J. Math. Kyoto Univ. (JMKYAZ)
16-3 (1976) 531–543

A necessary condition for well-posed Cauchy problems

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(Received Oct. 24, 1975)

§ 0. Introduction

Consider the partial differential equation

$$L[u]=\left(\frac{\partial}{\partial t}\right)^m u+\sum_{|\nu|+j\leq m,\, j<m} a_{\nu j}(x, t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j u=0 \tag{0.1}$$

in $\Omega=\boldsymbol{R}^n\times[0, T]$, $T>0$. Consider the Cauchy problem for this equation with given initial data at $t=0$.

Definition. *We say that the Cauchy problem for* (0.1) *is well-posed in* $\mathscr{D}_{L^2}^\infty$, *if for any given initial data* $\varphi_j(x)\in\mathscr{D}_{L^2}^\infty$, $j=0, 1,\dots, m-1$, *there exists a unique solution* $u(x, t)\in\mathscr{E}_t^m(\mathscr{D}_{L^2}^\infty)$, $0\leq t\leq T$, *which takes the given initial data at* $t=0$:

$$\left(\frac{\partial}{\partial t}\right)^j u(x, t)|_{t=0}=\varphi_j(x), \qquad j=0, 1,\dots, m-1\,.$$

It is an interesting problem to look for a necessary condition in order that the Cauchy problem be well-posed, and this problem was studied by many authors. The following theorem proved by S. Mizohata is one of the most important results.

We consider the characteristic equation

$$p(\lambda)=\lambda^m+\sum_{|\nu|+j=m} a_{\nu j}(x, t)\xi^\nu\lambda^j=0, \qquad \xi\in\boldsymbol{R}^n\,. \tag{0.2}$$

We denote the roots by $\lambda_j(x, t; \xi)$, $j=1, 2,\dots, m$. *If the Cauchy problem for* (0.1) *with initial time* $t=0$ *is well-posed in* $\mathcal{D}^\infty_{L^2}$, *then for any* $x\in \boldsymbol{R}^n$, $\xi\in \boldsymbol{R}^n$, *every root* $\lambda_j(x, t; \xi)$ *must be real-valued at* $t=0$:

$$\mathrm{Im}\,\lambda_j(x, 0; \xi)\equiv 0, \qquad x\in \boldsymbol{R}^n, \ \xi\in \boldsymbol{R}^n, \ j=1, 2,\dots, m.$$

It is natural to pose a question of whether every root $\lambda_j(x, t; \xi)$ must be real-valued not only at the initial time $t=0$ but also at an arbitrary time $t\in[0, T]$.

In this paper we treat only the equations whose coefficients depend only on t. Namely we consider

$$(0.1')\qquad L[u]=\left(\frac{\partial}{\partial t}\right)^m u+\sum_{|\nu|+j\leq m,\, j<m} a_{\nu j}(t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j u=0$$

in $\Omega=\boldsymbol{R}^n\times[0, T]$, $T>0$. We assume that $a_{\nu j}(t)\in C^{m-1}[0, T]$ in the case of $|\nu|+j=m$ and that $a_{\nu j}(t)\in C^0[0, T]$ in the other case. Consider the charateristic equation

$$(0.2')\qquad p(\lambda)=\lambda^m+\sum_{|\nu|+j=m} a_{\nu j}(t)\xi^\nu\lambda^j=0, \qquad \xi\in \boldsymbol{R}^n.$$

We denote the roots by $\lambda_j(t; \xi)$, $j=1, 2,\dots, m$. We define

$$(0.3)\qquad t_0(\xi)=\sup\{t;\ \mathrm{Im}\,\lambda_j(s; \xi)\equiv 0,\ 0\leq s\leq t,\ j=1, 2,\dots, m\},$$

where we regard ξ as a parameter. Our aim is to prove the

Theorem. *If for some* $\xi^0(\neq 0)\in \boldsymbol{R}^n$ *the following two conditions are both satisfied, then the Cauchy problem for* (0.1′) *with initial time* $t=0$ *is not well-posed in* $\mathcal{D}^\infty_{L^2}$.

Condition A: $t_0(\xi^0)<T$.

Condition B: *There exists* t_1; $t_0=t_0(\xi^0)<t_1<T$, *such that every characteristic root* $\lambda_j(t; \xi^0)$ *belongs to* $C^0[t_0, t_1]\cap C^{m-1}(t_0, t_1]$ *and satisfies*

$$(*)\qquad \mathrm{Im}\,\lambda_j(t; \xi^0)\equiv 0 \quad\text{or}\quad \mathrm{Im}\,\lambda_j(t; \xi^0)\neq 0, \qquad t_0<t\leq t_1.^{*)}$$

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*) Taking the definition of $t_0(\xi^0)$ into account, we may suppose that at least one of roots satisfies the latter.

Example. Consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \alpha t^p \frac{\partial^2 u}{\partial x^2} + (\text{lower order terms}) = 0,$$

Where α is a constant, p is a real non-negative constant. It follows from our theorem that the constant α must be real non-negative for the Cauchy problem to be well-posed.

Remark 1. Consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} + 2\sum_{j=1}^{n} a_j(t) \frac{\partial^2 u}{\partial x_j \partial t} + \sum_{j,k=1}^{n} a_{jk}(t) \frac{\partial^2 u}{\partial x_j \partial x_k}$$

$$+ (\text{lower order terms}) = 0,$$

in $\Omega = \boldsymbol{R}^n \times [0, T]$. Here we assume that $a_j(t)$, $a_{jk}(t)$ are all real-valued analytic functions. In this case we can see easily that the *condition B* is always satisfied. Therefore as a consequence of the theorem we can say that for the Cauchy problem to be well-posed $\lambda_j(t; \xi)$ must be real-valued:

$$\operatorname{Im} \lambda_j(t; \xi) \equiv 0, \qquad 0 \leq t \leq T, \quad \xi \in \boldsymbol{R}^n, \quad j = 1, 2.$$

Remark 2. We can replace the *condition B* by the following weaker one.

Condition B′: There exists t_1; $t_0 = t_0(\xi^0) < t_1 < T$, *such that every characteristic root* $\lambda_j(t; \xi^0)$ *belongs to* $C^0[t_0, t_1] \cap C^{m-1}(t_0, t_1]$. *Moreover we can choose* $\{j_1, j_2, \dots, j_p\}$, $p \geq 1$, *from* $\{1, 2, \dots, m\}$ *so that*

$$(**) \quad \begin{cases} \operatorname{Im} \lambda_{j_k}(t; \xi^0) > 0, \quad t_0 < t \leq t_1, \qquad k = 1, 2, \dots, p, \\ \operatorname{Im} \lambda_j(t; \xi^0) < \min\limits_{k=1,\dots,p} \operatorname{Im} \lambda_{j_k}(t; \xi^0), \quad t_0 < t \leq t_1, \quad j \notin \{j_1, \dots, j_p\}. \end{cases}$$

Now we state our plan of the proof. By the Fourier transform of (0.1′) with respect to the space variables we get

$$\hat{L}[v] = \left(\frac{\partial}{\partial t}\right)^m v + \sum_{|\nu| + j \leq m} a_{\nu j}(t)(i\xi)^\nu \left(\frac{\partial}{\partial t}\right)^j v = 0. \tag{0.4}$$

This is an ordinary differential equation with a parameter $\xi\in\boldsymbol{R}^n$. To each solution $v(t;\xi)$ of (0.4) we consider its energy:

$$E(t)=\sum_{j=1}^{m}\left|\,|\xi|^{m-j}\left(\frac{\partial}{\partial t}\right)^{j-1}v(t;\xi)\right|^2. \tag{0.5}$$

We know the following proposition, [4], cf. also [3].

Proposition. 1 (*Petrowski*). *In order that the Cauchy problem for* (0.1′) *be well-posed in* $\mathcal{D}_{L^2}^\infty$, *it is necessary and sufficient that there exist positive constants* C *and* p, *independent of* ξ, *such that* any solution of (0.4) satisfies

$$E(t)\leq C(1+|\xi|)^pE(0),\qquad 0\leq t\leq T. \tag{0.6}$$

Under the assumption of the theorem, we shall construct a sequence of solutions of (0.4) which violates the estimate (0.6) whatever the constants C and p may be.

We should remark that if we assume the smoothness of characteristic roots in $[0, t_0(\xi^0)]$, the proposition 2 in §2 can be obtained much more easily.*)

§1. A preparatory consideration

In this section we consider the behavior of the roots of algebraic equations perturbed by a small parameter. The following lemma is very simple but useful.

Lemma. We assume that the algebraic equation

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*) We shall explain briefly the notations used in this paper: $x=(x_1, x_2,\dots, x_n)$. $\xi=(\xi_1, \xi_2,\dots, \xi_n)$. $\nu=(\nu_1, \nu_2,\dots, \nu_n)$, ν_j are non-negative integers. $|\nu|=\sum_{j=1}^{n}|\nu_j|$. $\left(\frac{\partial}{\partial x}\right)^\nu=\left(\frac{\partial}{\partial x_1}\right)^{\nu_1}\left(\frac{\partial}{\partial x_2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\nu_n}$. $u(x)\in\mathcal{D}_{L^2}^\infty$ means that $u(x)$ and all its derivatives $\left(\frac{\partial}{\partial x}\right)^\nu u(x)$ (in distribution sense) belong to $L^2(\boldsymbol{R}^n)$. $\mathcal{D}_{L^2}^\infty$ provided with the semi-norm $p_m(u)=\sum_{|\nu|\leq m}\left\|\left(\frac{\partial}{\partial x}\right)^\nu u(x)\right\|$ is a Frechét space. $u(x,t)\in\mathcal{E}_t^m(\mathcal{D}_{L^2}^\infty)$ means that $t\longrightarrow u(x,t)\in\mathcal{D}_{L^2}^\infty$ is continuously differentiable up to order m. $\varphi(t)\in C^k[0,t_0]$ means that $\varphi(t)$, $0\leq t\leq t_0$, is continuously differentiable up to order k.

$$f(\lambda)=\lambda^m+a_1\lambda^{m-1}+\cdots+a_m=0 \tag{1.1}$$

has only real roots. Consider the equation perturbed by a complex number z:

$$f(\lambda)+z=\lambda^m+a_1\lambda^{m-1}+\cdots+(a_m+z)=0. \tag{1.2}$$

The discriminant $D(z)$ of this equation is a polynomial of z of degree $m-1$ and has only real roots.

Proof. From its definition,

$$D(z)=\begin{vmatrix} 1 & a_1 & \cdots & (a_m+z) & & \\ & 1 & a_1 & \cdots & (a_m+z) & \\ & & \cdots\cdots & \cdots & & \\ & & & 1 & a_1 \cdots & (a_m+z) \\ 0 & m & \cdots\cdots & a_{m-1} & & \\ & 0 & m & \cdots\cdots & a_{m-1} & \\ & & \cdots & \cdots\cdots & & \\ & & & 0 & m \cdots\cdots & a_{m-1} \end{vmatrix}.$$

This is a polynomial of z of degree $m-1$ of the form:

$$D(z)=m^m z^{m-1}+b_1 z^{m-2}+\cdots+b_{m-1}\,. \tag{1.3}$$

$D(z)=0$ means that the two equations:

$$f(\lambda)+z=0, \qquad f'(\lambda)=0$$

have at least one common root. If $\operatorname{Im} z\neq 0$, the first equation has no real root. On the other hand the second one has only real roots, because (1.1) has only real roots. Therefore $D(z)$ has only real roots. *q.e.d.*

Now we consider the equation of λ

$$f(\lambda;\, t)=\lambda^m+a_1(t)\lambda^{m-1}+\cdots+a_m(t)=0\,, \tag{1.1'}$$

$0 \leq t \leq t_0$, where $a_j(t) \in C^{m-1}[0, t_0]$, $j=1, 2, \dots, m$. we assume that for any $t \in [0, t_0]$, this equation has only real roots. It is a matter of course that the roots of (1.1′) are not necessarily differentiable with respect to t.

Consider the equation perturbed by z:

$$(1.2') \qquad f(\lambda;\, t)+z=\lambda^m+a_1(t)\lambda^{m-1}+\cdots+\{a_m(t)+z\}=0\,.$$

By the above lemma, we see that the discriminant $D(z;\, t)$ of this equation is a polynomial of z of degree $m-1$ and has only real roots. Therefore we may express $D(z;\, t)$ as follows:

$$D(z;\, t)=m^m\prod_{j=1}^{m-1}\{z-\beta_j(t)\}\,,$$

where $\beta_j(t)$ are real-valued.

Now we put $z=i\varepsilon$, $\varepsilon>0$. Then it follows from the above expression that

$$(1.4) \qquad |D(i\varepsilon;\, t)| \geq m^m\varepsilon^{m-1}\,.$$

This means that for any $t \in [0, t_0]$, the equation

$$(1.5) \qquad \lambda^m+a_1(t)\lambda^{m-1}+\cdots+a_m(t)+i\varepsilon=0$$

has distinct roots. We denote them by $v_j(t;\, \varepsilon)$, $j=1, 2, \dots, m$. $v_j(t;\, \varepsilon)$ belongs to $C^{m-1}[0, t_0]$.

We put

$$m(\varepsilon)=\min_{j\neq k}\inf_{0\leq t\leq t_0}|v_j(t;\, \varepsilon)-v_k(t;\, \varepsilon)|.$$

Then we have the following estimate:

$$(1.6) \qquad m(\varepsilon)\geq c_0\varepsilon^{\frac{m-1}{2}}, \qquad c_0 \text{ be a positive constant.}$$

Here we used the well-known relation

$$D(i\varepsilon;\, t)=(-1)^{\frac{m-1}{2}}\prod_{j<k}\{v_j(t;\, \varepsilon)-v_k(t;\, \varepsilon)\}^2\,.$$

Using (1.6), we obtain the following estimates:

$$(1.7) \qquad |v_j^{(k)}(t;\varepsilon)| \leq C\varepsilon^{-\frac{k}{2}(m-1)^2-\frac{k-1}{2}(m-1)}, \qquad 0\leq t\leq t_0,$$

where $v_j^{(k)}(t;\varepsilon)$ stands for $\frac{d^k}{dt^k}v_j(t;\varepsilon)$, C is a positive constant, $j=1, 2,\dots, m$, $k=1, 2,\dots, m-1$.

Proof of (1.7). For simplicity, we show this only in case of $k=1$.

$$f(\lambda;t)+i\varepsilon=\prod_{j=1}^{m}\{\lambda-v_j(t;\varepsilon)\}.$$

Differentiating with respect to t,

$$\frac{\partial}{\partial t}f(\lambda;t)\,|_{\lambda=v_j(t;\varepsilon)}=-v_j'(t;\varepsilon)\prod_{k\neq j}\{v_j(t;\varepsilon)-v_k(t;\varepsilon)\}.$$

Therefore by (1.6)

$$|v_j'(t;\varepsilon)|\leq C\varepsilon^{-\frac{(m-1)^2}{2}}. \qquad q.e.d.$$

The following inequalities are also useful in the next section.

$$(1.8) \qquad |\operatorname{Im} v_j(t;\varepsilon)|\leq 2\varepsilon^{\frac{1}{m}}, \qquad 0\leq t\leq t_0, \quad j=1, 2,\dots, m.$$

By Rouché's theorem, we can see this easily.

§ 2. Energy estimate in $[0, t_0(\xi^0)]$

In the equation (0.4) we take $\xi=\tau\xi^0$, $\tau>0$. Namely we consider the ordinary differential equation with a parameter τ

$$(2.1) \qquad \hat{L}[v]=\left(\frac{\partial}{\partial t}\right)^m v+\sum_{|\nu|+j\leq m}a_{\nu j}(t)(i\tau\xi^0)^\nu\left(\frac{\partial}{\partial t}\right)^j v=0.$$

We define the operator $\hat{L}_0$ by

$$(2.2) \quad \hat{L}_0=\left(\frac{\partial}{\partial t}\right)^m+a_1(t)(i\tau)\left(\frac{\partial}{\partial t}\right)^{m-1}+\cdots+a_m(t)\ (i\tau)^m+i(i\tau)^m\tau^{-\sigma},$$

where $a_j(t)=\sum_{|\nu|=j}a_{\nu,m-j}(t)(\xi^0)^\nu$, $j=1, 2,\dots, m$, σ is a positive constant which shall be determined later.

By the definition of $t_0=t_0(\xi^0)$, the characteristic equation

$$\lambda^m+a_1(t)\lambda^{m-1}+\cdots+a_m(t)=0$$

has only real roots always in $[0, t_0]$. So we can use the results prepared in the preceding section, taking $\varepsilon=\tau^{-\sigma}$. We denote by $\nu_j(t;\tau)$, $j=1, 2,\dots, m$, the roots of the equation

$$\lambda^m+a_1(t)\lambda^{m-1}+\cdots+a_m(t)+i\tau^{-\sigma}=0, \qquad 0\le t\le t_0. \tag{2.3}$$

We define the operators ∂_j and π_j by

$$\begin{aligned} &\partial_0=1, \quad \partial_j=\frac{\partial}{\partial t}-i\nu_j(t;\tau)\tau, \qquad j=1, 2,\dots, m, \\ &\pi_j=\partial_j\partial_{j-1}\cdots\partial_0, \qquad j=0, 1,\dots, m. \end{aligned} \tag{2.4}$$

Taking the estimate (1.7) into account, we put $\sigma=\dfrac{2}{m(m-1)}$. Then $\pi_m-\hat{L}_0$ can be represented as follows:

$$\pi_m-\hat{L}_0=\sum_{k=1}^{m-1}\left\{b_{k1}(t;\tau)\tau^k+\cdots+b_{kk}(t;\tau)\tau\left(\frac{\partial}{\partial t}\right)^{k-1}\right\}\pi_{m-k-1}, \tag{2.5}$$

where $|b_{kj}(t;\tau)|\le C\tau^{1-\frac{1}{m}}$, $j=1, 2,\dots, k$, $k=1, 2,\dots, m-1$.*)

Next we express $\left(\dfrac{\partial}{\partial t}\right)^j$ in terms of ∂_k, i.e.

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)^j=\partial_{k+j-1}\cdots\partial_k+c_1(t;\tau)\tau\partial_{k+j-2}\cdots\partial_k+\cdots \\ +c_{j-1}(t;\tau)\tau^{j-1}\partial_k+c_j(t;\tau)\tau^j, \end{aligned} \tag{2.6}$$

where $|c_l(t;\tau)|\le$ *constant*. Here we used (1.7) again.

By (2.5) and (2.6) we have

$$\pi_m-\hat{L}=c_1(t)\pi_{m-1}+c_2(t;\tau)\tau^{1+p}\pi_{m-2}+\cdots+c_m(t;\tau)\tau^{m-1+p}\pi_0, \tag{2.7}$$

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*) Hereafter we may use the the symbols C, M, ... in order to represent positive constants which can be chosen independently of τ.

where $p=1-\min\left\{\frac{1}{m}, \frac{2}{m(m-1)}\right\}$, $|c_1(t)|\leq constant$, $|c_j(t;\tau)|\leq constant$, $j=2, 3, \dots, m$.

By the relations

$$\tau^{m-j}\pi_{j-1}v=\tau^{-j\alpha}v_j, \qquad j=1, 2, \dots, m, \ \alpha=\frac{1-p}{m}, \tag{2.8}$$

we get the system of equations equivalent to (2.1):

$$\frac{\partial}{\partial t}\vec{v}=A\tau\vec{v}+B\tau^{1-\alpha}\vec{v}, \qquad 0\leq t\leq t_0, \tag{2.9}$$

where $\vec{v}={}^t(v_1, v_2, \dots, v_m)$,

$$A=\begin{pmatrix} iv_1 & & & \\ & iv_2 & & \\ & & \ddots & \\ & & & iv_m \end{pmatrix}, \qquad B=\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & 0 & 1 \\ c_m^* & \cdots & \cdots & c_1^* \end{pmatrix},$$

$|c_j^*(t;\tau)|\leq constant$, $j=1, 2, \dots, m$.

Now we put

$$S(t)=\sum_{j=1}^{m}|v_j(t;\tau)|^2. \tag{2.10}$$

We remark that by (1.8)

$$|\mathrm{Re}(iv_j)(t;\tau)|\leq 2\tau^{-\frac{2}{m^2(m-1)}}$$

Then

$$\frac{d}{dt}S(t)=2\sum_{j=1}^{m}\mathrm{Re}\left\{v_j\overline{\frac{\partial v_j}{\partial t}}\right\}$$

$$\geq -\{C_1\tau^{1-\rho}+C_2\tau^{1-\alpha}\}S(t), \quad \rho=\frac{2}{m^2(m-1)}.$$

Because $\alpha=\frac{1-p}{m}=\min\left\{\frac{1}{m^2}, \frac{2}{m^2(m-1)}\right\}\leq\rho$,

$$\frac{d}{dt}S(t)\geq -C\tau^{1-\alpha}S(t).$$

Integrating this inequality, we have

(2.11) $$S(0)\leq\exp\{C\tau^{1-\alpha}\}S(t_0).$$

Corresponding to (0.5), we put

(2.12) $$E(t)=\sum_{j=1}^{m}\left|\tau^{m-j}\left(\frac{\partial}{\partial t}\right)^{j-1}v(t)\right|^2.$$

$E(t)$ and $S(t)$ are equivalent in the following sense:

$$mE(t)\leq S(t)\leq M\tau^2E(t),\qquad 0\leq t\leq t_0,$$

where m and M are positive constants independent of τ. Here we used (2.6). Thus we have the

Proposition 2. *For any solution* $v(t;\tau)$ *of* (2.1) *the following energy estimate holds*:

(2.13) $$E(0)\leq\exp\{M_1\tau^{1-\alpha}\}E(t_0),\qquad \textit{for large } \tau,$$

where $\alpha=\min\left\{\frac{1}{m^2},\frac{2}{m^2(m-1)}\right\}$, M_1 *is a positive constant.*

§ 3. Proof of the theorem*)

Consider the equation (2.1) in $[t_0,t_1]$. Without any loss of generality we may assume that for some $k\geq 1$

(3.1) $$\begin{aligned}&-\mathrm{Im}\,\lambda_j(t)>0,\quad t_0<t\leq t_1,\quad j=1,2,\dots,k,\\&-\mathrm{Im}\,\lambda_j(t)\leq 0,\quad t_0<t\leq t_1,\quad j=k+1,\dots,m,\end{aligned}$$

where $\lambda_j(t)=\lambda_j(t;\xi^0)$ are roots of the characteristic equation (0.2′), which belong to $C^0[t_0,t_1]\cap C^{m-1}(t_0,t_1]$.

We define the operators $\tilde{\partial}_j$ and $\tilde{\pi}_j$ by

(3.2) $$\tilde{\partial}_0=1,\quad \tilde{\partial}_j=\frac{\partial}{\partial t}-i\lambda_j(t)\tau,\qquad j=1,2,\dots,m,$$

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*) The argument in this section is due to the idea of S. Mizohata, cf. [2].

$$\tilde{\pi}_j=\tilde{\partial}_j\tilde{\partial}_{j-1}\cdots\tilde{\partial}_0, \qquad j=0, 1,\dots, m.$$

Then $\tilde{\pi}_m-\hat{L}$ can be expressed as follows:

$$\tilde{\pi}_m-\hat{L}=\sum_{k=0}^{m-1}p_k(\tau;\, t)\tilde{\pi}_{m-k-1}, \qquad t_0<t\le t_1, \tag{3.3}$$

where $p_k(\tau;\, t)=\sum_{j=0}^{k}c_{kj}(t)\tau^{k-j}$. We should remark that $c_{kj}(t)$ are continuous in $(t_0, t_1]$ but not necessarily bounded. However if we put $M(t)=\{1+\max_{\substack{j=1,\dots,m\\ k=1,\dots,m-1}}|\lambda_j^{(k)}(t)|\}^{m-1}$, then

$$|c_{kj}(t)|\le CM(t), \qquad t_0<t\le t_1, \tag{3.4}$$

where C is a positive constant.

By the relations

$$\tau^{m-j}\tilde{\pi}_{j-1}v=\tau^{-j\alpha}v_j, \quad j=1, 2,\dots, m, \quad \alpha=\frac{1}{m}, \tag{3.5}$$

we get the system of equations equivalent to (2.1):

$$\frac{\partial}{\partial t}\vec{v}=A\tau\vec{v}+B\tau^{1-\alpha}\vec{v}, \tag{3,6}$$

where $\vec{v}={}^t(v_1, v_2,\dots, v_m)$,

$$A=\begin{pmatrix} i\lambda_1 & & & \\ & i\lambda_2 & & \\ & & \ddots & \\ & & & i\lambda_m \end{pmatrix}, \qquad B=\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & 0 & 1 \\ c_m & \cdots & \cdots & c_1 \end{pmatrix},$$

$|c_j(\tau;\, t)|=|\tau^{-(j-1)-1+j\alpha}p_{j-1}(\tau;\, t)|\le CM(t)$.

Now we set $2\delta(t)=\min_{j=1,\dots,k}\{-\operatorname{Im}\lambda_j(t)\}$. Because of (3.1), $\delta(t)>0$ for $t\in(t_0, t_1]$. We define $v_j^*(t)$ by

$$v_j^*(t)=v_j(t)\exp\left\{-\tau\int_{t_0}^{t}\delta(s)\mathrm{d}s\right\}. \tag{3.7}$$

Then we obtain

(3.6′) $$\frac{\partial}{\partial t}\tilde{v}^*=A^*\tau\tilde{v}^*+B\tau^{1-\alpha}\tilde{v}^*,\quad t_0<t\leq t_1,$$

where $\tilde{v}^*={}^t(v_1^*, v_2^*,\dots, v_m^*)$, $A^*=A-\delta(t)E$, E is a unit matrix. Here we remark that $\mathrm{Re}\{i\lambda_j(t)-\delta(t)\}\geq\delta(t)$, $j=1, 2,\dots, k$, and that $\mathrm{Re}\{i\lambda_j(t)-\delta(t)\}\leq-\delta(t)$, $j=k+1,\dots, m$.

We put

(3.8) $$S(t)=\sum_{j=1}^{k}|v_j^*(t)|^2-\sum_{j=k+1}^{m}|v_j^*(t)|^2.$$

Then we have

$$\frac{d}{dt}S(t)\geq\{2\delta(t)-CM(t)\tau^{-\frac{1}{m}}\}\tau\sum_{j=1}^{m}|v_j^*(t)|^2.$$

Define t_τ by

(3.9) $$t_\tau=\inf\{t;\ \delta(s)-CM(s)\tau^{-\frac{1}{m}}\geq 0\ \textit{for any}\ s\in[t, t_1]\}.$$

Then

$$\frac{d}{dt}S(t)\geq\delta(t)\tau\sum_{j=1}^{m}|v_j^*(t)|^2\geq\delta(t)\tau S(t),\qquad t_\tau\leq t\leq t_1.$$

Integrating this inequality, we get

(3.10) $$S(t_1)\exp\left\{-\tau\int_{t_\tau}^{t_1}\delta(s)ds\right\}\geq S(t_\tau).$$

Here we remark that

(3.11) $$t_\tau\longrightarrow t_0\qquad as\quad \tau\longrightarrow\infty.$$

Now we define a sequence of solutions of (2.1). Let $v(t;\tau)$ be the solution of (2.1) which satisfies

(3.12) $$v_1^*(t_\tau;\tau)=1,\quad v_j^*(t_\tau;\tau)=0,\qquad j=2, 3,\dots, m.$$

By (3.10) and (3.11) we have

(3.13) $$S(t_1;\tau)\geq e^{c_1\tau},\qquad c_1\ \textit{be a positive constant.}$$

Here $S(t;\tau)$ stands for $S(t)$ corresponding to $v(t;\tau)$. Because $S(t_1;\tau) \leq E(t_1;\tau)$, we get

$$E(t_1;\tau) \geq e^{c_1\tau}. \tag{3.14}$$

At last we consider the equation (2.1) in $[t_0, t_\tau]$. It is easy to obtain the estimate

$$E(t_0;\tau) \leq E(t_\tau;\tau)\exp\{C_1(t_\tau - t_0)\tau\}.$$

Therefore by (2.13) we get

$$E(0;\tau) \leq E(t_\tau;\tau)\exp\{M_1\tau^{1-\alpha} + C_1(t_\tau - t_0)\tau\}.$$

If we remark that $M(t_\tau) \leq C\tau^{\frac{1}{m}}$, we have

$$E(t_\tau;\tau) \leq C_2\exp\{C_3(t_\tau - t_0)\tau\}.$$

Therefore

$$E(0;\tau) \leq C\exp\{M_1\tau^{1-\alpha} + M_2(t_\tau - t_0)\tau\}. \tag{3.15}$$

Since $t_\tau \to t_0$ as $\tau \to \infty$ and $\alpha > 0$, (3.14) and (3.15) can not be compatible with the estimate (0.6) whatever the constants C and p we may take.

Thus we have completed the proof of the theorem.

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References

[1] Lax P. D.: Asymptotic solutions of oscillatory initial value problems, *Duke Math. J.*, **24**, 627-646 (1957).

[2] Mizohata S.: Some remarks on the Cauchy problem, *J. Math. Kyoto Univ.* **1-1**, 109-127 (1961).

[3] Mizohata S: The theory of partial differential equations, *Iwanami, Tokyo* (1965). *Cambridge Univ. Press* (1973).

[4] Petrowski I. G.: Uber das Cauchysche Problem fur ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Functionen, *Bull. Univ. Etat, Moskow*, 1-74 (1938).