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The distribution of cercles de remplissage for functions having spiral asymptotic values

By

P. M. GAUTHIER and J. S. HWANG*

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1. Introduction

Let f be a function defined in the unit disc $D=\{z: |z|<1\}$. For $z, z' \in D$, we denote by $\rho(z, z')$ the non-Euclidean distance: $\rho(z, z') = 2^{-1} \log[(1+a)/(1-a)]$, where $a=|(z'-z)/(1-\bar{z}z')|$. We call $\rho(z, z')$ the ρ-distance between z and z'. A sequence $\Delta(n)$ of discs in D is a sequence of cercles de remplissage for f provided that the ρ-diameters of $\Delta(n)$ tend to zero, and the images $f(\Delta(n))$ cover all of the Riemann sphere, with the possible exception of two sets $E(n)$ and $G(n)$ whose spherical diameters tend to zero as $n\to\infty$. The sequence $\{z_n\}$ of centres of the $\{\Delta(n)\}$ is called a sequence of ρ-points for f.

Our first theorem states that if f approaches some value quickly on a spiral, then f has many sequences of ρ-points.

Theorem 1. *Let f be a function meromorphic in D. Suppose that for some spiral σ, f satisfies*

(1) $|f(z)-w| \leq \exp(-1/(1-|z|))^{\alpha+\varepsilon}$, *for* $z \in \sigma$, *where* $\alpha \geq 1$, $\varepsilon > 0$, *and* w *is a complex number. Then either* $f \equiv w$ *or in each Stolz angle of opening* π/α, f *has a sequence of ρ-points.*

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The same conclusion holds if we relax the condition that f approach w rapidly along σ and replace it by a geometric condition on σ and the condition that f omit some value different from w. To state this result precisely, we call a spiral a bounded hyperbolic spiral, if for any point $e^{i\theta}$ on $C=\{z: |z|=1\}$ and any segment L in D with one vertex at $e^{i\theta}$, the sequence of points $\{z_n\}$, where $\bigcup_{n=1}^{\infty}\{z_n\}=\sigma\cap L$, $|z_n|\nearrow 1$, satisfies $\limsup_{n\to\infty}\rho_n<\infty$, where $\rho_n=\rho(z_n, z_{n+1})$.

Theorem 2. *Let f be meromorphic in D and omit some value v. Suppose that for some bounded hyperbolic spiral σ, f tends to a value w different from v along σ. Then either $f\equiv w$, or in each Stolz angle, f has a sequence of ρ-points.*

Our theorem 1 was conjectured in [7] and proved for the case that w is an omitted value. Gavrilov [9] has obtained a version of theorem 2, but only for Stolz angles containing radii.

2. The necessary lemmas. We first need a Lemma which describes the relation between normal functions and sequences of ρ-points.

Lemma 1. [6, *Theorem* 3]. *A meromorphic function is normal in D if and only if it possesses no sequence of ρ-points.*

Gavrilov [9] has introduced a different way to characterize the behavior of sequence of ρ-points. A sequence of points $\{z_n\}$ in D is called a sequence of P-points for a meromorphic function $f(z)$, if for each $c>0$ and each subsequence $\{z_{n_k}\}$, the function $f(z)$ assumes every value, with at most two exceptions, infinitely often in

$$\bigcup_{k=1}^{\infty} D_k, \quad \text{where} \quad D_k=\{z: \rho(z_{n_k}, z)<c\}, \qquad k=1, 2,\dots.$$

Lemma 2. [6, *Theorem* 1]. *Let $f(z)$ be a meromorphic function in D. Then a sequence of points $\{z_n\}$ in D is a sequence of P-points for $f(z)$ if and only if there is a sequence of points $\{w_n\}$ in D and a* *positive number c such that*

$$\rho(z_n, w_n) \longrightarrow 0, \quad as \quad n \longrightarrow \infty \quad and \quad \chi(f(z_n), f(w_n)) > c,$$
$$n = 1, 2, \dots,$$

where $\chi(a, a')$ *is the chordal distance between* a *and* a'.

Lemma 3. [6, *Theorem* 4]. *Let* $f(z)$ *be a meromorphic function in* D. *Then a sequence of points in* D *is a sequence of* ρ-*points for* $f(z)$ *if and only if it is a sequence of* P-*points for* $f(z)$.

We also need a theorem of Pick [10, Theorem 15.1.3].

Lemma 4. *Let* $h(z)$ *be a function holomorphic and bounded by* 1 *in* D. *Then* $\rho(h(z), h(z')) \leq \rho(z, z')$, $z, z' \in D$.

With the help of the above four lemmas, we are able to prove our main lemma which will be necessary in the proof of Theorems 1 and 2.

Lemma 5. *Let* $z = g(\xi)$ *be a conformal mapping from the unit disk* $D_\xi = \{\xi : |\xi| < 1\}$ *onto a Stolz angle* Δ *in* D. *If* $f(z)$ *has no sequence of* ρ-*points in* Δ, *then the function* $F(\xi) = f(g(\xi))$ *is normal in* D_ξ.

Proof. According to Lemma 1, it is sufficient to prove that $F(\xi)$ has no sequence of ρ-points in D_ξ. Suppose on the contrary, $\{\xi_n\}$ is a sequence of ρ-points in D_ξ. Then by Lemma 3, the same sequence $\{\xi_n\}$ is also a sequence of P-points for $F(\xi)$. It follows from Lemma 2, there is a sequence of points $\{w_n\}$ in D_ξ and a positive number c such that

$$\rho(\xi_n, w_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \quad \text{and} \quad \chi(F(\xi_n), F(w_n)) > c,$$
$$\text{for all} \quad n.$$

Let $a_n = g(\xi_n)$ and $b_n = g(w_n)$, then by virtue of Lemma 4, we have

$$\rho(a_n, b_n) \le \rho(\xi_n, w_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \quad \text{and} \quad \chi(f(a_n), f(b_n)) > c,$$

for all n.

Again, by Lemmas 2 and 3, we can see that the sequence $\{a_n\}$ is a sequence of ρ-points for $f(z)$ in Δ which contradicts the hypothesis. This completes the proof of Lemma 5.

The proof of theorem 1 depends upon the following uniqueness theorem for normal meromorphic functions f determined by a special sequence of Jordan arcs and the well-known Nevanlinna characteristic function, $T(r, f)$.

As introduced by Rung [12], a sequence of Jordan arcs $\{\gamma_n\}$ in D is called a PHD-sequence (positive hyperbolic diameter), if it satisfies

$$\liminf_{n \to \infty} HD(\gamma_n) > 0,$$

where $HD(\gamma) = \sup\{\rho(a, b)\}$, $a, b \in \gamma$.

Lemma 6. [12, *Theorem* 1 *and Corollary* 1]. *Let* $f(z)$ *be a function normal meromorphic in* D, *and suppose that for some PHD-sequence* $\{\gamma_n\}, f(z)$ *satisfies*

$$|f(z) - w| \le \exp(-k/(1-|z|)^{1+\varepsilon}), \quad \textit{for} \quad z \in \gamma_n, \qquad n = 1, 2, \dots,$$

where $\varepsilon > 0$, $k > 0$, w *is a complex number. Then* $f(z) \equiv w$.

Lemma 7. [3, p. 14]. *Let* L *be a segment in* D *with one end point at* $z=1$ *and let* $I = [0, 1)$. *Then for any two points* $a \in L$ *and* $b \in I$, *we have* $\rho(a, b) \ge \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, *where* θ *is the subtended angle between* L *and* I.

The following two Lemmas will be needed in the proof of Theorem 2.

Lemma 8. [3, *Theorem* 1]. *Let* $f(z)$ *be normal meromorphic and* *omit one value w in D. Suppose that $\{z_n\}$ is a sequence of points in D which converges to a point $e^{i\theta}\in C$. If there exists a positive number k such that for every n, $\rho_n\leq k$, and if $\lim_{n\to\infty} f(z_n)=w$, then $f(z)$ has the angular limit w at $e^{i\theta}$.*

Lemma 9. [2, p. 431 *and* 12, *Lemma* 1]. *Let $f(z)$ be holomorphic in D and tend to zero along a spiral σ. If $f(z)\not\equiv 0$, then there is a spiral σ' such that f tends to infinity along σ'.*

Our proof of this lemma is different from both of [2] and [13], but similar to that of [4, Theorem 2]. For completeness we include the proof.

Proof. If $f(z)$ were bounded, then by the well-known Koebe's Lemma [5, p. 42], we would have $f(z)\equiv 0$. Hence $f(z)$ must be unbounded in D. Choose a sequence $\{k_n\}$ of numbers such that $k_n\to\infty$, as $n\to\infty$. Clearly the set of all points $z\in D$ at which $|f(z)|>k_1$ is open and not empty. Let D_1 be some component of this set, then at all boundary points of D_1 that lie in D, we have $|f(z)|=k_1$. If the closure $\overline{D_1}\subset D$, then by the maximum principle, we would have $|f(z)|<k_1$, for each $z\in D_1$ which is absurd. It follows that the boundary of D_1 contains a point of C, and therefore contains all points of C due to the fact that $f(z)$ tends to zero along the spiral σ.

Now, we are going to prove that $f(z)$ is unbounded in D_1. Suppose on the contrary that $f(z)$ is bounded in D_1. Let $z=g(\xi)$ map D_ξ conformally onto the universal covering surface D_1^* of D_1. Then we have $|g(\xi)|<1$ in D_ξ. Consider the function $F(\xi)=f(g(\xi))$ which is bounded holomorphic in D_ξ. We want to prove that $|F(\xi)|\leq k_1$ in D_ξ. Let $e^{i\theta}$ be a point for which $g(\xi)$ has the radial limit $g(e^{i\theta})$ at $e^{i\theta}$. Since every point on C is not an accessible boundary point of D_1, we must have $|g(e^{i\theta})|<1$. It follows that the radial limit $|F(e^{i\theta})| =|f(g(e^{i\theta}))|=k_1$. This shows that almost everywhere on $0\leq\theta<2\pi$, $|F(e^{i\theta})|=k_1$. By virtue of the Poisson representation [5, Theorem 5.3], we would have $|F(\xi)|<k_1$ in D_ξ, so that $|f(z)|<k_1$, in D_1. This contradiction establishes that $f(z)$ is unbounded in D_1.

Similarly, let D_2 be a component of the set of all points $z \in D_1$ and $|f(z)| > k_2$. Then again, $f(z)$ is unbounded in D_2. Proceeding in this manner, we obtain a sequence of nested regions $D_1 \supset D_2 \supset \cdots$ such that for each $n = 1, 2, \ldots,$

$$|f(z)| > k_n, \qquad z \in D_n.$$

Now, we take $z_1 \in D_1$, $z_2 \in D_2 - \{z_1\}, \ldots, z_n \in D_n - \{z_1, \ldots, z_{n-1}\}, \ldots,$ and join z_1 to z_2 by means of a Jordan arc $J_1 \subset D_1, \ldots,$ join z_n to z_{n+1} by means of a Jordan arc $J_n \subset D_n$. We thus find a path

$$\sigma' = \bigcup_{n=1}^{\infty} J_n$$

in D such that $f(z)$ tends to infinity along σ'. Clearly σ' is also a spiral. This completes the proof of Lemma 9.

3. Proof of Theorem 1. Suppose that Δ is a Stolz angle of opening π/α with one vertex on C, say at $z=1$. Let $z = g(\xi)$ map the unit disk D_ξ conformally onto Δ so that $g(1)=1$. Then $g(\xi)$ can be written as

$$z = g(\xi) = 1 + (\xi - 1)^{1/\alpha} h(\xi), \tag{2}$$

where $h(\xi)$ is holomorphic at the point $\xi = 1$ and $h(1) \neq 0$.

Now, let $F(\xi) = f(g(\xi))$, $\xi \in D_\xi$. If $f(z)$ has no sequence of ρ-points in Δ, then by virtue of Lemma 5, $F(\xi)$ is normal in D_ξ. In this case, it is sufficient to prove that $f(z) \equiv w$.

Clearly, from equation (2), there are two positive numbers k and k', and a Stolz angle Δ_ξ with one vertex at the point $\xi = 1$ such that

$$k(1-|\xi|) \leq (1-|z|)^\alpha \leq k'(1-|\xi|), \qquad \text{for} \quad \xi \in \Delta_\xi. \tag{3}$$

The intersection $\sigma \cap \Delta$ is a sequence of Jordan arcs which we denote by $\{J_n\}$. Let $\gamma_n = g^{-1}(J_n)$, then by the given condition (1) and the inequality (3), we have

$$|F(\xi) - w| \leq \exp\{-1/[k'(1-|\xi|)]^{1+\varepsilon/\alpha}\}, \qquad \text{for} \quad \xi \in \gamma_n.$$

According to Lemma 6, $F(\xi)$ will be identically equal to w provided $\{\gamma_n\}$ is a *PHD*-sequence. To see this, by applying Lemma 7, we can see

$$\lim_{n\to\infty} HD(J_n) \geq \frac{1}{2} \log\tan\left(\frac{\pi+2\pi/\alpha}{4}\right) > 0,$$

and therefore by Lemma 4, we find that

$$\liminf_{n\to\infty} HD(\gamma_n) \geq \liminf_{n\to\infty} HD(J_n) > 0.$$

This concludes that $F(\xi)\equiv w$ and so is $f(z)\equiv w$.

4. Proof of Theorem 2. The first part of this proof will be the same as that of Theorem 1, except for the opening angle of a Stolz angle Δ. Instead of the fixed opening π/α in Theorem 1, we now consider an arbitrarily small one π/m, provided m is sufficiently large. By use of the same argument, we also have the following two equations:

$$z = g(\xi) = 1 + (\xi - 1)^{1/m} h(\xi), \tag{2}$$

where again $h(\xi)$ is holomorphic at $\xi=1$ and $h(1)\neq 0$, and

$$K(1-|\xi|) \leq (1-|z|)^m \leq K'(1-|\xi|), \qquad \text{for} \quad \xi \in \Delta_\xi, \tag{3}$$

where again $0<K<K'$ and Δ_ξ is a Stolz angle at $\xi=1$.

Without loss of generality, we may assume that L is a segment in Δ with one endpoint at $z=1$, for which the preimage $g^{-1}(L)$ lies on the real segment I in D_ξ. By hypothesis, we know that the sequence $\{z_n\}$, where $\bigcup_{n=1}^{\infty} \{z_n\} = \sigma \cap L$, satisfies

$$\rho_n \leq K'' < \infty, \quad \text{where} \quad \rho_n = \rho(z_n, z_{n+1}). \tag{4}$$

Let $x_n = g^{-1}(z_n)$, then $x_n \in I$ and $x_n \to 1$, as $n\to\infty$. By virtue of (3), we can see that $x_n = 1-(1-|z_n|)^m/K_n$, where $K \leq K_n \leq K'$. It follows from (4) that we have the following estimate.

$$(5)\qquad \rho'_n=\rho(x_n,\, x_{n+1})=\frac{1}{2}\log\frac{(1-x_nx_{n+1})+(x_{n+1}-x_n)}{(1-x_nx_{n+1})-(x_{n+1}-x_n)}$$

$$=\frac{1}{2}\log\frac{(1-|z_n|)^m[K_{n+1}-(1-|z_{n+1}|)^m]}{(1-|z_{n+1}|)^m[K_n-(1-|z_n|)^m]}$$

$$\leq\frac{m}{2}\log\frac{1-|z_n|}{1-|z_{n+1}|}+\frac{1}{2}\log\frac{K'}{K}$$

$$\leq\frac{m}{2}\log\frac{(1-|z_n|)(1+|z_{n+1}|)}{(1+|z_n|)(1-|z_{n+1}|)}+K^*,\ \text{where } K^*=\frac{1}{2}\log\frac{K'}{K},$$

$$=m\,\rho(|z_n|,\,|z_{n+1}|)+K^*\leq m\rho_n+K^*\leq mK''+K^*<\infty\,.$$

From the conditions, we know that $F(\xi)$ omits the value w and $\lim F(x_n)=w$. Moreover, from Lemma 5, we also know that $F(\xi)$ is normal in D_ξ. By virtue of (5) and Lemma 8, we find that $F(\xi)$ has the angular limit w at the point $\xi=1$. This in turns implies that there is a subangle which again is denoted by Δ such that $f(z)$ tends to the limit w as $z\to 1$ uniformly in Δ.

Now, let us consider the function $f^*(z)=(f(z)-w)/(w-v)(f(z)-v)$ which is holomorphic in D and tends to zero along σ. If $f(z)\not\equiv w$, then by the last Lemma 9, there would be another spiral σ' such that $f^*(z)$ tends to infinity along σ'. However, this is impossible since we have just proved that $f(z)$ tends to the limit w as $z\to 1$ uniformly in Δ. This establishes that $f(z)$ has a sequence of ρ-points in Δ.

5. Remarks. The first remark we want to make is a variation of theorem 2. To do this, we say that a spiral σ is monotonic if it is monotonic both in argument and modulus. A function f holomorphic in D is said to belong to the Valiron class $V(\sigma)$ if f is unbounded in D but bounded on σ. The following lemma is well known and can be proved in the same way as lemma 9.

Lemma 9′. *Let $f\in V(\sigma)$ for some spiral σ. Then there is a spiral σ' such that f tends to infinity along σ'.*

If σ is monotone bounded hyperbolic, it can be shown [8] that

σ' is also bounded hyperbolic. We thus have:

Theorem 2'. *Let $f \in V(\sigma)$ for some monotone bounded hyperbolic spiral σ. Then in each Stolz angle, f has a sequence of ρ-points.*

It is easy to construct bounded hyperbolic spirals; for if σ is monotone it can be shown [8] that one need only check that σ is bounded hyperbolic along radii.

In theorem 2 the condition that f omit some value is essential. To see this, we need only apply a theorem of Roth [11] which enables us, for any spiral σ, to construct a meromorphic function f having an asymptotic value on σ and approaching the same value in some Stolz angle. Then f has no sequence of ρ-points in this angle.

Finally, we are going to construct a function which omits one value w and tends to this value w along some bounded hyperbolic spiral. It is sufficient to deal with the case that $w=\infty$.

Example. There exists a function $w=f(z)$ holomorphic in D which tends to the value ∞ on a bounded hyperbolic spiral σ. Moreover there is a Stolz angle in which $f(z)$ tends to the value ∞.

Consider the function $\prod_{j=1}^{\infty}\{1-(z/(1-(1/n_j)))^{n_j}\}$, $n_j/n_{j-1}\to\infty$, $n_1>1$, studied by Bagemihl, Erdös, and Seidel [1]. The zeros of $f(z)$ occur on each circle $|z|=1-1/n_j$. About each zero z_{jk} $(k=1, 2,\dots, n_j)$ of $f(z)$ on the circle $|z|=1-1/n_j$ $(j=1, 2,\dots)$, we consider the discs Γ_{jk}, $(k=1, 2,\dots, n_j)$ of Euclidean radius $r_j=1/(j^2 n_j)$ and Euclidean center z_{jk}. Bagemihl, Erdös, and Seidel showed that if we remove these discs, then in the remaining subset of D, $f(z)$ tends uniformly to ∞ as z approaches the circumference $|z|=1$.

Let D_{jk} be the disk of non-Euclidean center z_{jk} and non-Euclidean radius

$$R_j=\rho\left(z_{jk}, \frac{(|z_{jk}|+r_j)z_{jk}}{|z_{jk}|}\right)=\rho\left(z_{jk}, (1-1/n_j+r_j)\frac{z_{jk}}{1-1/n_j}\right)=$$

$\rho(1-1/n_j, 1-1/n_j+r_j)$, since the non-Euclidean metric is invariant under rotations about the origin. Hence $R_j=\rho(1-1/n_j, 1-1/n_j+r_j)$, $r_j\to 0$, and so by direct computation we have

$$\lim R_j = 0, \quad \text{as} \quad j \longrightarrow \infty. \tag{6}$$

Our choice of R_j assures us that $D_{jk} \supset \Gamma_{jk}$, and so $f(z)$ tends to ∞ uniformly as z tends to the circumference C from within the disc D with all the D_{jk} removed.

Since $n_j/n_{j-1} \to +\infty$, again by direct computation we find that

$$\lim \rho(1 - 1/n_{n-1}, 1 - 1/n_j) = \infty, \quad \text{as} \quad j \longrightarrow \infty. \tag{7}$$

Let z_{jk}, $z_{j(k+1)}$ be two adjacent zeros on $|z| = 1 - 1/n_j$. We wish to show that

$$\liminf A_j > 0, \quad \text{as} \quad j \longrightarrow \infty, \tag{8}$$

where $A_j = \rho(z_{jk}, z_{j(k+1)})$. Since A_j does not depend on k, it is sufficient to consider $A_j = \rho(z_{j0}, z_{j1}) = \rho(1 - 1/n_j, (1 - 1/n_j)\exp(i2\pi/j))$.

$$\rho(z_{j0}, z_{j1}) \geq \rho(z_{j0}, a_j) - \rho(a_j, z_{j1}), \tag{9}$$

where $\mathrm{Re}(a_j) = z_{j0}$ and $\arg a_j = 2\pi/n_j$. Let

$$g(x) = \frac{x \sin(2(1-x))}{\cos(2(1-x))}.$$

Then $a_j = z_{j0} + ig(z_{j0})$. $g'(1) = -2\pi$. Since $g'(1) \neq 0$, this means that $g(x)$ is not tangent to the real axis at $x = 1$, and so

$$\liminf \rho(x, g(x)) > 0, \quad \text{as} \quad x \longrightarrow 1.$$

Hence

$$\liminf \rho(z_{j0}, a_j) > 0, \quad \text{as} \quad j \longrightarrow \infty. \tag{10}$$

Since the non-Euclidean metric is invariant under rotations about the origin, it follows that

$$\rho(z_{j1}, a_j) = \rho(z_{j0}, |a_j|) = \rho(z_{j0}, (z_{j0}^2 + g(z_{j0})^2)^{1/2}).$$

Hence

(11) $$\lim_{j\to\infty} \rho(z_{ji}, a_j) = \lim_{x\to 1} \rho(x, (x^2+g(x)^2)^{1/2}),$$

provided the latter exists. It is easily shown that

(12) $$\lim \frac{1-(x^2+g(x)^2)^{1/2}}{1-x} = 1, \text{ as } x \longrightarrow 1 .$$

It follows from (11) and (12) that

(13) $$\lim \rho(z_{j1}, a_j) = 0, \quad \text{as} \quad j \longrightarrow \infty.$$

From (10), (13), and (9) we have $\liminf \rho(z_{j0}, z_{j1}) > 0$, as $j \to \infty$. Hence (8) holds.

From (6), (7), and (8), it follows that a bounded hyperbolic spiral can be constructed along which $f(z)$ tends to ∞.

Let $a = \liminf \rho(z_{j0}, z_{j1})$, as $j \to \infty$ and let $H(S, r) = \{z \in D, \rho(S, z) \leq r\}$, $S \subset D$, $r \geq 0$. It follows from (8) that $a > 0$. Let R be the radius $\arg z = 0$, and let α_0 be the upper boundary of $H(R, a/2)$. It follows from (8) and (6) that $H(\alpha_0, a/3)$ meets none of the disks Γ_{jk} for all sufficiently large j. Hence $f(z)$ tends to ∞ in $H(\alpha_0, a/3)$. Since α_0 is a circular arc which is non-tangential at $z=1$, there is a Stolz angle Δ for which $\Delta \subset H(\alpha_0, a/3)$. Then $f(z)$ tends to ∞ in Δ, and the claims of the Example have been verified.

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE MONTRÉAL

DEPARTMENT OF MATHEMATICS
MCMASTER UNIVERSITY

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