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Liouville's theorem on a transcendental equation $\log y = y/x$

By

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Abstract. The purpose of this note is to give an algebraic proof to Liouville's theorem that any solution of a transcendental equation $\log y = y/x$ is not an elementary transcendental function of x([5, pp. 526–531]).

§0. **Introduction.** Let K be an algebraically closed field of characteristic 0. We shall suppose that H is a differential field whose field of constants is K. Consider a differential equation

(1) $$y' = A$$

and a homogeneous differential equation

(2) $$y' = By,$$

where A, $B \in H$. Suppose that F is a differential extension of H whose field of constants is K. Then, Kolchin [2, pp. 801–803] proved the following two theorems (Cf. Ostrowski [6], Kolchin [3, p. 1156], Risch [7, p. 172]):

1. Suppose that an element η of F satisfies (1). Then, η is algebraic over H if and only if $\eta \in H$.

2. Suppose that an element ζ of F satisfies (2). Then, ζ is algebraic over H if and only if there exists such a positive integer k that $\zeta^k \in H$.

Take a transcendental element θ over H. Let us define $\theta' = A$.

Then, $H(\theta)$ is a differential extension of H. Suppose that any element of H does not satisfy (1). Then, the field of constants in $H(\theta)$ is K.

Take a transcendental element ρ over H. Let us define $\rho'=B\rho$. Then, $H(\rho)$ is a differential extension of H. Suppose that for each positive integer k any element of H different from 0 does not satisfy $y'=kBy$. Then, the field of constants in $H(\rho)$ is K(Cf. Remark 1).

Any algebraic extension of H is a differential extension of H. Its field of constants is K, because K is algebraically closed.

Suppose that M_1 is a differential field whose field of constants is K, and that M_2 is a differential extension of M_1. Then, M_2 will be called a *primitive extension* of M_1 if the following two conditions are satisfied:

(i) The field of constants in M_2 is K.

(ii) There exists a finite system of elements $\mu_1,\dots,\mu_r$ of M_2 which satisfies the following two conditions:

$(\text{ii})_1$ For each $i(1\leqq i\leqq r)$, μ_i is a solution of either $y'=A_i$ or $y'=C_i'y$, where A_i, $C_i\in M_1$.

$(\text{ii})_2$ M_2 is an algebraic extension of $M_1(\mu_1,\dots,\mu_r)$ of finite degree.

We shall suppose that M is a differential field whose field of constants is K. A finite chain of extending differential fields $L_0\subset L_1\subset\cdots\subset L_n$ will be called a *Liouville chain* over M if the following two condition sare satisfied:

(i) For each $i(1\leqq i\leqq n)$, L_i is a primitive extension of L_{i-1}.

(ii) L_0 is an algebraic extension of M of finite degree.

A differential extension L of M is called a *Liouville extension* of M if there exists in L a Liouville chain over M which ends with L.

Take a transcendental element x over K. Let us define $x'=1$ and $a'=0$ for any element a of K. Then, $K(x)$ is a differential field whose field of constants is K. Kolchin [2, p. 771] proved that every differential field of characteristic 0 has a universal extension. We shall take and fix a universal extension Ω of $K(x)$. An element z of Ω is called an *elementary transcendental function* of x over K if there exists a Liouville extension of $K(x)$ in Ω which contains z.

Let u, v be elements of Ω. Suppose that $v' \neq 0$. Then we write $u = l(v)$ if $u'v = v'$.

Liouville [4, pp. 91–94] proved the following theorem:

Let $p_1, \dots, p_n$ be algebraic functions of x over K different from 0, and $\alpha_1, \dots, \alpha_n$, β be elements of K. Suppose that $\sum \alpha_i p_i'/p_i = \beta$. Then, $\beta = 0$.

As a corollary to this theorem we see that $l(p)$ is transcendental over $K(x)$ for any algebraic function p of x over K different from a constant (Cf. Rosenlicht [8, p. 22]).

Theorem. *Any solution of a transcendental equation $l(y) = y/x$ is not an elementary transcendental function of x over K.*

This theorem can be stated in the following form:

Any nontrivial solution of a differential equation $x(y-x)y' = y^2$ is not an elementary transcendental function of x over K.

Remark 1. Kolchin [1] proved that there exists a Picard-Vessiot extension for any linear homogeneous ordinary differential equation over a differential field of characteristic 0 with an algebraically closed field of constants.

Remark 2. Liouville ([4], [5]) treated $\int u dx$ only in the case where $u = v'/v$ and $\int u dx = \log v$. It seems that to him $\log v$ is a transcendental function of v defined by $\log v = -\sum (1-v)^n/n$ $(1 \leqq n < \infty)$ rather than a solution of a differential equation $vy' = v'$ in a fixed differentiation signed by the prime. He claimed that $\log v$ satisfies a differential equation $v\dot{y} = \dot{v}$ in *any* differentiation signed by the dot. Liouville's proof of Theorem [5, pp. 526–531] is not an algebraic one.

Remark 3. Liouville [5, pp. 536–539] stated the following theorem: Suppose that f is an algebraic function of x, y, and that $f_x \neq 0$ and $f_y \neq 0$. Then, any solution of a transcendental equation $\log y = f(x, y)$ is not an elementary transcendental function of x unless it is a constant.

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§1. Integral and exponential extensions. In this section we shall prepare several lemmas. We shall suppose that N is a Liouville extension of H, where H is a differential field whose field of constants is K.

Definition 1. *N will be called an integral extension of H if there exists an element θ of N which satisfies the following two conditions:*

(i) *θ is transcendental over H, and $N=H(\theta)$.*

(ii) *θ is a solution of $y'=A$, where $A\in H$.*

Definition 2. *N will be called an exponential extension of H if there exists an element ρ of N which satisfies the following two conditions:*

(i) *ρ is transcendental over H, and $N=H(\rho)$.*

(ii) *ρ is a solution of $y'=By$, where $B\in H$.*

Through this section $H(\theta)$ and $H(\rho)$ will denote an integral and an exponential extension of H respectively.

If an elment Q of $H[\theta]$ divides Q', then $Q\in H$. Let R be an element of $H[\theta]$, and u be an elment of $H(\theta)$ different from 0. Suppose that $u'=Ru$. Then, $R\in H$ and $u\in H$.

Suppose that an element S of $H[\rho]$ divides S'. Then, $S=b\rho^m$, where $b\in H$ and m is a nonnegative integer. Let T be an element of $H[\rho]$, and v be an element of $H(\rho)$ different from 0. Suppose that $v'=Tv$. Then, $T\in H$ and $v=c\rho^t$, where t is an integer and $c\in H$.

Lemma 1. *Suppose that an element u of $H(\rho)$ satisfies $u'=a$, where $a\in H$. Then, $u\in H$.*

Proof. Suppose that $u=Q/P$, where P, $Q\in H[\rho]$ and $(P, Q)=1$. The leading coefficient of P is assumed to be one. Then, $PQ'-P'Q=aP^2$.

Hence, $P|P'$ and $P=\rho^s$, where s is a nonnegative integer. We have $Q'-sBQ=a\rho^s$. Suppose that

$$Q=\sum_{i=0}^{n} b_{n-i}\rho^i \qquad (b_0\neq 0,\ b_i\in H,\ 0\leqq i\leqq n).$$

Then, $b'_k+(n-k-s)Bb_k=0$ for any $k(0\leqq k\leqq n)$ different from $n-$s. Hence, $b_k=0$ for such k. Since $b_0\neq 0$, we have $n=s$. Suppose that $s>0$. Then, $b_n=0$. This is a contradiction to the assumption that $(P, Q)=1$. Hence, $n=s=0$.

Lemma 2. *Suppose that two elements u, v of $H(\theta)$ satisfy $u'=uv'$, and that $u\neq 0$. Then, $u\in H$.*

Proof. Suppose that $u=Q/P$, $v=S/R$ where P, Q, R, $S\in H[\theta]$ and that $(P, Q)=(S, R)=1$. The leading coefficient of R is assumed to be one. Then, $R^2(PQ'-P'Q)=PQ(RS'-R'S)$. Suppose that X is an irreducible factor of R and that $R=X^tT$, where $(X, T)=1$, $t>0$. The leading coefficient of X is assumed to be one. Then, $X^{t+1}T^2\cdot(PQ'-P'Q)=PQ\{XTS'-S(tX'T+XT')\}$. Hence, $X|P$ or $X|Q$ or $X|X'$. Suppose that $X|P$ and that $P=X^sP_0$, where $(X, P_0)=1$, $s>0$. Then,

$$X^tT^2\{XP_0Q'-Q(sX'P_0+XP'_0)\}=P_0Q\{XTS'-S(tX'T+XT')\}.$$

Hence, $X|X'$, because $(X, Q)=(X, S)=1$. Suppose that $X|Q$. Then we also have $X|X'$. Hence, in any case $X|X'$, and $X=1$. This is a contradiction. Hence, $R=1$. We have $u\in H$, because $u'=S'u$.

Lemma 3. *Suppose that two elements u, v of $H(\rho)$ satisfy $u'=uv'$, and that $u\neq 0$. Then, $v\in H$.*

Proof. Let us replace θ by ρ in the previous proof. Then, the proof goes to a conclusion that $X|X'$. In this case we have $X=\rho$. Hence, $R=\rho^m$, and $\rho^m(PQ'-P'Q)=PQ(S'-mBS)$. Suppose that $m>0$. Then, $\rho|P$ or $\rho|Q$ or $\rho|(S'-mBS)$. Suppose that $\rho|(S'-mBS)$, and that

$$S=\sum_{i=0}^{n} b_{n-i}\rho^i \qquad (b_0\neq 0,\ b_i\in H,\ 0\leqq i\leqq n).$$

Then, $b'_n - mBb_n = 0$. Hence, $b_n = 0$. This is a contradiction to the assumption that $(R, S) = 1$. Hence, ρ does not divide $S' - mBS$. Suppose that $\rho | P$, and that $P = \rho^r P_0$, where $(\rho, P_0) = 1$ and $r > 0$. Then, $\rho^m\{P_0 Q' - Q(P'_0 + rBP_0)\} = P_0 Q(S' - mBS)$, and $\rho | Q$. This contradicts to the assumption that $(P, Q) = 1$. Suppose that $\rho | Q$. Then we also meet a contradiction. Hence, $m = 0$ and $R = 1$. We have $v \in H[\rho]$, and $v' \in H[\rho]$. Hence, $v' \in H$, because $u' = uv'$. We have $v \in H$ by Lemma 1.

§2. **Proof of Theorem.** By the definition of $l(y)$, y is a solution of $l(y) = y/x$ if and only if it is a nontrivial solution of

$$x(y - x)y' = y^2. \tag{3}$$

Suppose that this equation has a nontrivial solution in a Liouville extension N of $K(x)$, where N is a subfield of Ω. Then such a solution is transcendental over $K(x)$, since $l(p)$ is transcendental over $K(x)$ for any algebraic function p of x over K different from a constant. Let M be the algebraic closure of N in Ω. Then, the field of constants in M is K. To each element u of M we can correspond a nonnegative integer $n(u)$ which satisfies the following two conditions:

(i) In M there exists such a Liouville chain $L_0 \subset L_1 \subset \cdots \subset L_{n(u)}$ over $K(x)$ that $L_{n(u)} \ni u$.

(ii) Suppose that $H_0 \subset H_1 \subset \cdots \subset H_m$ is a Liouville chain over $K(x)$ in M, and that $H_m \ni u$. Then, $m \geqq n(u)$.

For each nonnegative integer n, let $M(n)$ denote a subset $\{u \in M; n(u) = n\}$ of M. Suppose that $n > 0$. Then, to each element u of $M(n)$ we can correspond a positive integer $r_n(u)$ which satisfies the following two conditions:

(iii) In M there exists such a Liouville chain $L_0 \subset L_1 \subset \cdots \subset L_n$ that $L_n \ni u$ and the transcendental degree of L_n over L_{n-1} is $r_n(u)$.

(iv) Suppose that $H_0 \subset H_1 \subset \cdots \subset H_n$ is a Liouville chain over $K(x)$ in M, and that $H_n \ni u$. Then, the transcendental degree of H_n over H_{n-1} is not less than $r_n(u)$.

Suppose that $r_n(u) = r$. Then, there exist r elements $\mu_1, \dots, \mu_r$ of L_n which satisfy the following three conditions:

(v) u is algebraic over $L_{n-1}(\mu_1,\dots,\mu_r)$.

(vi) For each $i(1 \leqq i \leqq r)$, μ_i satisfies either $\mu_i' = A_i$ or $\mu_i' = C_i'\mu_i$, where A_i, $C_i \in L_{n-1}$.

(vii) $\mu_1,\dots,\mu_r$ are algebraically independent over L_{n-1}.

Let Γ be a subset of M consisting of all nontrivial solutions of (3) in M. Then, Γ is not empty by our assumption. There exists an element y of Γ which satisfies the following two conditions:

(viii) $n(y) = \min\{n(u);\ u \in \Gamma\}$.

(ix) $r_n(y) = \min\{r_n(u);\ u \in \Gamma \cap M(n)\}$, where $n = n(y)$.

We shall take such an element y of Γ. Suppose that $r_n(y) = r$. Then, there exist r elements $\mu_1,\dots,\mu_r$ of M which satisfy the three conditions (v)–(vii) if we replace u by y. Let L denote $L_{n-1}(\mu_1,\dots,\mu_{r-1})$ and μ denote μ_r. Then, $L(\mu)$ is either an integral extension of L or an exponential extension of L. Over $L(\mu)$, y satisfies an irreducible algebraic equation $f(y) = 0$. We shall suppose that

$$f = \sum_{i=0}^{m} \alpha_{m-i} y^i \qquad (\alpha_0 = 1,\ \alpha_i \in L(\mu),\ 1 \leqq i \leqq m).$$

We have $m \neq 1$. In fact suppose that $m = 1$. Then, $y \in L(\mu)$. It satisfies $y' = (y/x)'y$. If $L(\mu)$ is an integral extension of L, then $y \in L$ by Lemma 2. If $L(\mu)$ is an exponential extension of L, then $y/x \in L$ by Lemma 3. In any case we meet a contradiction. Differentiating $f = 0$, we have $f_x + y'f_y = 0$, where

$$f_x = \sum_{i=0}^{m} \alpha_{m-i}' y^i, \qquad f_y = \sum_{i=0}^{m} i\alpha_{m-i} y^{i-1}.$$

By (3) we have an identity $x(y-x)f_x + y^2 f_y = \{my + (x\alpha_1' - \alpha_1)\}f$ in y, since f is irreducible. Hence,

$$(\alpha_k/x^k)' = \alpha_{k-1}'/x^{k-1} + (\alpha_1/x)'(\alpha_{k-1}/x^{k-1}), \qquad 2 \leqq k \leqq m, \tag{4}$$

$$\alpha_m' + (\alpha_1/x)'\alpha_m = 0. \tag{5}$$

Let β_k denote α_k/x^k for each $k(1 \leqq k \leqq m)$. Then,

$$\beta_k' = \beta_{k-1}' + \left(\frac{k-1}{x} + \beta_1'\right)\beta_{k-1}, \quad 2 \leqq k \leqq m, \tag{6}$$

$$(7) \qquad 0=\beta'_m+\left(\frac{m}{x}+\beta'_1\right)\beta_m.$$

Suppose that $L(\mu)$ is an exponential extension of L. Then, by Lemma 3, we have $\alpha_1\in L$ because of (5). Hence, by Lemma 1, we obtain $\alpha_k\in L$, $2\leqq k\leqq m$, inductively from (4). This is a contradiction to the assumption on y. Hence, $L(\mu)$ is an integral extension of L and $\mu'=A$, where $A\in L$. By Lemma 2, we have $\alpha_m\in L$ and $\beta'_1\in L$ from (5). By (3), $L(y)$ is a differential field. By the assumption on y, it is transcendental over L. Hence, μ is algebraic over $L(y)$. We have $\mu\in L(y)$, because $\mu'=A$. Let us express μ in the form Q/P, where P, $Q\in L[y]$ and $(P, Q)=1$. The leading coefficient of P is assumed to be one. Differentiating $\mu=Q/P$, we have $Ax(y-x)P^2=PQ^*-P^*Q$, where $P^*=x(y-x)P_x+y^2P_y$ and the notation Q^* has the same meaning as P^*. Hence, $P|P^*$. Let us express P in the form

$$\sum_{i=0}^{s}a_{s-i}y^i \qquad (a_0=1,\ a_i\in L,\ 1\leqq i\leqq s).$$

Then, $P^*=\{sy+(xa'_1-a_1)\}P$. Suppose that S is an irreducible factor of P, and that $P=S^hR$, where $(S, R)=1$ and $h>0$. The leading coefficient of S is assumed to be one. Then,

$$(8) \qquad x(y-x)(hS_xR+SR_x)+y^2(hS_yR+SR_y)=\{sy+(xa'_1-a_1)\}SR.$$

An irreducible algebraic equation $S(y)=0$ has a solution z in M, since M is algebraically closed. Suppose that $z\neq 0$. Then, $z\in\Gamma$ by (8). We have either $n(z)<n(y)$ or $r_n(z)<r_n(y)$. This is a contradiction. Hence, $S=y$, and $P=y^s$. We obtain $Q-\mu y^s=0$. This algebraic equation in y over $L(\mu)$ is irreducible because $(Q, y)=1$. Suppose that $s>\deg Q$. Then, $s>0$, and the constant term c in Q is not 0. We have $f=y^s-\mu^{-1}Q$, and $\alpha_m=c/\mu$. Since $\alpha_m\in L$, this is a contradiction. Suppose that $s=\deg Q$. Then, $s>0$ because $\mu\notin L$. We have $f=(b-\mu)^{-1}\cdot(Q-\mu y^s)$, where b is the leading coefficient of Q. Hence, $\alpha_m=c/(b-\mu)$. This is also a contradiction. Hence, $s<\deg Q$. We have $f=b^{-1}(Q-\mu y^s)$, and $s>0$ because $\alpha_m\in L$. Hence, $\beta_k\in L$ for any $k(1\leqq k\leqq m)$ different from $m-s$. We shall express β_{m-s} in the form $c_0\mu+c_1$, where $c_0 = -b^{-1}x^{s-m}$ and $c_1 \in L$. First suppose that $s < m-1$. Set $k = m-s$ in (6). Then, $\beta'_{m-s} = \beta'_{m-s-1} + \{\beta'_1 + (m-s-1)/x\}\beta_{m-s-1}$. The right hand member is an element of L. Hence, $\beta'_{m-s} \in L$. Set $k = m-s-1$ in (6). Then, $\beta'_{m-s+1} = \beta'_{m-s} + \{\beta'_1 + (m-s)/x\}\beta_{m-s}$. Hence, $c_0\{\beta'_1 + (m-s)/x\} = 0$. We have $\beta'_1 + (m-s)/x = 0$ because $c_0 \neq 0$. Secondly suppose that $s = m-1$. Set $k=2$ in (6). Then, $\beta'_2 = \beta'_1 + (\beta'_1 + 1/x)\beta_1$. Hence, $(\beta'_1 + 1/x)c_0 = 0$ because $\beta'_1 \in L$. In any case we have

$$\beta'_1 + \frac{j}{x} = 0, \tag{9}$$

where j is a positive integer less than m. Integrating this equation, we get $\beta_1 = b_1 - jl(x)$, where $b_1 \in K$. By (6) and (9) we have

$$\beta'_k = \frac{1}{x}\{-j + \sum_{i=1}^{k-1}(i-j)\beta_i\}, \qquad 2 \leqq k \leqq m.$$

Integrating this equation inductively, we obtain

$$\beta_k = \sum_{i=0}^{k} c_{ki}\{l(x)\}^i, \qquad 2 \leqq k \leqq m,$$

where $c_{ki} \in K$, $0 \leqq i \leqq k$. On the other hand, we have $\beta'_m + \beta_m(m-j)/x = 0$ from (7) and (9). Integrating this equation, we obtain $\beta_m = b_2 x^{j-m}$, where $b_2 \in K$. Since f is irreducible, $\beta_m \neq 0$. Hence, we meet a contradiction, because $l(x)$ is not an algebraic function of x over K.

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Added in proof. On Liouville's general theorem stated in Remark 3, cf. Rosenlicht [8].