infty}\frac{\sigma_2(dt)}{t-z}$$

is an entire function. Then $f(z)$ satisfies the estimate

$$\log|f(z)|\leqq a|z|$$

for every sufficiently large $|z|$.

Proof. This lemma is essentially due to M. G. Krein ([3], Lemma 4.2). Let σ denote one of σ_1 and σ_2. Put

$$\varphi(z)=\int_{-\infty}^{\infty}\frac{\sigma(dt)}{t-z}.$$

If we change the variables as

$$\zeta=\frac{z-i}{z+i}, \quad e^{i\theta}=\frac{t-i}{t+i},$$

then

$$\varphi(z)=\phi(\zeta)=\frac{1-\zeta}{2i}\int_{-\infty}^{\infty}\frac{i-e^{i\theta}}{e^{i\theta}-\zeta}\,\tau(d\theta)\,,$$

where $\tau(d\theta)=\sigma(dt)$. Hence

$$|\phi(\zeta)|\leqq\frac{4\mathrm{var}\,\sigma}{2(1-|\zeta|)}=\frac{2\mathrm{var}\sigma}{1-|\zeta|}$$

for $|\zeta|\leqq 1$ and so

$$\log^+|\phi(\zeta)|\leqq c-\log(1-|\zeta|),$$

where $c=\log^+(2\,\mathrm{var}\,\sigma)$. Thus we have

$$\int_0^1\int_0^{2\pi}\log^+|\phi(re^{i\theta})|rdrd\theta\leqq(c+3/4)\pi\,.$$

The similar argument is possible also in $\boldsymbol{C}_-$, hence we obtain

$$\int_{\boldsymbol{C}}\frac{\log^+|\varphi(z)|}{(|z|+1)^4}\,dxdy<\infty\,.$$

From the above estimate and the condition (1.3), it is easy to see that

$$K=\int_{\boldsymbol{C}}\frac{\log^+|f(z)|}{(|z|+1)^4}\,dxdy<\infty\,.$$

Let $B(a, r)$ denote the closed disk with its center at a and its radius r. Since $\log^+|f(z)|$ is subharmonic, we have an inequality

$$\log^+|f(z)|\leqq\frac{1}{\pi r^2}\int_{B(z,r)}\log^+|f(\zeta)|\,dxdy\,.$$

Noting for any $\zeta\in B(z, r)$, $1\leqq\frac{(1+r+|z|)^4}{(1+|\zeta|)^4}$, we have for any z

$$\log^+|f(z)|\leqq\frac{(1+r+|z|)^4}{\pi r^2}\int_{B(z,r)}\frac{\log^+|f(\zeta)|}{(1+|\zeta|)^4}\,dxdy$$

$$\leqq\frac{(1+r+|z|)^4}{\pi r^2}\int_{\boldsymbol{C}}\frac{\log^+|f(\zeta)|}{(1+|\zeta|)^4}\,dxdy$$

$$= \frac{(1+r+|z|)^4}{\pi r^2} K .$$

Hence, putting $r=|z|$, we have for every sufficiently large $|z|$,

$$\log|f(z)| \leqq c|z|^2 .$$

From the assumption (1.3) and the definition of $f(z)$, we see that $\log^+|f(z)|$ is dominated by $a|z|$ for every sufficiently large $|z|$ along any ray different from the real line. Since we have proved that f is an entire function of at most order 2, Lemma 1.3 results from the Phragmén-Lindelöf theorem immediately.

Lemma 1.4. *Let $f(z)$ be a nontrivial entire function of minimal exponential type. Then for any positive ε, there exists a divergent sequence $\{r_n\}$ such that $r_n/r_{n+1} \to 1$ as $n \to \infty$ and $\log|f(z)| \geqq -\varepsilon|z|$ for $|z| = r_n$.*

As for the proof, refer to Theorem 3.7.1 of Boas, Jr. [4].

Lemma 1.5. *Let H be a $K-B$ space and σ be a measure on $\boldsymbol{R}^1$ such that H is contained isometrically in $L^2(\sigma)$. Let h be an element of $L^2(\sigma)$ which is orthogonal to H and g be an entire function of $L^2(\sigma)$. Then there exists an entire function $F(w)$ satisfying*

$$f(w)F(w) = \int_{-\infty}^{\infty} \frac{f(t)g(w) - g(t)f(w)}{t-w} \overline{h(t)}\sigma(dt) \tag{1.4}$$

for every f of H.

Proof. Let $\phi(f)(w)$ denote the right hand side of (1.4). Taking any two elements f_1 and f_2 of H, we have

$$f_1(w)\{f_2(t)g(w) - g(t)f_2(w)\}/(t-w)$$

$$= f_2(w)\{f_1(t)g(w) - f_1(w)g(t)\}/(t-w)$$

$$+ g(w)\{f_1(w)f_2(t) - f_2(w)f_1(t)\}/(t-w) ,$$

where the last term belongs to H. Hence the identity

$$f_1(w)\phi(f_2)(w)=f_2(w)\phi(f_1)(w)$$

follows. Choosing f_1 and f_2 so that they may not vanish at w, we see that $F(w)=\phi(f)(w)/f(w)$ is an entire function independent of f of H. This completes the proof.

We remark here that the following three statements are equivalent.

(1) Every element of H is of minimal exponential type.

(2) The characteristic function E of H is of minimal exponential type.

(3) $\Delta(\lambda)$, which was defined in $(H.4)$, has the estimate $\log\Delta(\lambda)=o(|\lambda|)$ as $|\lambda|\to\infty$.

This comes from the formulas (1.1), (1.2) and the identity $\Delta(\lambda)=J_\lambda(\lambda)$.

Our proof of the theorem depends entirely on the methods used by de Branges [2]. It is, however, possible to simplify the proof by consulting with L. D. Pitt [5].

Theorem. *Let H_1 and H_2 be $K-B$ spaces whose all elements are of minimal exponential type. If H_1 and H_2 are contained isometrically in a space $L^2(\sigma)$, then either H_1 contains H_2 or H_2 contains H_1.*

Proof. Let $\Delta_k(\lambda)$ be the square of the norm of the linear functional $H_k\ni f\to f(\lambda)$ for $k=1, 2$. Put $\rho(\lambda)=\max\{\Delta_1(\lambda), \Delta_2(\lambda)\}$ and choose a measure τ on $\boldsymbol{R}^1$ such that $\int_{-\infty}^{\infty}\rho(t)\tau(dt)=1$. Then for any $f\in H_k$, we have

$$\int_{-\infty}^{\infty}|f(t)|^2\sigma(dt)\leqq\int_{-\infty}^{\infty}|f(t)|^2\sigma(dt)+\int_{-\infty}^{\infty}|f(t)|^2\tau(dt)$$
$$\leqq 2\int_{-\infty}^{\infty}|f(t)|^2\sigma(dt),$$

for we have $|f(t)|^2\leqq\Delta_k(t)\|f\|^2\leqq\rho(t)\|f\|^2$. Thus the two measures σ and $\sigma+\tau$ define equivalent norms in both H_1 and H_2. So we may assume that σ possesses the continuous part, and hence both H_1 and

H_2 are not dense in $L^2(\sigma)$. For each $g \in H_2$ and $h_1 \in H_1^\perp$ with $\|g\| \leqq 1$ and $\|h_1\| \leqq 1$, we may define an entire function $F(w)$ by Lemma 1.5 such that

$$f(w)F(w)=\int_{-\infty}^{\infty}\frac{f(t)g(w)-g(t)f(w)}{t-w}\overline{h_1(t)}\sigma(dt)$$

holds for every f of H_1. Since g and f are of minimal exponential type, we see that, by Lemma 1.3, $f(w)F(w)$ is also of minimal exponential type. On the other hand, by Lemma 1.4, there exists a divergent sequence $\{r_n\}$ such that $r_n/r_{n+1} \to 1$ as $n \to \infty$ and $\log|f(z)| \geqq -\varepsilon|z|$ for $|z|=r_n$. Hence we have

$$\log^+|F(z)| \leqq \varepsilon|z|$$

for $|z|=r_n$. Applying the maximum principle to F, we find that F is of minimal exponential type.

Similarly, for $f \in H_1$ and $h_2 \in H_2^\perp$ with $\|f\| \leqq 1$ and $\|h_2\| \leqq 1$, we may define $G(w)$ such that

$$(1.5) \qquad g(w)G(w)=\int_{-\infty}^{\infty}\frac{g(t)f(w)-g(w)f(t)}{t-w}\overline{h_2(t)}\sigma(dt)$$

holds for every g of H_2. For the same reason as above, G is of minimal exponential type. By the Schwarz inequality in $L^2(\sigma)$, we have

$$|f(z)F(z)| \leqq \frac{1}{|y|}\{|f(z)|+|g(z)|\}$$

$$|g(z)G(z)| \leqq \frac{1}{|y|}\{|f(z)|+|g(z)|\},$$

where $y=\operatorname{Im} z$. Hence we have

$$|y| \leqq |F(z)|^{-1}+|G(z)|^{-1}.$$

This implies that

$$\min\{|F(z)|,\ |G(z)|\} \leqq 2|y|^{-1}.$$

Therefore it follows from Lemma 1.2 that either F or G vanishes identically. Thus we may suppose that $F(z)=0$ for each $g\in H_2$ and $h_1\in H_1^\perp$. Unless H_2 is contained in H_1, there must exists a $g\in H_2$ with $g\not\in H_1$. Since $F(z)=0$, we have

$$\text{(1.6)}\qquad \frac{f(w)}{g(w)}\int_{-\infty}^{\infty}\frac{w}{w-t}\,g(t)\overline{h_1(t)}\sigma(dt)=\int_{-\infty}^{\infty}\frac{w}{w-t}f(t)\overline{h_1(t)}\sigma(dt).$$

Because $g\not\in H_1$, we may choose h_1 with

$$0\neq\int_{-\infty}^{\infty}g(t)\overline{h_1(t)}\sigma(dt)=\lim_{w\to\infty}\int_{-\infty}^{\infty}\frac{w}{w-t}\,g(t)\overline{h_1(t)}\sigma(dt),$$

and

$$0=\int_{-\infty}^{\infty}f(t)\overline{h_1(t)}\sigma(dt)=\lim_{w\to\infty}\int_{-\infty}^{\infty}\frac{w}{w-t}f(t)\overline{h_1(t)}\sigma(dt),$$

where lim should be taken along a ray not coinciding with the real line. By (1.6) we see that

$$\text{(1.7)}\qquad \lim_{w\to\infty}\left|\frac{f(w)}{g(w)}\right|=0.$$

On the other hand, by (1.5) we have

$$|G(w)|\leqq\left|\frac{f(w)}{g(w)}\right|\left|\int_{-\infty}^{\infty}\frac{g(t)\overline{h_2(t)}}{t-w}\sigma(dt)\right|+\left|\int_{-\infty}^{\infty}\frac{f(t)\overline{h_2(t)}}{t-w}\sigma(dt)\right|.$$

Hence from (1.7) and the dominated convergence theorem it follows that

$$\lim_{w\to\infty}|G(w)|=0$$

for any f of H_1. Since G is of minimal exponential type, G must vanish identically. Hence we have

$$\int_{-\infty}^{\infty}\frac{w}{w-t}f(t)\overline{h_2(t)}\sigma(dt)=\frac{f(w)}{g(w)}\int_{-\infty}^{\infty}\frac{w}{w-t}\,g(t)\overline{h_2(t)}\sigma(dt).$$

So, by (1.7) it is evident that

$$\int_{-\infty}^{\infty}f(t)\overline{h_2(t)}\sigma(dt)=0$$

holds for every $h_2 \in H_{\frac{1}{2}}$ and $f \in H_1$. This implies that f coincides with some element of H_2 almost everywhere with respect to σ. Recalling we have assumed that σ has the continuous part, we may conclude that f itself belongs to H_2. This completes the proof.

In conclusion, we remark a corollary of our theorem. We define a reflection operator R by

$$Rf(z)=f(-z).$$

Corollary. *Let H_1 and H_2 be $K-B$ spaces contained isometrically in a space $L^2(\sigma)$. Assume that every element of H_1 and H_2 satisfies a growth condition*

$$\log|f(z)|=o(|z|^2) \tag{1.8}$$

as $|z|\to\infty$. If both H_1 and H_2 have nontrivial domains of R, then either $H_1 \subset H_2$ or $H_2 \subset H_1$.

Proof. Let $\tilde{H}_k$ $(k=1, 2)$ be a pre-Hilbert space which is equal as set to the domain of R in H_k and whose inner product is defined by

$$(f, f)_{\tilde{H}_k}=(f, f)_{H_k}+(Rf, Rf)_{H_k}.$$

Since R is a closed operator, $\tilde{H}_k$ becomes a Hilbert space. It is obvious that each $\tilde{H}_k$ satisfies the axioms $(H.1)\sim(H.4)$. Hence $\tilde{H}_k$ turns to a $K-B$ space contained isometrically in $L^2(\tau)$, where $\tau(dt)=\sigma(dt)+\sigma(-dt)$. In each space $\tilde{H}_k$, R works as a unitary operator. In this case, de Branges [2] proved that there exists a unique $K-B$ space $(\tilde{H}_k)_+$ such that $f(z)\to f(z^2)$ is an isometric transformation from $(\tilde{H}_k)_+$ onto the even elements of $\tilde{H}_k$. (see Problem 182, p. 168.) We may assume that there exists a nontrivial even element of $\tilde{H}_k$ for the following reason. Take any nontrivial element f of $\tilde{H}_k$. If f is odd, then $f(0)=0$. So we may take $f(z)/z$ of $\tilde{H}_k$ in place of f, which is an even function. Otherwise, we have only to put $g=f+Rf$. Since $(\tilde{H}_k)_+$ is contained isometrically in $L^2(\nu)$, where $\nu(dt)=\sigma(\sqrt{|dt|})$, and the elements are of minimal exponential type by the assumption (1.8), we may conclude from our theorem that either $(\tilde{H}_1)_+ \subset (\tilde{H}_2)_+$ or

$(\tilde{H}_2)_+ \subset (\tilde{H}_1)_+$. Therefore there must exist a nontrivial function belonging to both H_1 and H_2. Since, for any f of H_k, $\log^+|f/E_k|$ is dominated by a harmonic function on $\boldsymbol{C}_+$ (see de Branges [2], p. 50), so is $\log^+|E_1/E_2|$. We have used here the fact that, for a holomorphic function f on $\boldsymbol{C}_+$, $\log^+|f|$ has a harmonic majorant on $\boldsymbol{C}_+$ if and only if it is the quotient of two bounded holomorphic functions on $\boldsymbol{C}_+$ (see P. L. Duren [6]). Now the ordering theorem of de Branges gives our corollary.

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