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Openness of a family of torsion free sheaves

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In this article we shall show that some conditions on coherent sheaves are open (cf. Definition 1.4). Some of our results will play an important role in the forthcoming paper [3].

§ 1. Definitions and preliminaries

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism with a noetherian scheme S and let $\mathcal{O}_X(1)$ be an f-very ample invertible $\mathcal{O}_X$-module. In this situation

Definition 1.1. 1) *Let* $(\alpha)=(\alpha_1,\dots,\alpha_{r-1})$ *be a sequence of rational numbers. A coherent module* F *of* rank r *on the fibre* X_s *over a geometric point* s *of* S *is said to be of type* (α) *(with respect to* $\mathcal{O}_X(1)$*) if and only if it is torsion free and for all quotient coherent sheaves* E *of* F *of* rank t $(1 \leqq t \leqq r-1)$, *the following inequalities hold;*

$$d(F, \mathcal{O}_{X_s}(1))/r-\alpha_t \leqq d(E, \mathcal{O}_{X_s}(1))/t,$$

where $d(\cdot, \mathcal{O}_{X_s}(1))$ *denotes the degree with respect to* $\mathcal{O}_{X_s}(1)$.

2) *Let* $(\beta)=(\beta_1,\dots,\beta_{r-1})$ *be a sequence of rational numbers. A coherent* $\mathcal{O}_{X_s}$*-module* F *of* rank r *with a geometric point* s *of* S *is said to be of cotype* (β) *(with respect to* $\mathcal{O}_X(1)$*) if and only if it is torsion free and for all coherent subsheaves* E *of* F *of* rank t $(1 \leqq t \leqq r-1)$, *the following inequalities hold;*

$$d(E, \mathcal{O}_{X_s}(1))/t \leqq d(F, \mathcal{O}_{X_s}(1))/r + \beta_t.$$

3) *A coherent $\mathcal{O}_{X_s}$-module F of* rank r *with a geometric point s of S is said to be stable* (*or, semi-stable*) (*with respect to $\mathcal{O}_X(1)$*) *if and only if it is torsion free and for all proper subsheaves E of* rank t ($1 \leqq t \leqq r$), *the inequalities*

$$P_E(m) = \chi(E(m))/t < P_F(m) = \chi(F(m))/r$$

(*or, $\leqq$, resp.*) *hold for all large m, where* $\chi(F(m)) = \sum_i (-1)^i \dim H^i(X_s, F(m))$.

A relation between "type" and "cotype" will be found in Lemma 1.2 of [2].

The following is due to A. Grothendieck ([1] Lemma 2.5).

Lemma 1.2. *Let S be a noetherian scheme, $f: X \to S$ be a projective morphism with an f-very ample invertible sheaf $\mathcal{O}_X(1)$ and let F be a coherent $\mathcal{O}_X$-module. Assume that A is a set of coherent quotient sheaves of the sheaves $F \otimes k(s)$ with geometric points s of S. If the dimensions of the fibres of X over S are not greater than n, then for an E in A, we can write*

$$\chi(E(m)) = a_E m^n/n! + b_E m^{n-1}/(n-1)! + \textit{terms of degree} \leqq n-2.$$

Then the set $\{a_E | E \in A\}$ is bounded and the set $\{b_E | E \in A\}$ is bounded below. Moreover if $\{b_E | E \in A\}$ is bounded, then $A' = \{E_{(n)} = E/E_n | E \in A\}$ is a bounded family, where for a coherent $\mathcal{O}_{X_s}$-module E, E_n is the coherent subsheaf of E defined as follows; for each open set U of X_s, $\Gamma(U, E_n) = \{a \in \Gamma(U, E) | \dim \mathrm{Supp}(a) < n\}$.

If S is connected and f is smooth and geometrically integral, then the dimensions of the fibres of X over S are constant n and $a_E = th$, where t is the rank of E and h is the degree of X_s with respect to $\mathcal{O}_{X_s}(1)$. Moreover, $b_E = d(E, \mathcal{O}_{X_s}(1)) - d(K_{X_s}, \mathcal{O}_{X_s}(1))t/2$ with the canonical sheaf K_{X_s} of X_s. On the other hand, if E is a torsion free coherent $\mathcal{O}_{X_s}$-module, then $E_n = 0$. Thus the above lemma implies

Corollary 1.2.1. *Let $f: X \to S$ be a smooth, projective, geometrically integral morphism with an f-very ample invertible sheaf $\mathcal{O}_X(1)$ and let S be noetherian.*

1) *If a set A of coherent sheaves of* rank r *on fibres of X over geometric points of S is bounded, then there exists a sequence $(\alpha) = (\alpha_1, \ldots, \alpha_{r-1})$ of rational numbers such that every F in A is of type (α) with respect to $\mathcal{O}_X(1)$.*

2) *Assume that A is as above and B is a set of torsion free, coherent, quotient sheaves of members of A. Then B is a bounded family if and only if the set $\{d(E, \mathcal{O}_{X_s}(1)) | E \in B\}$ is bounded above.*

The following, which will be used frequently in the sequel, is shown immediately from Nakayama's lemma.

Lemma 1.3. *Let A and B be local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively and let $\phi: A \to B$ be a local homomorphism. Suppose that B is noetherian and M is a finite B-module which is A-flat. Then M is a free B-module if and only if $M \otimes_A A/\mathfrak{m}$ is a free $B/\mathfrak{m}B$-module.*

Corollary 1.3.1. *Let $g: Y \to S$ be a morphism of noetherian schemes, F be an S-flat coherent $\mathcal{O}_Y$-module and let $r = \min\{\mathrm{rank}_{k(y)} F \otimes_{\mathcal{O}_Y} k(y) | y \in Y\}$. Assume that for all points s of S, Y_s is reduced. Then $U = \{y \in Y | \mathrm{rank}_{k(y)} F \otimes_{\mathcal{O}_Y} k(y) = r\}$ is an open set of Y. Moreover $F|_U$ is a locally free $\mathcal{O}_U$-module.*

Proof. Since $Y - U = \mathrm{Supp}(\overset{r+1}{\Lambda} F)$ and since $\overset{r+1}{\Lambda} F$ is coherent, U is open. By replacing Y by U, we may assume that $\mathrm{rank}_{k(y)} F \otimes_{\mathcal{O}_Y} k(y)$ is constant. Pick a point s of S and consider $F_s = F \otimes_{\mathcal{O}_S} k(s)$. Our assumption says that F_s has the same rank r at every point of Y_s. Thus F_s is a locally free $\mathcal{O}_{Y_s}$-module because Y_s is reduced. By virtue of Lemma 1.3, F is free at every point of Y_s. F is therefore locally free. q.e.d.

For convenience sake let us introduce the following definition.

Definition 1.4. *Let $\boldsymbol{P}$ be a property of a coherent module on a smooth projective variety. $\boldsymbol{P}$ is said to be open if for every smooth, projective, geometrically integral scheme X over a locally noetherian scheme S and for every S-flat coherent $\mathcal{O}_X$-module F, there exists an open set U of S such that for every algebraically closed field k, $\{s \in S(k) | F \otimes_{\mathcal{O}_X} k(s)$ has the property $\boldsymbol{P}\} = U(k)$, where for a scheme Y, $Y(k)$ denotes the set of k-valued points of Y.*

It is clear that in the above definition we may assume that S is noetherian and connected.

§ 2. Openness of a family of torsion free sheaves

In this section we shall show that some of the conditions on coherent sheaves considered in § 1 are open. Let us begin with the following.

Proposition 2.1. *The property that a coherent module is torsion free is open.*

Using 9.4.8 of E.G.A., IV, we can prove the proposition without assuming smoothness and projectivity. We shall give here another proof whose technique will be used several times in [3]. In order to do it, let us characterize torsion free sheaves on a smooth quasi-projective variety.

Lemma 2.2. *Let Y be a smooth quasi-projective variety of dimension n and let F be a coherent $\mathcal{O}_Y$-module. Assume that there exists a resolution of F by locally free $\mathcal{O}_Y$-modules of finite rank;*

$$0 \longrightarrow E_n \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} E_0 \xrightarrow{f_0} F \longrightarrow 0 .$$

Then F is torsion free if and only if $\dim B(\ker(f_i)) \leqq n-i-3$ and $\dim B(F) \leqq n-2$, where for a coherent $\mathcal{O}_Y$-module G, $B(G)$ is the set of pinch points of G, that is, $B(G) = \{y \in Y | \operatorname{rank}_{k(y)} G \otimes_{\mathcal{O}_Y} k(y) > \operatorname{rank} G\}$ ($B(G)$ is closed in Y by Corollary 1.3.1).

Proof. Assume that F is torsion free. It is clear that $\dim B(F)$

$\leqq n-2$ because Y is smooth. Pick a point y of Y of dimension $n-i-2$, then the stalk F_y of F at y is a torsion free $\mathcal{O}_{Y,y}$-module. Thus $\operatorname{depth}_{\mathcal{O}_{Y,y}}(F_y)\geqq 1$, which implies that $\operatorname{hd}_{\mathcal{O}_{Y,y}}(F_y)\leqq i+1$ because $\mathcal{O}_{Y,y}$ is a regular local ring of dimension $i+2$. Thus $\ker(f_i)$ is locally free at y. Hence $\ker(f_i)$ is locally free at every point of dimension $n-i-2$, whence $\dim B(\ker(f_i))\leqq n-i-3$. Conversely assume that $\dim B(\ker(f_i))\leqq n-i-3$ and $\dim B(F)\leqq n-2$. Let T be the torsion part of F. We have to show $T=0$. Assume the contrary and let y be the generic point of an irreducible component of $\operatorname{Supp}(T)$. If the dimension of y is $n-i-2$, then the assumption says that $i\geqq 0$ and $\ker(f_i)$ is free at y, whence $\operatorname{hd}_{\mathcal{O}_{Y,y}}(F_y)\leqq i+1$. Thus we have that $\operatorname{depth}_{\mathcal{O}_{Y,y}}(F_y)\geqq 1$. On the other hand, all the elements of the maximal ideal of $\mathcal{O}_{Y,y}$ are zero divisors of F_y. This is a contradiction. Hence we know that $T=0$. q.e.d.

Lemma 2.3. *Let $f: X\to S$ be a projective morphism of noetherian schemes with an f-very ample invertible sheaf $\mathcal{O}_X(1)$. If F is an S-flat coherent $\mathcal{O}_X$-module, then there exists a surjective homomorphism $\phi: E\to F$ with a locally free $\mathcal{O}_X$-module E.*

Proof. Since $f_*(F(m))$ is locally free and the natural homomorphism $\phi': f^*f_*(F(m))\to F(m)$ is surjective if m is sufficiently large (E.G.A., III, 7.9.10), $\phi=\phi'\otimes\mathcal{O}_X(-m)$ and $E=f^*f_*(F(m))\otimes_{\mathcal{O}_X}\mathcal{O}_X(-m)$ are the desired couple. q.e.d.

Now we can prove the proposition.

Proof of Proposition 2.1. Let $f: X\to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes and let F be an S-flat $\mathcal{O}_X$-module. Without losing any generality, we may assume that S is connected and noetherian. Then X is connected and the dimensions of the fibres of f are constant n. By virtue of Lemma 2.3 we obtain a surjective homomorphism $\phi_0: E_0\to F$ with a locally free $\mathcal{O}_X$-module E_0. Since f is flat, E_0 is S-flat, whence $\ker(\phi_0)$ is also S-flat. Thus, using Lemma 2.3 again, we get an exact sequence

$E_1 \xrightarrow{\phi_1} E_0 \xrightarrow{\phi_0} F \longrightarrow 0$ with a locally free $\mathcal{O}_X$-module E_1. Repeating this procedure, we obtain an exact sequence

$$E_{n-1} \xrightarrow{\phi_{n-1}} E_{n-2} \xrightarrow{\phi_{n-2}} \cdots \xrightarrow{\phi_1} E_0 \xrightarrow{\phi_0} F \longrightarrow 0$$

with locally free $\mathcal{O}_X$-modules $E_{n-1}, \dots, E_0$. Set $E_n = \ker(\phi_{n-1})$. Since $\ker(\phi_i)$'s are S-flat, for all points s of S,

$$*) \qquad 0 \longrightarrow E_{n,s} \xrightarrow{\phi_{n,s}} E_{n-1,s} \xrightarrow{\phi_{n-1,s}} \cdots \xrightarrow{\phi_{1,s}} E_{0,s} \xrightarrow{\phi_{0,s}} F_s \longrightarrow 0$$

is exact, where $E_{i,s} = E_i \otimes_{\mathcal{O}_S} k(s)$, $F_s = F \otimes_{\mathcal{O}_S} k(s)$ and $\phi_{i,s} = \phi_i \otimes k(s)$. Since X_s is a non-singular variety of dimension n, $E_{n,s}$ is locally free. Thus we know that E_n is locally free by virtue of Lemma 1.3 because E_n is S-flat. Let r_i (or, r) be the rank of $\ker(\phi_{i,s})$ (or, F, resp.), then they are independent of s because S is connected and $\ker(\phi_i)$'s are S-flat. Put $B(\ker(\phi_i)) = \{x \in X | \operatorname{rank}_{k(x)} \ker(\phi_i) \otimes_{\mathcal{O}_X} k(x) > r_i\}$ and $B(F) = \{x \in X | \operatorname{rank}_{k(x)} F \otimes_{\mathcal{O}_X} k(x) > r\}$. Then they are closed in X. By virtue of a theorem of Chevalley (E.G.A., IV, 13.1.3), $C_i = \{s \in S | \dim(f^{-1}(s) \cap B(\ker(\phi_i))) > n-i-3\}$ and $C = \{s \in S | \dim(f^{-1}(s) \cap B(F)) > n-2\}$ are closed in S. By the exact sequence (*), Lemma 2.2 and the fact that $\ker(\phi_i) \otimes_{\mathcal{O}_S} k(s) \cong \ker(\phi_{i,s})$, we know that the open set $U = S - C \cup (\bigcup_{i=1}^{n} C_i)$ is the desired one. q.e.d.

The properties "type (α), "stable" and "semi-stable" are closed under generalizations.

Lemma 2.4. *Let R be a discrete valuation ring and let $f: X \to S = \operatorname{Spec}(R)$ be a smooth, projective, geometrically integral morphism with an f-very ample invertible sheaf $\mathcal{O}_X(1)$. If F is an S-flat coherent $\mathcal{O}_X$-module and if for a geometric point s of S over the generic point of S, $F \otimes_{\mathcal{O}_S} k(s)$ is not of type (α) (stable or semi-stable), then for all geometric points s_0 of S over the closed point of S, $F \otimes_{\mathcal{O}_S} k(s_0)$ is not of type (α) (stable or semi-stable, resp.).*

Proof. If one notes the invariance of Hilbert polynomials and degrees under a flat deformation, our assertion is easily proved (see [2]

Lemma 3.3).

By virtue of the above lemma, in order to show the openness of type (α), stability or semi-stability, it is sufficient to show that they are constructible.

Lemma 2.5. *Let $f: X \to S$ be a projective, geometrically integral morphism of noetherian schemes and let F and G be coherent $\mathcal{O}_X$-modules such that $F \otimes_{\mathcal{O}_S} k(s)$ and $G \otimes_{\mathcal{O}_S} k(s)$ are torsion free $\mathcal{O}_{X_s}$-modules for all $s \in S$. Then $U = \{s \in S \mid \mathrm{Hom}_{\mathcal{O}_{X_s}}(F \otimes_{\mathcal{O}_S} k(s), G \otimes_{\mathcal{O}_S} k(s)) \neq 0\}$ is a constructible set of S.*

Proof. If S' is a subscheme of S, then for $X' = X_{S'}$, $F' = F \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ and $G' = G \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$, $\{s' \in S' \mid \mathrm{Hom}_{\mathcal{O}_{X'_{s'}}}(F' \otimes_{\mathcal{O}_{S'}} k(s'), G' \otimes_{\mathcal{O}_{S'}} k(s')) \neq 0\} = U \cap S'$. Thus we have only to show that U contains a non-empty open set of S or it is rare in S under the assumption that S is integral (E. G. A., 0_{III}, 9.2.3). Let z be the generic point of S. Since F is coherent, the canonical homomorphism $\mathcal{H}om_{\mathcal{O}_X}(F, G) \otimes_{\mathcal{O}_S} k(z) \to \mathcal{H}om_{\mathcal{O}_{X_z}}(F \otimes_{\mathcal{O}_S} k(z), G \otimes_{\mathcal{O}_S} k(z))$ is an isomorphism, whence $\mathcal{H}om_{\mathcal{O}_X}(F, G) \otimes_{\mathcal{O}_S} k(z)$ is torsion free. Then there exists a non-empty open set V of S such that for all points s of V, $\mathcal{H}om_{\mathcal{O}_X}(F, G) \otimes_{\mathcal{O}_S} k(s)$ is torsion free and $\mathcal{O}_X$, F are flat over V (E. G. A., IV, 9.4.8, 11.1.1). As in the proof of Proposition 2.1, we obtain an exact sequence $E'' \to E' \to F_V \to 0$ with locally free $\mathcal{O}_{X_V}$-modules E', E''. This provides us with the exact sequence of $\mathcal{O}_{X_V}$-modules;

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_{X_V}}(F_V, G_V) \longrightarrow \mathcal{H}om_{\mathcal{O}_{X_V}}(E', G_V) \longrightarrow \mathcal{H}om_{\mathcal{O}_{VX}}(E'', G_V).$$

Thus for all points s of V, we have the following exact commutative diagram;

$$\begin{array}{ccccccc} & & \mathcal{H}om_{\mathcal{O}_{X_V}}(F_V, G_V)_s & \xrightarrow{j} & \mathcal{H}om_{\mathcal{O}_{X_V}}(E', G_V)_s & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X_V}}(E'', G_V)_s \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X_s}}(F_s, G_s) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X_s}}(E'_s, G_s) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X_s}}(E''_s, G_s) \end{array}$$

where the suffix "s" denotes $\otimes_{\mathcal{O}_S} k(s)$. Since E' and E'' are locally free, v and w are isomorphisms. On the other hand, since F is flat over V and since f is geometrically integral, F is locally free on an open set W of X with $f(W)\supset V$ (see Corollary 1.3.1). Thus u induces an isomorphism on a non-empty open set $W\cap X_s$ in X_s. Moreover our assumption on V forces $\mathscr{H}om_{\mathcal{O}_{X_V}}(F_V, G_V)\otimes_{\mathcal{O}_V}k(s)$ to be torsion free. Thus u is injective and hence so is j. Five lemma says that u is surjective, whence it is an isomorphism. Now there exists a non-empty open set V' of V such that $\mathscr{H}om_{\mathcal{O}_X}(F, G)_{V'}\cong\mathscr{H}om_{\mathcal{O}_{X_{V'}}}(F_{V'}, G_{V'})$ is flat over V'. Then $V''=\{s\in V'|H^0(X_s, \mathscr{H}om_{\mathcal{O}_{X_{V'}}}(F_{V'}, G_{V'})\otimes_{\mathcal{O}_{V'}}k(s))\neq 0\}=\{s\in V'|\mathrm{Hom}_{\mathcal{O}_{X_s}}(F\otimes_{\mathcal{O}_S}k(s), G\otimes_{\mathcal{O}_S}k(s))\neq 0\}$ is closed in V'. If $V''=V'$, then U contains the open set V' and if $V''\neq V'$, then U is a subset of the rare set $V''\cup(X-V')$. q.e.d.

The proof of the following is similar to that of Lemma 3.2 of [2].

Lemma 2.6. *Let X be a non-singular projective variety. If a torsion free coherent $\mathcal{O}_X$-module F of* rank r *is not of cotype $(\beta)=(\beta_1,\dots,\beta_{r-1})$, then there exists a coherent $\mathcal{O}_X$-submodule E of* rank t *with some $1\leqq t\leqq r-1$ such that* (i) *E is of cotype $(\beta_1-\beta_t,\dots,\beta_{t-1}-\beta_t)$,* (ii) *$F/E$ is torsion free and* (iii) *$d(E, \mathcal{O}_X(1))/t>d(F, \mathcal{O}_X(1))/r+\beta_t$.*

Similarly we have

Lemma 2.7. *Let X be a non-singular projective variety. If a torsion free coherent $\mathcal{O}_X$-module of* rank r *is not stable (or, semi-stable), then it has a coherent subsheaf E of* rank t $(1\leqq t\leqq r-1)$ *such that* (i) *E is stable,* (ii) *F/E is torsion free and* (iii) *$P_E(m)=\chi_E(m)/t\geqq P_F(m)=\chi_F(m)/r$ (or, $>$, resp.) for large m.*

Proof. Let us prove our lemma by induction on r. Since F is not stable (or, semi-stable), there exists a coherent subsheaf E_1 of rank t_1 $(1\leqq t_1\leqq r-1)$ such that $P_{E_1}(m)\geqq P_F(m)$ (or, $>$, resp.) for large m. Since the inverse image of the torsion part of F/E_1 by the canonical homomorphism $F\to F/E_1$ has the same property as E_1, we may assume that F/E_1 is torsion free. If E_1 is stable, then there exists nothing to prove. Assume the contrary, then our induction assumption provides us with a coherent subsheaf E of E_1 of rank t $(1 \leqq t \leqq t_1 - 1)$ such that (a) E is stable, (b) E_1/E is torsion free and (c) $P_E(m) \geqq P_{E_1}(m)$ for large m. Since both F/E_1 and E_1/E are torsion free, so is F/E. Moreover $P_E(m) \geqq P_{E_1}(m) \geqq P_F(m)$ (or, $P_E(m) \geqq P_{E_1}(m) > P_F(m)$, resp.) for large m. Therefore E is the desired subsheaf of F. q.e.d.

Now we are ready to prove our main theorem.

Theorem 2.8. A) *If* (1) $(\alpha)=(\alpha_1,\dots,\alpha_{r-1})$, $\alpha_t = t\beta_{r-t}/(r-t)$, *with an ascending sequence of rational numbers* $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_{r-1}$ *or if* (2) $(\alpha)=(\alpha_1,\dots,\alpha_{r-1})$ *is an ascending sequence of rational numbers, then the property* $\boldsymbol{P}$ *that a coherent sheaf is of type* (α) *is open.*

(B) *The property* $\boldsymbol{P}$ *that a coherent sheaf is stable* (*or, semi-stable*) *is open.*

Proof. Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes with an f-very ample invertible sheaf $\mathcal{O}_X(1)$ and let F be an S-flat coherent $\mathcal{O}_X$-module of rank r. We may assume that S is connected and noetherian. By virtue of Proposition 2.1 we may also assume that $F \otimes_{\mathcal{O}_S} k(s)$ is a torsion free $\mathcal{O}_{X_s}$-module for every geometric point s of S. It is sufficient to show that $\boldsymbol{P}$ is constructible by virtue of Lemma 2.4.

A) First of all let us show that the case (2) can be reduced to the case (1). In fact as was shown in the proof of Lemma 2.5 there exists a finite set $\{S_i \hookrightarrow S\}$ of subschemes of S such that $S = \bigcup_i S_i$ and for all geometric points s of S_i, $\mathcal{H}om_{\mathcal{O}_{X_{S_i}}}(F_{S_i}, \mathcal{O}_{X_{S_i}}) \otimes_{\mathcal{O}_{S_i}} k(s)$ is isomorphis to $(F \otimes_{\mathcal{O}_S} k(s))^\vee = \mathcal{H}om_{\mathcal{O}_{X_s}}(F \otimes_{\mathcal{O}_S} k(s), \mathcal{O}_{X_s})$. Taking a finer decomposition of S if necessary, we may assume that there exists an S_i-flat coherent $\mathcal{O}_{X_{S_i}}$-module $F^{(i)}$ such that for all geometric points s of S_i, $F^{(i)} \otimes_{\mathcal{O}_{S_i}} k(s)$ is isomorphic to $(F \otimes_{\mathcal{O}_S} k(s))^\vee$. On the other hand, we know that on a non-singular projective variety, a coherent sheaf E is of type (α) if and only if $E^\vee$ is of type (β) with $\beta_t = t\alpha_{r-t}/(r-t)$ ([2] Lemma 1.5). Thus the case (1) implies that there exists a constructible set U_i of S_i such that for all algebraically closed fields

k, $\{s\in S_i(k)|F\otimes_{\mathcal{O}_S}k(s)$ is of type $(\alpha)\}$ is just $U_i(k)$. Then the set $\bigcup_i U_i$ is the desired constructible set in S.

Now let us prove (A) in the case (1) by induction on r. If $r=1$, there is nothing to prove. Since a coherent sheaf E on a non-singular projective variety is of type (α) if and only if it is of cotype (β), we have only to show that the property that a coherent sheaf is of cotype (β) is open under the assumption $\beta_1\leqq\beta_2\leqq\cdots\leqq\beta_{r-1}$. Let Φ be the family of classes of coherent sheaves E on the fibres of X/S such that E is a subsheaf of $F\otimes_{\mathcal{O}_S}k(s)$ with some geometric point s of S and it has the properties (i), (ii) and (iii) in Lemma 2.6 for $F\otimes_{\mathcal{O}_S}k(s)$, (β) and $\mathcal{O}_{X_s}(1)$. Taking Corollary 1.2.1 and the properties (ii) and (iii) into account we know that Φ is bounded. Thus there exist an S-scheme S' of finite type and an S'-flat coherent $\mathcal{O}_{X_{S'}}$-module E' such that

(2.8.1) for every $\mathcal{O}_{X_s}$-module E in Φ, there exists a geometric point s' of S' lying over s with $E'\otimes_{\mathcal{O}_{S'}}k(s')\cong E\otimes_{k(s)}k(s')$.

By virtue of the invariance of ranks and degrees under flat deformations and our induction assumption, we may assume, replacing S' by an open set of it, that (2.8.1) holds and that for all geometric points s' of S', (a) the rank t of $E'\otimes_{\mathcal{O}_{S'}}k(s')$ is less than r, (b) $E'\otimes_{\mathcal{O}_{S'}}k(s')$ is torsion free, (c) it is of cotype $\beta_1-\beta_t,\dots,\beta_{t-1}-\beta_t$ and (d) $d(E'\otimes_{\mathcal{O}_{S'}}k(s'),\mathcal{O}_{X_{s'}}(1))/t>d(F\otimes_{\mathcal{O}_S}k(s),\mathcal{O}_{X_s}(1))/r+\beta_t$, with s' lying over s. Put $F'=F_{S'}$. For an algebraically closed field k, set $U(k)=\{s\in S(k)|\mathrm{Hom}_{\mathcal{O}_{X_{s'}}}(E'\otimes_{\mathcal{O}_{S'}}k(s'),F'\otimes_{\mathcal{O}_{S'}}k(s'))\neq 0$ with some s' lying over $s\}$ and set $V(k)=\{s\in S(k)|F\otimes_{\mathcal{O}_S}k(s)$ is not of cotype $(\beta)\}$. Then Lemma 2.6 implies that $V(k)$ is contained in $U(k)$. Pick a point s in $U(k)$ and a non-zero homomorphism $\phi\colon E'\otimes_{\mathcal{O}_{S'}}k(s')\to F'\otimes_{\mathcal{O}_{S'}}k(s')$ with some s' lying over s. Assume that the rank of $E'\otimes_{\mathcal{O}_{S'}}k(s')$ (or, $\phi(E'\otimes_{\mathcal{O}_{S'}}k(s'))$) is t (or, t', resp.). Then the conditions (c) and (d) imply that for $E_1=E'\otimes_{\mathcal{O}_{S'}}k(s')$, $E_2=\phi(E_1)$ and $F_1=F'\otimes_{\mathcal{O}_{S'}}k(s')$,

$$d(E_2,\mathcal{O}_{X_{s'}}(1))/t'\geqq d(E_1,\mathcal{O}_{X_{s'}}(1))/t-t'(\beta_{t'}-\beta_t)/(t-t')>$$

$$d(F_1,\mathcal{O}_{X_{s'}}(1))/r+\beta_t-t'(\beta_{t'}-\beta_t)/(t-t')\geqq$$

$$d(F_1,\mathcal{O}_{X_{s'}}(1))/r+\beta_{t'}.$$

Thus $F'\otimes_{\mathcal{O}_{S'}}k(s')$ is not of cotype (β), whence we have $V(k)=U(k)$. Now Lemma 2.5 says that there exists a constructible set Z' of S' such that for all algebraically closed fields k, $Z'(k)=\{s'\in S'(k)|\mathrm{Hom}_{\mathcal{O}_{X_{s'}}}(E'\otimes_{\mathcal{O}_{S'}}k(s'), F'\otimes_{\mathcal{O}_{S'}}k(s'))\neq 0\}$, whence the image of $Z'(k)$ in $S(k)$ is $U(k)$. By a theorem of Chevalley the image Z of Z' in S is also constructible. It is clear that $S-Z$ is the desired constructible set.

(B) Let Ψ be the family of classes of coherent sheaves E on the fibres of X/S such that (1) E is a subsheaf of $F\otimes_{\mathcal{O}_S}k(s)$ with some geometric point s of S, (2) it is stable, (3) $P_E(m)\geqq P_{F\otimes_{\mathcal{O}_S}k(s)}(m)$ (or, $>$, resp.) for large m, and (4) $F\otimes_{\mathcal{O}_S}k(s)/E$ is torsion free. Since the condition (3) implies that $\{d(E, \mathcal{O}_{X_s}(1))|E\in\Psi\}$ is bounded below, the conditions (1) and (4) assert that Ψ is bounded (Corollary 1.3.1). Then as in the proof of (A) we can find an S-scheme S' of finite type and an S'-flat coherent $\mathcal{O}_{X_{S'}}$-module E' such that for every $\mathcal{O}_{X_s}$-module E in Ψ, there exists a geometric point s' of S' lying over s with $E'\otimes_{\mathcal{O}_{S'}}k(s')\cong E\otimes_{k(s)}k(s')$ and that for all geometric points s' of S', (a) $\mathrm{rank}(E'\otimes_{\mathcal{O}_{S'}}k(s'))<r$, (b) $E'\otimes_{\mathcal{O}_{S'}}k(s')$ is stable and (c) $P_{E'\otimes_{\mathcal{O}_{S'}}k(s')}(m)\geqq P_{F\otimes_{\mathcal{O}_S}k(s)}(m)$ (or, $>$, resp.) for large m with s' lying over s. Put $F'=F_{S'}$. Then $Z'=\{s'\in S'|\mathrm{Hom}_{\mathcal{O}_{X_{s'}}}(E'\otimes_{\mathcal{O}_{S'}}k(s'), F'\otimes_{\mathcal{O}_{S'}}k(s'))\neq 0\}$ is a constructible set. And, as in the proof of (A), we know that for the image Z of Z' in S, $S-Z$ is the desired constructible set for stable sheaves (or, semistable sheaves, resp.). q.e.d.

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