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Finsler manifolds modeled on a Minkowski space

By

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As is well known, the tangent space at any point of a Riemannian manifold is a Euclidean space. On the other hand, in Finsler manifolds and modern Banach manifolds, the tangent space at any point is a Minkowski space, but the tangent spaces at two distinct points are, in general, not the same Minkowski space. Hence it seems significant to us to study the manifolds with the property such that the tangent spaces at arbitrary points of them are congruent (isometrically linearly isomorphic) to a single Minkowski space. We will call a Finsler manifold with this property as a Finsler manifold modeled on a Minkowski space.

The main purpose of the present paper is to develop the theory of Finsler manifolds modeled on a Minkowski space and to give some examples of them.

First, we shall introduce a Minkowski norm and linear Lie groups leaving the Minkowski norm invariant. In the section 2, we shall define the notion of $\{V, H\}$-manifolds, where V is a Minkowski space and H is a linear Lie group leaving the Minkowski norm invariant. We shall show that the $\{V, H\}$-manifold offers an example of a Finsler manifold modeled on a Minkowski space. In the section 3, it will be proved that a $\{V, H\}$-manifold is a generalized Berwald space defined by Hashiguchi [7], and also a generalized Berwald space is a Finsler manifold modeled on a Minkowski space. We shall consider, in the section 4, a condition for a Finsler manifold to be a Finsler manifold modeled on a Minkowski space. The last two sections will be devoted to give some examples of the $\{V, H\}$-manifold, which are found among completely parallelizable manifolds and so-called Randers spaces.

§ 1. Minkowski spaces

Let V be an n-dimensional Minkowski space, that is to say, an n-dimensional linear space on which a Minkowski norm is defined. Here a Minkowski norm on the linear space V is a real valued function on V, whose value at a point ξ we denote by $\|\xi\|$, with properties:

(1) $$\|\xi\| \geqq 0 .$$

(2) $$\|\xi\| = 0 \quad \text{if and only if} \quad \xi = 0 .$$

(3) $$\|\xi_1 + \xi_2\| \leqq \|\xi_1\| + \|\xi_2\| .$$

(4) $$\|k\xi\| = k\|\xi\| \qquad \text{for} \quad k > 0 .$$

Moreover we assume in this paper that

(5) The Minkowski norm $\|\xi\|$ can be represented explicitly by

$$\|\xi\| = f(\xi^1, \xi^2, \dots, \xi^n)$$

for any vector $\xi = \xi^1 e_1 + \xi^2 e_2 + \cdots + \xi^n e_n \ (= \xi^\alpha e_\alpha)$ where $\{e_\alpha\}$ is a given basis of V, and the function $f(\xi^1, \xi^2, \dots, \xi^n)$ is 3-times continuously differentiable at $\xi \neq 0$. For brevity we write $f(\xi^1, \xi^2, \dots, \xi^n)$ as $f(\xi^\alpha)$ or $f(\xi)$ hereafter.

Lemma. *In a Minkowski space V, let us define a set G by*

$$G = \{T \mid T \in GL(n, R), \ \|T\xi\| = \|\xi\| \qquad \text{for any} \quad \xi \in V\}$$

where $GL(n, R)$ is the real general linear Lie group of degree n. Then the set G is a Lie group.

Proof. If $T_1, T_2 \in G$, then $\|T_1 T_2 \xi\| = \|T_2 \xi\| = \|\xi\|$, that is, $T_1 T_2 \in G$. And if $T \in G$, then there exists $T^{-1} \in GL(n, R)$ and $\|\xi\| = \|TT^{-1}\xi\|$

$=\|T^{-1}\xi\|$ holds good, that is $T^{-1}\in G$. Therefore G is a subgroup of $GL(n, R)$. Next, for $T_1, T_2,\dots, T_k,\dots\in G$ where $\lim_{k\to\infty} T_k=T\in GL(n, R)$, the relation $\lim_{k\to\infty}\|T_k\xi\|=\|\xi\|$ holds good. On the other hand, the condition (5) tells us that $\lim_{k\to\infty}\|T_k\xi\|=\|T\xi\|$. Thus $T\in G$. Then G is a closed set in $GL(n, R)$. Hence G is a Lie group [11].

§ 2. $\{V, H\}$-manifolds

Let V be a Minkowski space and G be a Lie group given in the preceding section, and H be a Lie subgroup of G. Suppose that an n-dimensional C^∞-manifold M admits the H-structure in the sense of a G-structure ([6], [15]). Then we can introduce a Finsler metric on M by regarding the tangent space at each point of M as the Minkowski space V such as the following:

Let $\{U_A\}$ be a coordinate neighborhood system, $\{X_1^{(A)}, X_2^{(A)},\dots, X_n^{(A)}\}$ be an n-frame of U_A adapted to the H-structure, and v be any vector in the tangent space $T_p(M)$ at a point p of M. Then we can express v as $v=\xi^\alpha X_\alpha^{(A)}$, and define the length $\|v\|$ of v by the equation $\|v\|_A=f(\xi^\alpha)$. This definition does not depend on the choice of the local coordinate system and the adapted frame. Because, for $p\in U_A\cap U_B$ and for an arbitrary n-frame $\{X_\alpha^{(B)}\}$ of U_B, we have $X_\alpha^{(A)}=T_\alpha^\beta X_\beta^{(B)}$ where the matrix $(T_\beta^\alpha)\in H$. If we put $v=\eta^\beta X_\beta^{(B)}$ in U_B, we have $\eta^\beta=T_\alpha^\beta\xi^\alpha$. Now, owing to the property of H, we have

$$\|v\|_B=f(\eta^\alpha)=f(T_\beta^\alpha\xi^\beta)=f(\xi^\alpha)=\|v\|_A\,.$$

So $\|v\|$ is well-defined over M. This definition tells us that the tangent space at each point of M is the Minkowski space V.

On the other hand, we set

$$X_\alpha^{(A)}=\lambda_{\alpha(A)}^i\frac{\partial}{\partial x_{(A)}^i}\,, \tag{2.1}$$

then $\lambda_{\alpha(A)}^i$ are n linearly independent contravariant vectors in U_A. Hereafter we abbreviate them as $X_\alpha=\lambda_\alpha^i\partial/\partial x^i$. Of course, in U_A, we can express v as

$$v=v^i\partial/\partial x^i=\xi^\alpha X_\alpha=\lambda_\alpha^i\xi^\alpha\partial/\partial x^i\,,$$

then we have $\xi^{\alpha}=\mu^{\alpha}_{j}v^{j}$, where we put

$$(2.2)\qquad (\lambda^{j}_{\alpha})^{-1}=(\mu^{\alpha}_{j}).$$

Thus we have

$$(2.3)\qquad \|v\|=f(\mu^{\alpha}_{j}v^{j}).$$

Consequently, it is observed that the function

$$(2.4)\qquad F(x,\ y)=f(\mu^{\alpha}_{j}(x)y^{j})$$

just gives a Finsler metric on M ([9], [10], [13]). We sum up these facts in the following theorem:

Theorem 1. *Let V be an n-dimensional Minkowski space and H be a linear Lie group leaving the Minkowski norm invariant. If an n-dimensional C^{∞}-manifold M admits the H-structure, then the Minkowski norm can be induced in the tangent space at each point of M, and it gives M a Finsler metric with the form* (2.4).

A Finsler metric defined in Theorem 1 is called a $\{V, H\}$-*Finsler metric* and a manifold admitting a $\{V, H\}$-Finsler metric is called a $\{V, H\}$-*manifold* hereafter. If a Minkowski space V is given, the Lie group G is uniquely determined. But, since the Lie subgroup H of G can be arbitrarily chosen, various kinds of H-structure may be considered. These $\{V, H\}$-manifolds, however, possess commonly the property that the tangent space at each point is congruent to the Minkowski space V.

Now, let M be a Finsler manifold with a metric function $F(x, y)$. The tangent space $T_p(M)$ at each point $p=(x_0)$ of M can be regarded as a Minkowski space, where the norm $\|v\|$ of any vector $v=v^i\partial/\partial x^i \in T_p(M)$ is given by $\|v\|=F(x_0, v)$. Therefore, $T_p(M)$ can be called the tangent Minkowski space at p. For arbitrary distinct two points p and q of M, $T_p(M)$ is not congruent to $T_q(M)$ in general, nevertheless, if $T_p(M)$ and $T_q(M)$ are always congruent mutually, then M is called *a Finsler manifold modeled on a Minkowski space*. A Finsler space is not necessarily a Finsler manifold modeled on a Minkowski space, but a $\{V, H\}$-manifold is a Finsler manifold modeled on a Minkowski space, the model Minkowski space of which is V itself.

Remark. Let E be an n-dimensional Euclidean space, and $O(n)$ be the orthogonal group of degree n, then an $\{E, O(n)\}$-manifold is nothing but a Riemannian manifold. Conversely, if a $\{V, H\}$-manifold is a Riemannian manifold, then V is necessarily a Euclidean space and H is a Lie subgroup of $O(n)$.

§ 3. Generalized Berwald spaces

Recently, M. Hashiguchi [7] defined the notion of a generalized Berwald space and investigated it in detail. Following his work, a Finsler space is said to be *a generalized Berwald space* if it is possible to introduce a metrical Finsler connection in such a way that the connection coefficients Γ^i_{jk} depend on position only. Now, we turn to the relation between the Finsler manifold modeled on a Minkowski space and the generalized Berwald space. In the following, we use the notation ∂_i and $\dot{\partial}_i$ instead of $\partial/\partial x^i$ and $\partial/\partial y^i$ respectively.

Theorem 2. *In a $\{V, H\}$-manifold, let $\Gamma^i_{jk}(x)$ be a G-connection relative to the H-structure, then the metric tensor of the $\{V, H\}$-Finsler metric is covariant constant with respect to the Finsler connection $\{\Gamma^i_{jk}(x), \Gamma^i_{jk}(x)y^j, C^i_{jk}\}$, that is, a $\{V, H\}$-manifold is a generalized Berwald space, where we put $C^i_{jk}=\frac{1}{2}g^{il}\dot{\partial}_l g_{jk}$.*

Proof. Let $\mathfrak{h}$ be the Lie algebra of H. For any $A=(A^\alpha_\beta)\in\mathfrak{h}$, we have $f((\exp tA)\xi)=f(\xi)$ because of $(\exp tA)\in H$. Hence $\left[\frac{d}{dt}f((\exp tA)\xi)\right]_{t=0}=0$ holds, which leads us to

$$\frac{\partial f(\xi)}{\partial \xi^\gamma}A^\gamma_\beta\xi^\beta=0\,. \tag{3.1}$$

On the other hand, from the theory of G-structure [15], it is well-known that there exists a G-connection relative to the H-structure, which satisfies, for any vector $v=v^i\,\partial/\partial x^i=v^\alpha X_\alpha$,

$$\nabla^*_v X_\alpha = v^\beta \Gamma^\gamma_{\beta\alpha} X_\gamma \quad \text{and} \quad v^\beta \Gamma^\gamma_{\beta\alpha} \in \mathfrak{h}$$

with respect to the adapted frame $\{X_\alpha\}$, where ∇^* means the covariant differentiation with respect to the G-connection. Then (3.1) leads us to $\partial f(\xi)/\partial\xi^\gamma v^m \mu^\beta_m \Gamma^\gamma_{\beta\alpha}\xi^\alpha=0$ for any vector v. Hence we have

$$\partial f(\xi)/\partial\xi^\gamma \mu^\beta_i \Gamma^\gamma_{\beta\alpha}\xi^\alpha=0\,. \tag{3.2}$$

Setting $\nabla^*_{\frac{\partial}{\partial x^i}} \dfrac{\partial}{\partial x^j}=\Gamma^k_{ji}\dfrac{\partial}{\partial x^k}$, we have

$$\nabla^*_v X_\alpha = v^j(\partial_j\lambda^i_\alpha + \Gamma^i_{mj}\lambda^m_\alpha)\frac{\partial}{\partial x^i} = v^j \nabla^*_j \lambda^i_\alpha \frac{\partial}{\partial x^i}\,.$$

From the fact $v^\beta \Gamma^\gamma_{\beta\alpha} X_\gamma = v^m \mu^\beta_m \Gamma^\gamma_{\beta\alpha}\lambda^i_\gamma \dfrac{\partial}{\partial x^i}$, we have $v^j\nabla^*_j\lambda^i_\alpha = v^m\mu^\beta_m\Gamma^\gamma_{\beta\alpha}\lambda^i_\gamma$ for any v. Then we have $\nabla^*_m\lambda^i_\alpha=\mu^\beta_m\Gamma^\gamma_{\beta\alpha}\lambda^i_\gamma$, that is,

$$\mu^\beta_m \Gamma^\gamma_{\beta\alpha} = \mu^\gamma_h \nabla^*_m \lambda^h_\alpha\,. \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\partial f(\xi)/\partial\xi^\gamma \mu^\gamma_l \nabla^*_i \lambda^l_\alpha \xi^\alpha = 0\,. \tag{3.4}$$

Now, we denote by ∇_k the covariant differentiation with respect to the connection $\{\Gamma^i_{jk}(x),\ \Gamma^i_{lk}(x)y^l\}$, e.g., for a tensor T^i_j,

$$\nabla_k T^i_j = \partial_k T^i_j - \dot{\partial}_m T^i_j \Gamma^m_{lk} y^l + \Gamma^i_{mk} T^m_j - \Gamma^m_{jk} T^i_m\,. \tag{3.5}$$

Then we have $\nabla_i y^j=0$. If the components of the tensor T^i_j are functions of position alone, $\nabla_k T^i_j$ becomes to equal to $\nabla^*_k T^i_j$. Using the fact that λ^i_α and μ^α_i are functions of position only, we have also $\nabla_i \mu^\alpha_j = -\mu^\alpha_l \nabla^*_i \lambda^l_\gamma \mu^\gamma_j$. Hence, differentiating (2.4) covariantly and using (3.4), we obtain

$$\nabla_i F(x,\ y) = -\partial f(\xi)/\partial\xi^\gamma \mu^\gamma_l \nabla^*_i \lambda^l_\alpha \xi^\alpha = 0\,.$$

Since Γ^i_{jk} depend on position alone, the commutation formula $\nabla_i\dot{\partial}_j = \dot{\partial}_j\nabla_i$ holds. Thus we obtain

(3.6) $$\nabla_i g_{jk} = \dot{\partial}_j \dot{\partial}_k (F \nabla_i F) = 0\,.$$

This shows us that the Finsler space admits a metrical Finsler connection $\{\Gamma^i_{jk}(x), \Gamma^i_{lk}(x)y^l, C^i_{jk}\}$, where $C^i_{jk} = \frac{1}{2} g^{il} \dot{\partial}_l g_{jk}$. Hence the Finsler space is a generalized Berwald space in the sense of Hashiguchi ([7], [8]). Thus the proof is complete.

If a G-connection relative to the H-structure is symmetric, that is, $\Gamma^i_{jk} = \Gamma^i_{kj}$, then owing to Theorem of Matsumoto ([9], [10]), we obtain that the G-connection $\Gamma^i_{jk}(x)$ is the Cartan's connection $\overset{*}{\Gamma}{}^i_{jk}$ itself, and the manifold is a Berwald space. If in addition to this assumption, we assume that the connection $\Gamma^i_{jk}(x)$ is flat in the sense of affinely connected manifold, then as is well known, the manifold is a locally Minkowski space. Hence we obtain

Theorem 3. *Let M be a $\{V, H\}$-manifold. If a G-connection relative to the H-structure is symmetric, then M is a Berwald space with respect to the $\{V, H\}$-Finsler metric. If the G-connection is flat, then M is a locally Minkowski space.*

Moreover we obtain

Theorem 4. *Let M be a connected Finsler manifold and admit a linear connection $\Gamma^i_{jk}(x)$ with respect to which M is a generalized Berwald space. Then M is a Finsler manifold modeled on a Minkowski space.*

Proof. From our assumption and the proof of Theorem 2, we have

$$\nabla_k g_{ij} = \dot{\partial}_i \dot{\partial}_j (F \nabla_k F) = 0\,.$$

Contracting this equation with y^i and y^j, we have

(3.7) $$\nabla_k F = \partial_k F(x, y) - \dot{\partial}_m F(x, y) \Gamma^m_{lk}(x) y^l = 0\,.$$

On the other hand, for arbitrary distinct two points p and q in M, we can take a piecewise differentiable curve C joining p and q. We represent the curve C by $C=\{x(t);\ 0\leqq t\leqq 1\}$ where $x(0)=p$ and $x(1)=q$, and take a vector v in $T_p(M)$. Then we denote by $Y(t)$ the vector field on C which is given by parallel displacement of v with respect to the linear connection $\Gamma^i_{jk}(x)$ along the curve C. Of course $Y(0)=v$. If we put $Y(1)=\tilde{v}$ and define a mapping T: $T_p(M)\to T_q(M)$ by $T(v)=\tilde{v}$, then T is a linear isomorphism. Moreover we see

$$\frac{d}{dt}F(x(t),\ Y(t))=\partial_i F(x,\ Y)\frac{dx^i}{dt}+\dot{\partial}_m F(x,\ Y)\frac{dY^m}{dt}$$

$$=(\partial_i F(x,\ Y)-\dot{\partial}_m F(x,\ Y)\Gamma^m_{li}(x)Y^l)\frac{dx^i}{dt}$$

$$=0\,.$$

Hence the length $\|Y(t)\|$ of $Y(t)$ with respect to the given Finsler metric is constant on the curve C. Thus the mapping T maps a vector v in $T_p(M)$ isometrically to $\tilde{v}$ in $T_q(M)$. That is to say, T is an isometrically linearly isomorphic mapping of $T_p(M)$ onto $T_q(M)$. Hence $T_p(M)$ and $T_q(M)$ are congruent. Since the manifold is connected, M is a Finsler manifold modeled on a Minkowski space.

§ 4. Finsler manifolds

For the Finsler metric $F(x,\ y)=f(\mu^\alpha_j(x)y^j)$ on a $\{V,\ H\}$-manifold, we have

$$g_{jk}=\frac{1}{2}\frac{\partial^2 f^2(\xi)}{\partial\xi^\alpha\partial\xi^\beta}\mu^\alpha_j(x)\mu^\beta_k(x)\,,$$

$$C_{ijk}=\frac{1}{2}\dot{\partial}_k g_{ij}=\frac{1}{4}\frac{\partial^3 f^2(\xi)}{\partial\xi^\alpha\partial\xi^\beta\partial\xi^\gamma}\mu^\alpha_i(x)\mu^\beta_j(x)\mu^\gamma_k(x)\,,$$

where we put $\xi^\alpha=\mu^\alpha_i(x)y^i$. Now we denote the metric tensor and C-tensor in the Minkowski space V by $g^*_{\alpha\beta}$ and $C^*_{\alpha\beta\gamma}$ respectively. Then we have

$$g_{jk}=g^*_{\alpha\beta}(\mu^\alpha_m(x)y^m)\mu^\alpha_j(x)\mu^\beta_k(x)\,, \tag{4.1}$$

$$C_{ijk}=C^*_{\alpha\beta\gamma}(\mu^\alpha_m(x)y^m)\mu^\alpha_i(x)\mu^\beta_j(x)\mu^\gamma_k(x)\,. \tag{4.2}$$

Differentiating (4.1) with respect to x^h, we have

$$\partial_h g_{jk} = \lambda^r_\sigma \partial_h \mu^\sigma_m (2C_{jkr} y^m + g_{rk}\delta^m_j + g_{jr}\delta^m_k). \tag{4.3}$$

On the other hand (3.6) shows us

$$\partial_h g_{jk} = \Gamma^r_{mh}(2C_{jkr} y^m + g_{rk}\delta^m_j + g_{jr}\delta^m_k).$$

Hence, by (4.3), we have

$$(2C_{jkr} y^m + g_{rk}\delta^m_j + g_{jr}\delta^m_k)(\lambda^r_\sigma \partial_h \mu^\sigma_m - \Gamma^r_{mh}) = 0.$$

Now, let us use the notation ∇^*_j defined in the preceding section and the notation $|$ defined by E. Cartan. Since the vectors μ^σ_m depend on position only, we have

$$\lambda^r_\sigma \nabla^*_h \mu^\sigma_m (2C_{jkr} y^m + g_{kr}\delta^m_j + g_{jr}\delta^m_k) = 0. \tag{4.4}$$

Contracting (4.4) with y^j and y^k, we have

$$g_{rl}\lambda^r_\sigma \nabla^*_h \mu^\sigma_m y^l y^m = 0. \tag{4.5}$$

This equation can be rewritten easily in the form

$$g_{rl}\lambda^r_\sigma \mu^\sigma_{m|h} y^l y^m = 0. \tag{4.6}$$

Thus we obtain

Theorem 5. *In a $\{V, H\}$-manifold, the n linearly independent covariant vectors $\mu^\sigma_i(x)$ defined by* (2.1) *and* (2.2) *satisfy the equation $g_{rl}\lambda^r_\sigma \nabla^*_h \mu^\sigma_m y^l y^m = 0$, where ∇^* means the ordinary covariant differentiation with respect to a G-connection relative to the H-structure. Moreover this equation can be rewritten as $g_{rl}\lambda^r_\sigma \mu^\sigma_{m|h} y^l y^m = 0$, where $|$ means the Cartan's covariant differentiation with respect to the $\{V, H\}$-Finsler metric.*

Now let us consider the converse of this theorem. Let M be an n-dimensional connected Finsler manifold. We assume here that M is covered by a coordinate neighborhood system $\{U_A\}$ each of which admits n linearly independent covariant vectors $\mu^{\alpha}_{i}(x)$ satisfying (4.6) where $(\mu^{\alpha}_{h})=(\lambda^{h}_{\alpha})^{-1}$ and $|$ means the Cartan's covariant differentiation with respect to the given Finsler metric. In each U_A, we shall calculate $A_i \equiv \partial_i F(x, y)+\dot{\partial}_r F(x, y)\partial_i \lambda^r_\sigma \mu^\sigma_m y^m$. Since $F(x, y)=(g_{ij}y^i y^j)^{1/2}$, A_i are rewritten as

$$A_i=\frac{1}{2F}(\partial_i g_{lm} y^l y^m+2g_{lr}y^l \partial_i \lambda^r_\sigma \mu^\sigma_m y^m)$$

$$=\frac{1}{2F}(\partial_i g_{lm}-2g_{lr}\lambda^r_\sigma \partial_i \mu^\sigma_m)y^l y^m .$$

Owing the fact $g_{lm|i}=0$ and (4.6), we have

$$2FA_i=(\overset{*}{\Gamma}{}^r_{li}g_{rm}-g_{lr}\overset{*}{\Gamma}{}^r_{mi}-2g_{lr}\lambda^r_\beta \mu^\beta_{m|i})y^l y^m$$

$$=-2g_{lr}\lambda^r_\beta \mu^\beta_{m|i}y^l y^m$$

$$=0 .$$

Thus we obtain

$$\partial_i F(x, y)+\dot{\partial}_r F(x, y)\partial_i \lambda^r_\sigma \mu^\sigma_m y^m=0 . \tag{4.7}$$

Now on putting $\xi^\alpha=\mu^\alpha_m(x)y^m$, we can set

$$F(x, y)=F_A(x^i, \lambda^i_\sigma(x)\xi^\sigma)\equiv f_A(x, \xi) .$$

Then (4.7) means $\partial f_A(x, \xi)/\partial x^i=0$. Thus we have $F(x, y)=f_A(\mu^\alpha_i(x)y^i)$ in each given coordinate neighborhood U. This implies that the coordinate neighborhood U_A of M is a Finsler space modeled on a Minkowski space $V_{(A)}$ whose norm function is given by $f_A(\xi^\alpha)$. Next, take another given coordinate neighborhood U_B such that $U_A \cap U_B \neq \phi$. Then U_B is, similarly, a Finsler space modeled on a Minkowski space $V_{(B)}$ such that the norm function of $V_{(B)}$ is given by $f_B(\tilde{\xi}^\alpha)=F(\tilde{x}, \tilde{y})$ where we put $\tilde{\xi}^\alpha=\tilde{\mu}^\alpha_{i'}(\tilde{x})\tilde{y}^{i'}$. Since the Finsler metric is given globally on M, the tangent Minkowski space $T_p(M)$ at any point p of $U_A \cap U_B$ is congruent to $V_{(A)}$ and, at the same time, to $V_{(B)}$. Then $V_{(A)}$ and $V_{(B)}$ are mutually congruent. Hence, from our assumption, we have proved the following theorem:

Theorem 6. *If a Finsler manifold M is connected and is covered by a coordinate neighborhood system $\{U_A\}$, each of which admits n linearly independent covariant vectors $\mu_i^\alpha(x)$ satisfying the equation $g_{rl}\lambda_\sigma^r\mu_{m|h}^\sigma y^l y^m=0$, where $(\lambda_\alpha^i)=(\mu_i^\alpha)^{-1}$ and $|$ means the covariant differentiation defined by E. Cartan, then M is a Finsler manifold modeled on a Minkowski space.*

§ 5. Completely parallelizable manifolds

Let M be a completely parallelizable manifold, in other words, let M admit an $\{e\}$-structure. Then there exist n linearly independent covariant vector fields $\mu_i^\alpha(x)$ on M. We take them as $\mu_i^\alpha(x)$ defined in § 2. Since the Lie algebra of the Lie group $\{e\}$ is $\{0\}$, the G-connection relative to the $\{e\}$-structure is given uniquely by

$$\Gamma^i_{jk}=\lambda^i_\sigma\partial_j\mu^\sigma_k. \tag{5.1}$$

Since the Lie subgroup H is $\{e\}$ only, we can take any Minkowski space as V. Therefore, if we assign a Minkowski space V and denote its norm function by f, we can write the $\{V, \{e\}\}$-Finsler metric concretely as $F(x, y)=f(\mu_i^\alpha(x)y^i)$. Of course, from Theorem 2, we have $\nabla_k g_{ij}=0$ where ∇ means the covariant differentiation with respect to $\Gamma^i_{kj}(x)$ defined by (5.1) and $\Gamma^i_{lj}(x)y^l$. At the same time, we have that M is a generalized Berwald space. Hence we get

Theorem 7. *Let M be an n-dimensional completely parallelizable manifold and V be any Minkowski space. Then M admits a $\{V, \{e\}\}$-Finsler metric with respect to which M is a generalized Berwald space.*

If we take up, for an example, $\ell^{2p}(n)$ as the Minkowski space V, where p is a natural number. As is well known, the space $\ell^{2p}(n)$ is an n-dimensional Banach space whose norm is given by $\|\xi\|=(\sum_{\alpha=1}^{n}(\xi^\alpha)^{2p})^{1/2p}$. Then, the $\{\ell^{2p}(n), \{e\}\}$-Finsler metric can be written as

$$F(x, y)=(\sum_{\alpha=1}^{n}(\mu_i^\alpha(x)y^i)^{2p})^{1/2p}. \tag{5.2}$$

This is a concrete example of a $\{V, H\}$-Finsler metric.

§6. Special Randers space

Let S be a Minkowski space whose Minkowski norm is given by

$$\|\xi\|=\sqrt{\sum_{\alpha=1}^{n}(\xi^\alpha)^2}+k\xi^1, \tag{6.1}$$

where k is constant and $0<k<1$.

Now, it is easy to verify "*The linear Lie group G which leaves the Minkowski norm* (6.1) *invariant has the form* $1\times O(n-1)$.". We shall prove

Theorem 8. *Let M be an n-dimensional manifold. If M admits a* $\{1\times O(n-1)\}$*-structure, then M admits a Finsler metric such that*

$$F(x, y)=\sqrt{a_{ij}(x)y^iy^j}+kb_i(x)y^i, \tag{6.2}$$

where $a_{ij}(x)$ *is a Riemannian metric on M and* $b_i(x)$ *is a covariant vector field on M and satisfies* $a^{ij}b_ib_j=1$. *Conversely, if M admits a Finsler metric of* (6.2), *then M is a* $\{S, 1\times O(n-1)\}-$*manifold.*

Proof. Denote by $X_\alpha=\lambda_\alpha^i\partial/\partial x^i$ an adapted frame of the $\{1\times O(n-1)\}$-structure, then $\lambda_1^i(x)$ is a contravariant vector field on M. Put $(\lambda_\alpha^i)^{-1}=(\mu_i^\alpha)$ and $\sum_{\alpha=1}^{n}\mu_i^\alpha(x)\mu_j^\alpha(x)=a_{ij}$, then a_{ij} gives a Riemannian metric on M, because of the fact that $1\times O(n-1)\subset O(n)$. Then $a_{ij}\lambda_\alpha^i\lambda_\beta^j=\delta_{\alpha\beta}$ holds good. If we put $\mu_i^1=b_i$, then $b_i(x)$ is a covariant vector field on M and $a^{ij}b_ib_j=1$ is true. By applying Theorem 1 to our case, we obtain that M admits the $\{S, 1\times O(n-1)\}$-Finsler metric such that

$$F(x, y)=\sqrt{\sum_{\alpha=1}^{n}(\mu_i^\alpha(x)y^i)^2}+k\mu_i^1(x)y^i$$

$$=\sqrt{a_{ij}(x)y^iy^j}+kb_i(x)y^i.$$

Conversely, we assume that M admits a Finsler metric given by (6.2). Now we take $\mu_i^1(x)=b_i(x)$ and $\mu_i^\alpha(x)$ to form an orthogonal ennuple with respect to the Riemannian metric $a_{ij}(x)$. Then we have $a_{ij}(x) = \sum_{\alpha=1}^{n} \mu_i^\alpha(x)\mu_j^\alpha(x)$ and $a_{ij}(x)y^i y^j = \sum_{\alpha=1}^{n} (\mu_i^\alpha(x)y^i)^2$. Hence we have proved Theorem 8.

Lastly we add a remark. A Finsler metric given by

$$F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i,$$

where a_{ij} is a Riemannian metric and $b_i(x)$ is a covariant vector field, is called a Randers metric. The Finsler metric given in Theorem 8 is a Randers metric. A manifold with this special Randers metric is a generalized Berwald space (Theorem 2) and, at the same time, an $\{S, 1\times O(n-1)\}$-manifold (Theorem 8). Besides, Hashiguchi and the present author have obtained in their paper [8] some results about the relation between a generalized Berwald space and (α, β)-metric which is a generalization of the Randers metric.

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References

[1] L. Berwald, Untersuchung der Krümmung allgemeiner metrischer Raüme auf grund des in ihnen herrschenden Parallelismus, Math. z., **25** (1926), 40–73.

[2] L. Berwald, Über zweidimensionale allgemeine metrische Raüme, I, II, J. reine angew. Math., **156** (1927), 191–210, 211–222.

[3] E. Cartan, Les espaces de Finsler, Actualités 79, Paris (1934).

[4] J. Dowling, Finsler geometry on Sobolev manifolds, Proc. Symp. Pure Math. Amer. Math. Soc. held at Berkley, (1968) 1–10.

[5] J. Eells Jr., A setting for global analysis, Bull. Amer. Math. Soc. **72** (1966), 751–807.

[6] A. Fujimoto, Theory of *G*-structures, Publ. of the Study Group of Geometry, Vol. **1** (1972).

[7] M. Hashiguchi, On Wagner's generalized Berwald space, J. Korean Math. Soc., **12** (1975), 51–61.

[8] M. Hashiguchi and Y. Ichijyō, On some special (α, β)-metrics, To appear in Rep. Fac. Sci. Kagoshima Univ., (Math., Phys., Chem.), **8** (1975).

[9] M. Matsumoto, The theory of Finsler connections, Publ. of the Study Group of Geometry, Vol. **5** (1970).

[10] M. Matsumoto, Metric differential geometry, Syokabô (1975) (in Japanese).
[11] Y. Matsushima, Differentiable manifolds, Marcel Dekker (1972).
[12] S. Numata, On the curvature tensor S_{hijk} and the tensor T_{hijk} of generalized Randers spaces, Tensor **29** (1975), 35–39.
[13] H. Rund, The differential geometry of Finsler spaces, Springer (1957).
[14] A. E. Taylor, Introduction to functional analysis, John Willey & Sons (1959).
[15] K. Yano and S. Ishihara, Tangent and cotangent bundles, Marcel Dekker (1973).