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On additive functionals admitting exceptional sets

By

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§ 1. Statement of Theorem

Let X be a locally compact separable Hausdorff space and m be an everywhere dense positive Radon measure on X. We consider a Hunt process $\boldsymbol{M}=\{\Omega, \mathscr{M}, X_t, \zeta, P_x\}$ on X which is m-symmetric in the sense of [1]: $\int_X p_t f(x) g(x) m(dx) = \int_X f(x) p_t g(x) m(dx)$, $f, g \in \mathscr{B}^+$, $t>0$, where p_t is the transition function of $\boldsymbol{M}$ and $\mathscr{B}^+$ is the family of all non-negative Borel functions on X.

In this paper a set $B \subset X$ is said to be an *exceptional set* if B is almost polar in the sense of [1]: $P_m(\sigma_B<\infty)=0$ where $P_m(\cdot)=\int_X P_x(\cdot) m(dx)$ and $\sigma_B=\inf\{t>0; X_t \in B\}$. When the transition function of $\boldsymbol{M}$ has a density with respect to the basic measure m, then a set B is exceptional if and only if B is polar ([1; Theorem 4]). A set B is called a *proper exceptional set* if B is Borel, $m(B)=0$ and

$$P_x(X_t \in B \text{ or } X_{t-} \in B \text{ for some } t>0)=0, \qquad \forall x \in X-B.$$

Evidently any proper exceptional set is exceptional. Conversely any exceptional set is contained in a proper exceptional set ([1; Lemma 3]). A countable union of (proper) exceptional sets is again (proper) exceptional. The notion of a proper exceptional set B is useful in that we get again an m-symmetric Hunt process on $X-B$ if we restrict the process $\boldsymbol{M}$ to $X-B$ in an obvious manner. In the following, "q.e." means "except for an exceptional set".

Let us call a non-negative Borel measure μ on X *smooth* (with respect to $\boldsymbol{M}$) if μ satisfies the following conditions:

(μ.1) μ charges no exceptional set

(μ.2) there exists an increasing sequence $\{F_n\}$ of compact sets such that

$$P_x\left(\lim_{n\to\infty} \sigma_{X-F_n}<\zeta\right)=0 \qquad \text{q.e. } x \in X \tag{1.1}$$

$$\mu(F_n)<\infty \qquad n=1,2,\cdots \tag{1.2}$$

$$\mu\left(X-\bigcup_{n=1}^{\infty} F_n\right)=0. \tag{1.3}$$

Denote by S the family of all smooth measures.

S is clearly larger than the class of all positive Radon measures charging no exceptional set. As an example, consider the measure $|x|^{\alpha} dx$ on the n-space R^n ($n \geqq 2$). This is an element of S with respect to the Brownian motion for any α. Notice that this is *not* even smooth in the sense of Mckean-Tanaka when $\alpha \leqq -2$ ([3]). Nevertheless we show in this paper that the class S characterizes the class of all finite positive continuous additive functionals defined in a relaxed way.

Let us call an extended real valued function $A_t(\omega)$, $t \geqq 0$, $\omega \in \Omega$, an *additive functional* if $A_t(\omega)$ is an additive functional in the ordinary sense but with respect to the restricted process $M|_{X-B}$, B being some proper exceptional set. More specifically our requirement for $A_t(\omega)$ is the following:

(A.1) $A_t(\cdot)$ is $\mathcal{M}_t$-measurable, $\mathcal{M}_t$ being the smallest completed sub σ-field of $\mathcal{M}$ making X_s, $s \leqq t$, measurable.

(A.2) there exist a set $\Lambda \in \bigvee_t \mathcal{M}_t$ and an exceptional set $B \subset X$ such that $P_x(\Lambda) = 1$, $\forall x \in X-B$, and moreover, for each $\omega \in \Lambda$, $A\cdot(\omega)$ is right continuous and has left limit on $[0, \infty)$, $A_0(\omega)=0$, $|A_t(\omega)| < \infty$, $\forall t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$, $\forall t \geqq \zeta(\omega)$, and $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s\omega)$, $\forall t, s \geqq 0$. Here θ_s is the shift operator: $X_t(\theta_s\omega) = X_{t+s}(\omega)$. Λ and B in the above are called a *defining set* and an *exceptional set* of the additive functional A_t respectively.

Two additive functionals $A^{(1)}$ and $A^{(2)}$ are said to be *equivalent* if for each $t > 0$ $P_x(A_t^{(1)} = A_t^{(2)}) = 1$ q.e. $x \in X$. We can then find a common defining set Λ and a common proper exceptional set B of $A^{(1)}$ and $A^{(2)}$ such that $A_t^{(1)}(\omega) = A_t^{(2)}(\omega)$, $\forall t > 0$, $\forall \omega \in \Lambda$. By *PCAF*, we mean an additive functional which is non-negative and continuous on its defining set. The class of all PCAF's is denoted by A_c^+.

We now state our theorem. We denote by E_x the integration with measure P_x and put $E_\nu(\cdot) = \int_X E_x(\cdot)\nu(dx)$ for a measure ν on X. We further use the notations $\langle \nu, h \rangle = \int_X h(x)\nu(dx)$, $(fA)_t = \int_0^t f(X_s)dA_s$, $A \in A_c^+$.

Theorem. *Assume that the Dirichlet space on $L^2(X; m)$ of the process M is regular. Then the family of all equivalence classes of A_c^+ and the family S are in one-to-one correspondence. The correspondence is characterized by the following relation:*

$$\lim_{t \downarrow 0} \frac{1}{t} E_{h\cdot m}((f\cdot A)_t) = \langle f\mu, h \rangle \tag{1.4}$$

for any γ-excessive function h ($\gamma \geqq 0$) and any $f \in \mathcal{B}^+$.

See the next section for the definition of γ-excessive functions. As an example of this theorem, consider the above cited measure $\mu(dx) = |x|^{\alpha} dx$ on R^n ($n \geqq 2$). μ then corresponds by (1.4) to the PCAF $A_t = \int_0^t |X_s|^{\alpha} ds$ of the Brownian paths. When $\alpha \leqq -2$, this is not a PCAF in the ordinary sense because $P_0(A_t = \infty) = 1$, $t > 0$ ([3]). However this is always a PCAF in our sense because we can ignore the polar set $\{0\}$

as an exceptional set of A_t. The point is that, by admitting exceptional set in the definition of additive functionals, we can attain a larger and much simpler class of smooth measures than H. P. McKean and Tanaka [3], A. D. Wentzell [6] D. Revuz [4] to decide PCAF's. It should be also mentioned that M. L. Silverstein [5] already considered PCAF's in our sense in connection with Radon measures charging no exceptional set. But he described the relation in terms of the approximate Markov process or excessive functions and did not consider thr general characterization (1.4) (see Appendix).

In this paper we just follow the methods of McKean-Tanaka and Revuz to establish the relation (1.4) first for a Radon measure μ of finite energy integral and then for a general smooth measure. It turns out that the potential theoretic lemmas of § 2 relevant to the Dirichlet space $\mathscr{F}$ work quite effectively in carrying out the programm in the present general context.

Lemma 9 in § 3 states that, when μ is of finite energy integral, the relation (1.4) can be replaced by the same formula for any non-negative bounded $h \in \mathscr{F}$ and for $f=1$. This suggest us a more general relation

$$\lim_{t \downarrow 0} \frac{1}{t} E_{h \cdot m}(A_t) = \langle T, h \rangle \tag{1.5}$$

between an additive functional A not necessarily positive nor of bounded variation and an element T in the dual space of $\mathscr{F}$ not necessarily a signed measure. Such relation will be considered in a subsequent paper.

§ 2. Preparatory lemmas

We use those notions in [1] relevant to the Dirichlet form $\mathscr{E}$ on $L^2(X; m)$ of the symmetric process $\boldsymbol{M}$. We denote $\mathscr{D}[\mathscr{E}]$ by $\mathscr{F}$ and call $(\mathscr{F}, \mathscr{E})$ the Dirichlet space. The space $(\mathscr{F}, \mathscr{E})$ is assumed to be regular in the sense that $\mathscr{F} \cap C_0(X)$ is both uniformly dense in $C_0(X)$ and $\mathscr{E}_1$-dense in $\mathscr{F}$.

A positive Radon measure μ on X is said to be *of finite energy integral* if

$$\left| \int_X v(x) \mu(dx) \right| \leq C \sqrt{\mathscr{E}_1(v, v)}, \qquad v \in \mathscr{F} \cap C_0(X) \tag{2.1}$$

for some constant $C = C(\mu)$. There exists then for each $\alpha > 0$ a unique element $U_\alpha \mu \in \mathscr{F}$ such that

$$\mathscr{E}_\alpha(U_\alpha \mu, v) = \int_X \tilde{v}(x) \mu(dx) \qquad \forall v \in \mathscr{F} \tag{2.2}$$

where $\tilde{v}$ is any quasi-continuous version of v (see for instance [2; Theorem 1.5]). $\mathscr{E}_\alpha(U_\alpha \mu, U_\alpha \mu)$ is denoted by $\mathscr{E}_\alpha(\mu)$ and called (α-) *energy integral* of μ. $U_\alpha \mu$ is called (α-) *potential* of μ. We denote by S_0 the family of all positive Radon measures of finite energy integrals. We further put

(2.3) $$S_{00}=\{\mu \in S_0;\ \mu(X)=1,\ \|U_1\mu\|_\infty<\infty\}$$

Lemma 1. *The following conditions are equivalent for $B\subset X$:*

(i) *B is exceptional (with respect to $\boldsymbol{M}$)*

(ii) *$Cap(B)=0$*

(iii) $\mu(B)=0\ \forall\mu \in S_0$

(iv) $\mu(B)=0\ \forall\mu \in S_{00}$.

Proof. The relation (i)$\Leftrightarrow$(ii) is proven in [1; Theorem 6]. The relation (ii)$\Leftrightarrow$(iii) can be seen for instance from [2; Theorem 1.5 and Theorem 1.7]. Assume that a set B satisfies condition (iv). For any non-vanishing $\mu \in S_0$, put $\Gamma_n=\{x \in X;\ \widetilde{U_1\mu}(x)\leqq n\}$. Choose compact sets K_n increasing to X and set $\mu_n(E)=\mu(E\cap\Gamma_n\cap K_n)/\mu(\Gamma_n\cap K_n)$. Since $U_1\mu_n\leqq n/\mu(\Gamma_n\cap K_n)$ q.e. on Γ_n, the same inequality holds q.e. on X ([5; Corollary 3.15]). Hence $\mu_n \in S_{00}$ and $\mu(B)=\lim_{n\to\infty}\mu(\Gamma_n\cap K_n)\mu_n(B)=0$ getting the condition (iii). q.e.d.

Condition (1.1) in the definition of the smooth measure can be restated in an analytical term:

Lemma 2. *Let $\{F_n\}$ be an increasing sequence of closed sets. $\{F_n\}$ satisfies* (1.1) *if and only if*

(2.4) $$\lim_{n\to\infty} Cap\,(K-F_n)=0 \qquad \textit{for any compact set } K.$$

Proof. Condition (2.4) is equivalent to

(2.5) $$P_x\Big(\lim_{n\to\infty}\sigma_{G-F_n}=\infty\Big)=1 \text{ q.e. } \textit{for any relatively compact open set } G$$

on account of [1; formula (9)]. (1.1) implies (2.5) because of the quasi-left continuity of the Hunt process $\boldsymbol{M}$. To get (1.1) from (2.5), it suffices to choose a sequence $\{G_l\}$ of relatively compact open sets with $\bar{G}_l\subset G_{l+1}$, $G_l\uparrow X$, and observe the inequality $\sigma_{X-F_n}\geqq\sigma_{G_l-F_n}\wedge\sigma_{X-G_l}$. q.e.d.

Lemma 3. *A measure μ is smooth if and only if there exists an increasing sequence $\{F_n\}$ of closed sets such that* (1.1) *and* (1.3) *are satisfied and that $I_{F_n}\cdot\mu \in S_0$ for each n, I_{F_n} being the indicator of F_n.*

Proof. "If" part is clear from Lemma 1 and we have only to replace $\{F_n\}$ by $\{F_n\cap\bar{G}_n\}$ where $\{G_n\}$ is a sequence of relatively compact open sets such that $\bar{G}_n\subset G_{n+1}$, $G_n\uparrow X$. "Only if" part follows immediately from a Silverstein's lemma and Lemma 2. In fact it is implied in the proof of [5; Lemma 3.18] that, for any bounded Borel measure μ charging no exceptional set, there exists an increasing sequence $\{F_n\}$ of closed sets such that $I_{F_n}\cdot\mu \in S_0$, $n=1,2,\cdots$, $\lim_{n\to\infty}\mu(X-F_n)=0$ and $\lim_{n\to\infty}Cap(X-F_n)=0$. q.e.d.

We say that a function u on X is *α-excessive* if $u(x)\geqq 0$ q.e. and $e^{-\alpha t}p_t u(x)\uparrow u(x)$, $t\downarrow 0$, q.e. Then the restriction of u outside some proper exceptional set is Borel measurable and finely continuous ([1; Lemma 3]). Such function is said to be finely continuous q.e.

Lemma 4. (i) *A function $u\in\mathcal{F}$ is a quasi-continuous version of an α-potential if and only if u is α-excessive.*

(ii) *Suppose both u and v are α-excessive, $u\leqq v$ m-a.e. and $v\in\mathcal{F}$, then $u\in\mathcal{F}$ and $\mathcal{E}(u, u)\leqq\mathcal{E}(v, v)$. In particular u is a quasi-continuous version of an α-potential.*

For the proof of (i), see [2; Theorem 1.4] and [1; Theorem 6]. (ii) is due to Silverstein [5; Lemma 3.3].

Lemma 5. *For any $u\in\mathcal{F}$, $\nu\in S_0$, $T<\infty$ and $\varepsilon>0$, it holds that*

$$P_\nu\Big(\sup_{0\leqq t\leqq T}|\tilde{u}(X_t)|>\varepsilon\Big)\leqq\frac{e^T}{\varepsilon}\sqrt{\mathcal{E}_1(\nu)}\sqrt{\mathcal{E}_1(u, u)}$$

where $\tilde{u}$ is any quasi-continuous version of u.

Proof. Put $E=\{x\in X;|\tilde{u}(x)|>\varepsilon\}$, then $P_\nu\Big(\sup_{0\leqq t\leqq T}|\tilde{u}(X_t)|>\varepsilon\Big)=P_\nu(\sigma_E\leqq T)<e^T E_\nu(e^{-\sigma_E})$ since $\tilde{u}$ is fine continuous q.e. ([1; Theorem 6]). On the other hand, the function $p(x)=E_x(e^{-\sigma_E})$ is a quasi-continuous version of the (1-) equilibrium potential in view of Lemma 4 and [1; Lemma 2]. Hence

$$E_\nu(e^{-\sigma_E})=\int_X p(x)\nu(dx)=\mathcal{E}_1(p, U_1\nu)\leqq\sqrt{\mathcal{E}_1(\nu)}\sqrt{Cap(E)}\leqq\frac{1}{\varepsilon}\sqrt{\mathcal{E}_1(\nu)}\sqrt{\mathcal{E}_1(u, u)}.\quad\text{q.e.d.}$$

Lemma 6. *Let $\{u_n\}$ be a sequence of quasi-continuous functions belonging to $\mathcal{F}$. Suppose $\{u_n\}$ constitutes an $\mathcal{E}_1$-Cauchy sequence, then there exists a subsequence $\{n_k\}$ such that $P_x(u_{n_k}(X_t)$ converges uniformly in t on each compact subinterval of $[0, \infty))=1$ q.e. $x\in X$.*

Proof. By virtue of Lemma 5, there exists a subsequence $n_1<n_2<\cdots<n_k<\cdots$ independent of $T>0$ and $\nu\in S_0$ such that

$$P_\nu\Big(\sup_{0\leqq t\leqq T}|u_{n_k}(X_t)-u_{n_{k+1}}(X_t)|>\frac{1}{2^k}\Big)\leqq e^T 2^{-k}\sqrt{\mathcal{E}_1(\nu)},\quad T>0,\ \nu\in S_0.$$

This means $P_\nu(\Lambda)=0$ for $\Lambda=\Big\{\sup_{0\leqq t\leqq T}|u_{n_k}(X_t)-u_{n_{k+1}}(X_t)|>\frac{1}{2^k}\text{ i.o.}\Big\}$. Since ν is arbitrary, we get $P_x(\Lambda)=0$ q.e. $x\in X$ in view of Lemma 1. q.e.d.

For a Borel set $B\subset X$, we put $\mathcal{F}_{X-B}=\{u\in\mathcal{F};\tilde{u}=0$ q.e. *on* $B\}$. This is a closed subspace of $(\mathcal{F}, \mathcal{E}_\alpha)$. Denote by $\mathcal{H}_\alpha^B$ the orthogonal complement:

$$\mathcal{F}=\mathcal{F}_{X-B}\oplus\mathcal{H}_\alpha^B. \tag{2.6}$$

$P_{\mathscr{H}_\alpha^B}$ will stand for the projection operator on the space $\mathscr{H}_\alpha^B$. The next lemma relates this operator to the average by the hitting distribution:

$$H_\alpha^B u(x) = E_x(e^{-\alpha\sigma_B} u(X_{\sigma_B})). \tag{2.7}$$

Lemma 7. *Let u be a quasi-continuous function belonging to the space $\mathscr{F}$. Then $H_\alpha^B u$ is a quasi-continuous version of $P_{\mathscr{H}_\alpha^B} u$.*

In particular, if $u \in \mathscr{F}$ is α-excessive, then $H_\alpha^B u$ is an α-potential of a measure of S_0 supported by $\bar{B}$.

The proof of this lemma was first given in [2; Lemma 3.4] but a simpler proof has been presented in [5; Theorem 7.3].

§ 3. PCAF for a Radon measure of finite energy integral

Proposition 1. *For $\mu \in S_0$, there exists $A \in A_c^+$ such that the function of x $E_x\left(\int_0^\infty e^{-t} dA_t\right)$ is a quasi-continuous version of the potential $U_1\mu$.*

Proof. A version u of $U_1\mu$ can be choosen as follows: u is a non-negative finite Borel function on X and for some proper exceptional set B, $nR_{n+1}u(x) \uparrow u(x)$, $n \to \infty$, $^\forall x \in X - B$, and $u(x) = 0$, $^\forall x \in B$, where $\{R_\alpha, \alpha > 0\}$ is the resolvent of M. If we set

$$g_n(x) = \begin{cases} n(u(x) - nR_{n+1}u(x)) & x \in X - B \\ 0 & x \in B \end{cases}$$

then $g_n \cdot m \to \mu$ vaguely, $R_1 g_n(x) \uparrow u(x)$, $x \in X - B$, and moreover $R_1 g_n$ is $\mathscr{E}_1$-convergent to u.

We define an approximating 1-order PCAF $\tilde{A}_n$ by

$$\tilde{A}_n(t, \omega) = \int_0^t e^{-s} g_n(X_s(\omega)) ds.$$

Then for any $\nu \in S_{00}$,

$$E_\nu((\tilde{A}_n(+\infty) - \tilde{A}_l(+\infty))^2) \leq 2M_\nu \sqrt{\mathscr{E}_1(\mu)} \sqrt{\mathscr{E}_1(R_1 g_n - R_1 g_l, R_1 g_n - R_1 g_l)} \tag{3.1}$$

where $M_\nu = \| U_2 \nu \|_\infty$. In fact by setting $g_{n,l} = g_n - g_l$, $n > l$, the left hand side of (3.1) is equal to

$$\begin{aligned} 2E_\nu&\left(\int_0^\infty e^{-s} g_{n,l}(X_s) ds \int_s^\infty e^{-u} g_{n,l}(X_u) du\right) \\ &= 2E_\nu\left(\int_0^\infty e^{-2s} g_{n,l}(X_s) R_1 g_{n,l}(X_s) ds\right) = 2\langle \nu, R_2(g_{n,l} R_1 g_{n,l})\rangle = 2(U_2\nu, g_{n,l} R_1 g_{n,l}) \\ &\leq 2(U_2\nu, g_n R_1 g_{n,l}) \leq 2M_\nu(g_n, R_1 g_n - R_1 g_l) = 2M_\nu \mathscr{E}_1(R_1 g_n, R_1 g_n - R_1 g_l). \end{aligned}$$

Since $E_\nu(\tilde{A}_n(+\infty)/\mathscr{M}_t) = \tilde{A}_n(t) + e^{-t} E_{X_t}(\tilde{A}_n(+\infty)) = \tilde{A}_n(t) + e^{-t} R_1 g_n(X_t)$, we see that

(3.2) $$M_n(t)=\tilde{A}_n(t)+e^{-t}R_1g_n(X_t), \qquad 0\leqq t\leqq +\infty$$

is a martingale with respect to $(\mathcal{M}_t, P_\nu)$, $\nu \in S_{00}$. By Doob's inequality

$$P_\nu\Big(\sup_{0\leqq t\leqq\infty}|M_n(t)-M_l(t)|>\varepsilon\Big)\leqq\frac{1}{\varepsilon^2}E_\nu((\tilde{A}_n(+\infty)-\tilde{A}_l(+\infty))^2)$$

which, combined with (3.1), means that there exists a subsequence $\{n_k\}$ and

(3.3) $$P_\nu(M_{n_k}(t) \text{ converges uniformly on } [0, \infty))=1, \qquad \forall\nu \in S_{00}.$$

In view of (3.2), (3.3), Lemma 1 and Lemma 6, we can conclude that, by selecting a new subsequence if necessary, $P_x(\Lambda)=1$, $x \in X-\tilde{B}$, where

(3.4) $\Lambda=\{\omega \in \Omega;\ \tilde{A}_{n_k}(+\infty, \omega)<\infty,\ \tilde{A}_{n_k}(t, \omega)$ *converges uniformly in* t *on each finite interval of* $[0, \infty)\}$

and $\tilde{B}$ is some proper exceptional set containing B.

Let us put $\tilde{A}(t, \omega)=\lim_{n_k\to\infty}\tilde{A}_{n_k}(t, \omega)$ for $\omega \in \Lambda$ and $\tilde{A}(t, \omega)=0$ for $\omega \notin \Lambda$. We further put $A(t, \omega)=\int_0^t e^s d\tilde{A}(s, \omega)$. A is then a PCAF with Λ and $\tilde{B}$ being its defining set and exceptional set respectively.

In order to complete the proof, it suffices to show $E_\nu(\tilde{A}(+\infty))=\langle\nu, u\rangle$, $\forall\nu \in S_{00}$, on account of Lemma 1 and Lemma 4. Since $M_n(+\infty)=\tilde{A}_n(+\infty)$ is $L^2(P_\nu)$-convergent, so is the martingale $M_n(t)$. Hence $E_\nu(\tilde{A}(t))+e^{-t}\langle\nu, p_t u\rangle=\lim_{n\to\infty}E_\nu(M_n(t))$ $=\lim_{n\to\infty}E_\nu(\tilde{A}_n(+\infty))=\lim_{n\to\infty}\langle\nu, R_1g_n\rangle=\langle\nu, u\rangle$. By letting t tend to infinity and noting the bound $\langle\nu, p_t u\rangle=\mathcal{E}_1(U_1\nu, p_t u)\leqq\sqrt{\mathcal{E}_1(\nu)}\sqrt{\mathcal{E}_1(u, u)}$, we get the desired equality.

q.e.d.

Lemma 8. *Consider* μ *and* A *of the preceding proposition. Then for any* $\alpha>0$ *and bounded non-negative function* f, $E_x\Big(\int_0^\infty e^{-\alpha t}f(X_t)dA_t\Big)$ *is a quasi-continuous version of* $U_\alpha(f\cdot\mu)$.

Proof. It is sufficient to consider the case that $\alpha=1$ and $f=I_G$, G being an open set with $\mu(\partial G)=0$. Put

$$\phi(x)=E_x\Big(\int_0^\infty e^{-t}I_G(X_t)dA_t\Big), \qquad \psi(x)=E_x\Big(\int_0^\infty e^{-t}I_{X-G}(X_t)dA_t\Big),$$

then both ϕ and ψ are 1-excessive and $\phi+\psi=U_1\mu$. By Lemma 4, ϕ and ψ are quasi-continuous versions of α-potentials of some measures λ and $\nu \in S_0$ respectively.

We know from Lemma 7 and the equalities $\phi=H_1^G\phi$, $\psi=H_1^{X-G}\psi$ that Supp $[\lambda]$ $\subset\bar{G}$ and Supp $[\nu]\subset X-G$. Since $\mu=\lambda+\nu$, we have $\lambda=I_G\cdot\mu$. q.e.d.

Proposition 2. *For* $\mu \in S_0$, $A \in \mathbf{A}_c^+$ *of Proposition* 1 *is unique up to the equivalence.*

Proof. Suppose that $A^{(1)}, A^{(2)} \in A_c^+$ are associated with $\mu \in S_0$ in the manner of Proposition 1. Then

$$E_x\left(\int_0^\infty e^{-t}dA_t^{(1)}\right)=E_x\left(\int_0^\infty e^{-t}dA_t^{(2)}\right)=u(x), \qquad x \in X-B,$$

for some proper exceptional set B. By the same computation as in the proof of Proposition 1,

$$v_{ij}(x)=E_x\left(\int_0^\infty e^{-t}dA_t^{(i)}\int_0^\infty e^{-t}dA_t^{(j)}\right)=E_x\left(\int_0^\infty e^{-2t}u(X_t)dA_t^{(i)}\right), \quad x \in X-B,\ i=1,2.$$

Put $u_n=u\wedge n$. By virtue of Lemme 8, we have

$$\langle \nu, v_{ij}\rangle=\lim_{n\to\infty} E_\nu\left(\int_0^\infty e^{-2t}u_n(X_t)dA_t^{(i)}\right)=\lim_{n\to\infty}\langle \nu, \widetilde{U_2(u_n\cdot\mu)}\rangle=\langle U_2\nu\cdot u, \mu\rangle<\infty$$

for any $\nu \in S_{00}$. Hence

$$E_\nu\left(\left\{\int_0^\infty e^{-t}dA_t^{(1)}-\int_0^\infty e^{-t}A_t^{(2)}\right\}^2\right)=\langle \nu, v_{11}-2v_{12}+v_{22}\rangle=0, \qquad \nu \in S_{00},$$

from which follows $A^{(1)}\sim A^{(2)}$. q.e.d.

In the following, we denote $E_x\left(\int_0^\infty e^{-\alpha t}f(X_t)dA_t\right)$ by $U_A^\alpha f(x)$ for $A \in A_c^+$ and $f \in \mathscr{B}^+$.

Lemma 9. *For $\mu \in S_0$ and $A \in A_c^+$, the next conditions are equivalent each other:*

(i) $U_A^1 1$ *is a quasi-continuous version of* $U_1\mu$.

(ii) $(h, U_A^\alpha f)=\langle f\cdot\mu, R_\alpha h\rangle$, $\alpha>0$, $f, h \in \mathscr{B}^+$.

(iii) $E_{h\cdot m}((fA)_t)=\int_0^t \langle f\cdot\mu, p_s h\rangle ds$, $t>0$, $f, h \in \mathscr{B}^+$.

(iv) $\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}((fA)_t)=\langle f\cdot\mu, h\rangle$ *for any γ-excessive function h ($\gamma\geqq 0$) and* $f \in \mathscr{B}^+$.

(v) $\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}(A_t)=\langle \mu, \tilde{h}\rangle$, $h \in \mathscr{B}^+\cap\mathscr{F}$.

(vi) $\alpha(h, U_A^{\alpha+\gamma}f)\uparrow\langle f\cdot\mu, h\rangle$, $\alpha\uparrow\infty$, *for any γ-excessive function h ($\gamma\geqq 0$) and* $f \in \mathscr{B}^+$.

(vii) $\lim_{\alpha\to\infty}\alpha(h, U_A^\alpha 1)=\langle \mu, \tilde{h}\rangle$, $h \in \mathscr{B}^+\cap\mathscr{F}$.

Proof. (i) is equivalent to (ii) by Lemma 8. We can also see the equivalence of (ii) and (iii) by the uniqueness of the Laplace transform. The implications (iii)$\Rightarrow$(iv)$\Rightarrow$(vi)$\Rightarrow$(ii), (iii)$\Rightarrow$(v) and (ii)$\Rightarrow$(vii) are clear. Suppose that (v) is satisfied and put $c_t(x)=E_x(A_t)$. Then,

$$\int_0^t \langle\mu, p_s h\rangle ds = \lim_{u\downarrow 0}\frac{1}{u}\int_0^t (p_s h, c_u)ds = \lim_{u\downarrow 0}\frac{1}{u}\int_0^t (h, c_{u+s}-c_s)ds$$
$$=\lim_{u\downarrow 0}\frac{1}{u}\int_t^{t+u}(h, c_s)ds - \lim_{u\downarrow 0}\frac{1}{u}\int_0^u (h, c_s)ds = (h, c_t)$$

for any γ-excessive function h in $\mathscr{F}$. By taking the Laplace transform we get (ii) for h of this type and $f=1$, which is enough to obtain (i). In the same way we can derive the implication (vii)$\Rightarrow$(i). q.e.d.

Lemma 10. *Let $\mu \in S_0$ and $A \in A_c^+$ be related as in Lemma 9. Then for any closed set $F\subset X$*

$$\alpha E_{h\cdot m}\left(\int_0^{\sigma_{X-F}} e^{-(\alpha+\gamma)t} f(X_t)dA_t\right) \uparrow \int_{F^{(0)}} h(x)f(x)\mu(dx), \qquad \alpha\uparrow\infty,$$

where h is any γ-excessive function ($\gamma \geqq 0$), $f \in \mathscr{B}^+$ and $F^{(0)}$ is the fine interior of F: $F^{(0)} = \{x \in X: P_x(\sigma_{X-F}>0)=1\}$ ($\subset F$).

Proof. It suffices to give the proof when h is bounded excessive and belongs to $L^2(X; m)$ and $f \in \mathscr{B}^+$ is bounded. By Lemma 8, $U_A^\alpha f$ is a quasi-continuous version of $U_\alpha(f\mu)$. On account of Lemma 7,

$$\alpha E_{h\cdot m}\left(\int_0^{\sigma_{X-F}} e^{-\alpha t} f(X_t)dt\right)$$
$$=\alpha(h, U_A^\alpha f - H_\alpha^{X-F}U_A^\alpha f) = \alpha\mathscr{E}_\alpha(R_\alpha h, U_A^\alpha f - H_\alpha^{X-F}U_A^\alpha f)$$
$$=\alpha\mathscr{E}_\alpha(R_\alpha h - H_\alpha^{X-F}R_\alpha h, U_A^\alpha f) = \alpha\mathscr{E}_\alpha(R_\alpha^F h, U_\alpha(f\mu)) = \alpha\int_X R_\alpha^F h(x)f(x)\mu(dx)$$

which increases to $\int_{F^{(0)}} h(x)f(x)\mu(dx)$ as $\alpha\uparrow\infty$. Here

$$R_\alpha^F h(x) = E_x\left(\int_0^{\sigma_{X-F}} e^{-\alpha t}h(X_t)dt\right).$$ q.e.d.

§ 4. Proof of Theorem

Theorem is divided into two propositions.

Proposition 3. *Given $A \in A_c^+$, there exists a unique $\mu \in S$ such that* (1.4) *holds.*

Proof. For a given $A \in A_c^+$ with a proper exceptional set $N\subset X$, we put $\phi(x)=E_x\left(\int_0^\infty e^{-t}f(X_t)e^{-A_t}dt\right)$, $x \in X-N$, where f is a Borel function in $L^2(X; m)$ such that $f(x)>0$, $\forall x \in X$. Then $\phi(x)>0$, $\forall x \in X-N$.

It can be seen that

(4.1) $$U_A^1\phi(x) = R_1 f(x) - \phi(x), \qquad x \in X-N.$$

In particular $U_A^1\phi$ is a 1-excessive function dominated by $R_1 f \in \mathscr{F}$. Hence $U_A^1\varphi$ is by Lemma 4 a quasi-continuous function of $\mathscr{F}$ and so is ϕ by (4.1) again. Accordingly there exists a sequence $\{E_n\}$ of increasing closed sets such that $Cap(X-E_n)\downarrow 0$, $n\to\infty$, $N\subset\bigcap_{n\infty 1}^{\infty}(X-E_n)$ and $\phi|_{E_n}$ is continuous for each n.

Let us put

$$F_n=\left\{x\in E_n;\ \phi(x)\geqq\frac{1}{n}\right\}, \tag{4.2}$$

and prove that $\{F_n\}$ satisfies condition (1.1). To this end, set $B_n=\left\{x\in X-N;\ \phi(x)\leqq\frac{1}{n}\right\}$, and $\sigma=\lim_{n\to\infty}\sigma_n$. Since ϕ is fine continuous on $X-N$, we have for $x\in X-N$, $E_x\left(\int_{\sigma_n}^{\infty}e^{-t}f(X_t)e^{-A_t}dt\right)=E_x(e^{-\sigma_n}e^{-A_{\sigma_n}}\phi(X_{\sigma_n}))\leqq\frac{1}{n}$. By letting n tend to infinity, we can see $P_x(\sigma<\zeta)=0$, $x\in X-N$, in view of the strict positivity of f. Hence $\{F_n\}$ satisfies (1.1) because of the inclusion $X-F_n\subset(X-E_n)\cup B_n$ and [1; formula (9)].

Now put $A_n=I_{F_n}\cdot A$. On account of the inequality $U_{A_n}^1 1\leqq nU_A^1\phi$ ($\in\mathscr{F}$) and Lemma 4, there exists a unique $\mu_n\in S_0$ such that $U_{A_n}^1 1$ is a quasi-continuous version of the potential $U_1\mu_n$. But then

$$\mu_n=I_{F_n}\cdot\mu_l,\qquad n<l \tag{4.3}$$

because $U_{A_n}^1 1=U_A^1 I_{F_n}$ is a version of $U_1 I_{F_n}\cdot\mu_l$ by Lemma 8. We can now define a measure μ by $I_{F_n}\cdot\mu=I_{F_n}\cdot\mu_n$, $n=1,2,\cdots$, $\mu\left(X-\bigcup_n F_n\right)=0$. μ is smooth in view of Lemma 3.

It remains to show that A and μ are related by (1.4). By Lemma 9, we see for any $f\in\mathscr{B}^+$ and γ-excessive function h,

$$\langle f\mu, h\rangle=\lim_{n\to\infty}\lim_{\alpha\to\infty}\alpha(h,\ U_{A_n}^{\alpha+\gamma}f)=\lim_{\alpha\to\infty}\lim_{n\to\infty}\alpha(h,\ U_A^{\alpha+\gamma}(I_{F_n}\cdot f)).$$

By virtue of (1.1), we get $\langle f\mu, h\rangle=\lim_{\alpha\to\infty}\alpha(h,\ U_A^\alpha f)$ which is obviously equivalent to (1.4). q.e.d.

Proposition 4. *Given $\mu\in S$, there exists $A\in A_c^+$ uniquely up to the equivalence such that* (1.4) *holds.*

Proof. Consider $\mu\in S$ with $\{F_n\}$ satisfying the conditions of Lemma 3. From Proposition 1, Proposition 2 and Lemma 9, we see that there exists $A^{(n)}\in A_c^+$ uniquely up to the equivalence such that $A^{(n)}$ is related to $I_{F_n}\cdot\mu$ by (1.4). But then

$$A^{(n)}\sim I_{F_n}\cdot A^{(l)},\qquad n<l \tag{4.4}$$

because $I_{F_n}\cdot A^{(l)}$ is related by (1.4) to $I_{F_n}\cdot I_{F_l}\cdot\mu=I_{F_n}\cdot\mu$.

Choose a proper exceptional set N and a defining set Λ which are common to all $A^{(n)}$ such that, for any $\omega \in \Lambda$, $A_t^{(n)}(\omega)=(I_{F_n}\cdot A^{(n+1)})_t(\omega)$, $\forall t>0$, $n=1,2,\cdots$, and $\sigma(\omega)\left(=\lim_{n\to\infty}\sigma_{X-F}(\omega)\right)\geqq\zeta(\omega)$. Put for $\omega\in\Lambda$

$$\begin{cases}A_t(\omega)=A_t^{(n)}(\omega), & \sigma_{X-F_{n-1}}(\omega)\leqq t<\sigma_{X-F_n}(\omega),\ n=1,2,\cdots \\ A_t(\omega)=A_{\sigma(\omega)-}(\omega), & t\geqq\sigma(\omega).\end{cases}$$

Obviously $A\in A_c^+$. Since $A_t=A_t^{(n)}$ for $t<\sigma_{X-F_n}$, we see from Lemma 10,

$$\alpha E_{h\cdot m}\left(\int_0^{\sigma_{X-F_n}} e^{-(\alpha+r)t}f(X_t)dA_t\right)\uparrow\int_{F_n^{(0)}} h(x)f(x)\mu(dx), \qquad \alpha\uparrow\infty,$$

$F_n^{(0)}$ being the fine interior. Notice that the set $\bigcup_n F_n-\bigcup_n F_n^{(0)}$ is exceptional because of (1.1). By letting n tend to infinity, we get $\lim_{\alpha\to\infty}\alpha(h, U_A^{\alpha+r}f)=\langle f\cdot\mu, h\rangle$ proving that A is related to μ by (1.4).

The uniqueness of A is then clear because $I_{F_n}\cdot A$ is related to $I_{F_n}\cdot\mu$ in the manner of the preceding section for each n. q.e.d.

Appendix

In § 3, we saw that the function $U_\alpha\mu$ for $\mu\in S_0$ can be expressed as the potential of the associated $A\in A_c^+$. It is convenient to give simple conditions for a more general excessive function to be expressible this way. We state here a criterion of this type due to M. L. Silverstein.

Let $\{D_n\}$ be a sequence of relatively compact open sets such that $\bar{D}_n\subset D_{n+1}$, $n=1,2,\cdots$, and $D_n\uparrow X$, $n\to\infty$. We put $\tau_n=\sigma_{X-D_n}$, $n=1,2,\cdots$, and fix $\alpha>0$.

Proposition 5. *Let u be an α-excessive function on X satisfying the following conditions*:

(i) $\lim E_x(e^{-\alpha\tau_n}u(X_{\tau_n}))=0$, *$m$-a.e. $x\in X$.*

(ii) *there exists, for each n, an α-excessive function $v_n\in\mathcal{F}$ such that $u\leqq v_n$ q.e. on D_n. Then u is the α-potential of a unique $A\in A_c^+$, namely $u(x)=E_x\left(\int_0^\infty e^{-\alpha t}dA_t\right)$* q.e.

By taking $v_n(x)=C\cdot E_x(e^{-\alpha\sigma_{D_n}})$, $x\in X$, $n=1,2,\cdots$, we get

Corollary. *If u is a bounded α-excessive function satisfying the condition* (i) *above, then u is the α-potential of a unique $A\in A_c^+$.*

Let $A\in A_c^+$ and $\mu\in S$ be related to each other by our Theorem. By Lemma 3, we can see that the formula of Lemma 9 (ii) still holds for A and μ. Hence Proposition 5 follows from the next proposition.

Proposition 6. *Let u be the function of Proposition 5. Then there exists a unique $\mu \in S$ such that*

$$(h, u) = \langle f \cdot \mu, R_a h \rangle, \qquad f, h \in \mathscr{B}^+.$$

μ is moreover a positive Radon measure charging no exceptional set.

This proposition is almost the same as Theorem 9.3 of M. L. Silverstein [5] and can be proved using Lemma 7.

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