On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots

By

Jiro TAKEUCHI

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1. Introduction.

Consider a linear partial differential operator

\( P(x; D_x, D_t) = D_t^n + a_t(x; D_x) D_t^{n-1} + \cdots + a_m(x; D_x), (x, t) \in \mathbb{R}^1 \times [0, T] = \Omega \)

where \( a_t(x; D_x) \) \( (1 \leq i \leq m) \) is a linear partial differential operator in \( \mathbb{R}^1 \).

It is said that \( P(x; D_x, D_t) \) defined by (1.1) is non-kowalewskian if

\[ \max_{1 \leq j \leq m} \text{order } a_j(x; D_x) = b > 1. \]

Denote the homogeneous part of order \( j_b \) of \( a_j(x; D_x) \) by \( a^j(x; D_x) \).

\[ P^j(x; \xi, \tau) = \tau^m + a^j(x; \xi) \tau^{m-1} + \cdots + a^1(x; \xi) \]

is said to be the principal symbol of \( P(x; D_x, D_t) \). \( D_t = -i\frac{\partial}{\partial t}, D_x = -\frac{\partial}{\partial x} \).

Consider the forward and backward Cauchy problem

\[ \begin{cases} P(x; D_x, D_t)u(x, t) = f(x, t) & \text{on } \Omega \\ D_t^j u(x, t_0) = g_j(x), \quad j = 0, 1, \ldots, m-1 \quad \text{for any } t_0 \in [0, T]. \end{cases} \]  

As is well known, it is necessary for the forward and backward Cauchy problem (1.4) to be \( H^m \)-wellposed that the characteristic equation in \( \tau \) \( P^j(x; \xi, \tau) = 0 \) has the only real roots for any \( (x, \xi) \in \mathbb{R}^1 \times \mathbb{R}^1 \). (cf. Petrowskii [4] and Mizohata [3]). As a corollary it follows from \( H^m \)-wellposedness that \( b = \max \{ \text{order } a_j/j; 1 \leq j \leq m \} \) is an integer if we assume that \( b > 1 \).

Denote the characteristic roots by \( \lambda_j(x, \xi) \), i.e.

\[ P^j(x; \xi, \tau) = \prod_{j=1}^m (\tau - \lambda_j(x, \xi)). \]

From now on we only consider the case where \( b = 2 \).

We shall give sufficient conditions for the forward and backward Cauchy problem to have a unique solution in \( L^2(\mathbb{R}^1) \).

We assume the following conditions.
Condition (A). The characteristic roots $\lambda(x, \xi)$ are non-zero, real and distinct for $(x, \xi) \in \mathbb{R}^1 \times \mathbb{R}^n \setminus \{0\}$, more precisely,

$$
\inf_{j \neq k, m} \frac{\lambda_j(x, \xi)}{\lambda_k(x, \xi) - \lambda_m(x, \xi)} > 0, \quad (x, \xi) \in \mathbb{R}^1 \times \mathbb{R}^n \setminus \{0\}
$$

Condition (B). For each $j$,

$$
H_{j, \varphi}(x, \xi) = h_j(x, \xi)
$$

has a $C^m$ bounded real solution $\varphi_j(x, \xi)$ homogeneous of degree 0 in $\xi$. Here

$$
H_f g = \{f, g\} = \sum_{j=1}^n \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right)
$$

denotes the Poisson bracket and $H_f$ the Hamilton field,

$$
h_j(x, \xi) = \frac{1}{2} \sum_{m=1}^n \left[ l_{j, m}(x, \xi) (\lambda_j(x, \xi) I - M_\mu(x, \xi)) r_{j, m}(x, \xi) \right]
$$

$$
M_j(x, \xi) = \begin{pmatrix}
0 & |\xi|^2 & 0 \\
\vdots & \vdots & \vdots \\
-\alpha^{\mu}_j(x, \xi/|\xi|)|\xi|^2 & \cdots & \alpha^{\mu}_j(x, \xi/|\xi|)|\xi|^2 \\
\vdots & \vdots & \vdots \\
0 & \cdots & -\alpha^{\nu}_j(x, \xi/|\xi|)|\xi|^2 & \cdots & a_j(x, \xi/|\xi|)|\xi|^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha^{\mu}_j(x, \xi/|\xi|)|\xi|^2 & \cdots & -\alpha^{\nu}_j(x, \xi/|\xi|)|\xi|^2 & \cdots & a_j(x, \xi/|\xi|)|\xi|^2 \\
\end{pmatrix},
$$

$a_j(x, \xi)$ is the homogeneous part of degree $2j-1$ of $a_j(x, \xi)$,

$$
M_j(x, \xi) = M_j(x, \xi) + \frac{1}{2i} \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial \xi_j} M_j(x, \xi),
$$

$l_j(x, \xi$) (resp. $r_j(x, \xi$)) is a left (resp. right) null vector of $\lambda_j(x, \xi) I - M_\mu(x, \xi)$ which is homogeneous of degree 0 in $\xi$ such that $l_j(x, \xi)r_\mu(x, \xi) = \delta_\mu (\text{Kronecker's delta})$ and $f^{(\partial^m \partial \xi)}(x, \xi) = (iD_\xi)^m D_\xi^m f(x, \xi)$.

For the global existence theorem in $C^m$ class for (1.8), we refer the reader to Duistermaat-Hörmander [1, Theorems 6.4.2 and 6.4.3].

Our result is the following
Theorem 1.1. Assume that the conditions (A) and (B) hold. For \( f(t) = f(x, t) \in C([0, T]; H^q(R^t)) \) and \( g_j(x), \ldots, g_m(x) \in H^{2m}(R^t) \times H^{2(m-1)}(R^t) \times \cdots \times H^q(R^t) \), the forward and backward Cauchy problem (1.4) has a unique solution
\[
\text{(1.14)} \quad u(t) = u(x, t) \in C([0, T]; H^2) \cap C^1([0, T]; H^{2(m-1)}) \cap \cdots \cap C^{m-1}([0, T]; H^2)
\]
and energy inequality
\[
\text{(1.15)} \quad \|u(t)\| \leq C(T) \left\{ \|u(t_0)\|^2 + \left| \int_{t_0}^{t} \|f(s)\|^2 \, ds \right| \right\}, \quad t, t_0 \in [0, T]
\]
holds where
\[
\text{(1.16)} \quad \|u(t)\| = \sum_{j=1}^{m} \|D_j^{1-j}u(t)\|_{L^2}^{2(m-j)}
\]
and \( \| \cdot \| \) is \( H^q(R^t) \)-norm.

As a special case, consider an operator with constant leading coefficients as 2-evolution, that is, an operator whose principal part \( P(x; D_x, D_t) \) defined by (1.3) has constant coefficients. In this case Condition (B) reduces to a more explicit condition as follows.

Condition (B')
\[
\text{(1.17)} \quad \varphi_j(x, \xi) = \int_0^1 \left( e_{\xi}^{ij} \xi^{j} x \right) l_j(\xi) \Im M_i \left( x - t \frac{\nabla \xi^{ij}}{\nabla \xi^{i}} \right) \frac{\xi}{\nabla \xi^{ij}} r_j(\xi) \, dt
\]
is a bounded function on \( R^t \times R^t \setminus 0, j = 1, \ldots, m \).

As a corollary of Theorem 1.1 we have the following

Theorem 1.2. Let \( P(x; D_x, D_t) \) be an operator with constant leading coefficients as 2-evolution. Assume that the conditions (A) and (B') hold. Then the same assertion as Theorem 1.1 holds.

2. Reduction to a system and its diagonalization.

Let \( P(x; D_x, D_t) \) be a differential operator;
\[
\text{(2.1)} \quad P(x; D_x, D_t) = D^n_t + a_1(x; D_x) D^{n-1}_t + \cdots + a_m(x; D_x)
\]
on \( \Omega \)
where
\[
\text{(2.2)} \quad a_j(x; D_x) = \sum_{i=1}^{m} a_{ij} D_x^i, \quad a_{ij}(x) \in \mathbb{R}^m(R^t), \quad \text{(i.e. } b=2 \text{ in (1.2)}\]
Put
\[
\text{(2.3)} \quad a_j(x; \xi) = \sum_{i=1}^{m} a_{ij}(x) \xi^i, \quad s=0, 1, \ldots, 2j.
\]
We consider the Cauchy problem
\[
\text{(2.4)} \quad \begin{cases}
P(x; D_x, D_t) u(x, t) = f(x, t) & \text{on } \Omega \\
D_t^j u(x, t_0) = g_j(x), & j=0, 1, \ldots, m-1, \quad t_0 \in [0, T].
\end{cases}
\]
We put
\( u_j(x, t) = (A^2+1)^{-1} D_t^{-1} u(x, t), \quad j = 1, \ldots, m, \)

\( U(x, t) = \langle u_1(x, t), \ldots, u_m(x, t) \rangle. \)

Then we have a system of the following form

\[
\begin{align*}
D_t U(x, t) &= M(x; D_x) U(x, t) + F(x, t) \\
U(x, t_0) &= G(x).
\end{align*}
\]

Here \( M(x; D_x) = M_2 + M_1 + M_0 \) is a pseudodifferential operator of order 2, \( M_j \) is a pseudodifferential operator of homogeneous order \( j \) \((j = 1, 2)\), and \( M_0 \) is a pseudodifferential operator of order 0. The symbol \( \sigma(M_j) = \sigma(M_j(x, \xi)) \) of \( M_j(x; D_x) \) has the following form

\[
M_2(x; \xi) = \begin{bmatrix}
0 & |\xi|^2 & 0 \\
& & \ddots & \ddots & 0 \\
& & & |\xi|^2 \\
-\alpha_{2}^0(x, \xi/|\xi|) |\xi|^2 & \cdots & -\alpha_{2}^j(x, \xi/|\xi|) |\xi|^2 \\
\end{bmatrix},
\]

\[
M_1(x; \xi) = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots \\
\end{bmatrix},
\]

\[
-\alpha_{1}^j(x, \xi/|\xi|) |\xi|^2 & \cdots & -\alpha_{1}^j(x, \xi/|\xi|) |\xi|^2 \\
\end{bmatrix},
\]

\[
M_0(x; \xi) = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots \\
\end{bmatrix},
\]

\[
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots \\
\end{bmatrix},
\]

\[
-\alpha_{1}^j(x, \xi/|\xi|) |\xi|^2 & \cdots & -\alpha_{1}^j(x, \xi/|\xi|) |\xi|^2 \\
\end{bmatrix},
\]

\[
F(x, t) = \langle 0, \ldots, 0, f \rangle,
\]

\[
G(x) = \langle (A^2+1)^{-1} g_0(x), (A^2+1)^{-1} g_1(x), \ldots, g_{m-1}(x) \rangle.
\]

The Condition (A) implies that the system (2.7) is diagonalizable as follows.

**Proposition 2.1.** Under the Condition (A) there exist a diagonal pseudodifferential operator \( \mathcal{D}(x; D_x) \) of order 2 and a pseudodifferential operator \( N(x; D_x) \) of order 0 such that

\[
N(x; D_x) (D_t - M(x; D_x)) = (D_t - \mathcal{D}(x; D_x)) N(x; D_x), \quad \text{(mod. } S^3) \]

\[
|\det N(x; \xi)| \geq \delta > 0 \quad \text{for } (x; \xi) \in R^l \times R^l.
\]

**Proof.** At first consider the equation

\[
N(x; D_x) M(x; D_x) = \mathcal{D}(x; D_x) N(x; D_x), \quad \text{(mod. } S^3).
\]

We put

\[
N(x; \xi) = N_0(x; \xi) + N_\omega(x; \xi),
\]

\[
\mathcal{D}(x; \xi) = \mathcal{D}_0(x; \xi) + \mathcal{D}_\omega(x; \xi),
\]

\[
N_j(x; \xi), \mathcal{D}_j(x; \xi) \text{ are homogeneous of degree } j \text{ in } \xi.
\]

Then (2.14) implies that

\[
N_\omega(x; \xi) M_\omega(x; \xi) = \mathcal{D}_\omega(x; \xi) N_\omega(x; \xi).
\]
Since
\[(2.16) \quad \det(\tau I - M_0(x ; \xi)) = P(\tau, \xi) = \prod_{j=1}^{m}(\tau - \lambda_j(x, \xi)),\]
we have
\[(2.17) \quad \varpi_0(x ; \xi) = \begin{bmatrix}
\lambda_1(x, \xi) & & 0 \\
& \ddots & \\
0 & & \lambda_m(x, \xi)
\end{bmatrix}
\]
and
\[(2.18) \quad N_0(x ; \xi) = \begin{bmatrix}
l_1(x, \xi) \\
\vdots \\
l_m(x, \xi)
\end{bmatrix}.
\]
Here \(l_j(x, \xi)\) is a left nullvector of \(\lambda_j(x, \xi)I - M_0(x ; \xi)\) which is homogeneous of degree 0 in \(\xi\) such that
\[(2.19) \quad |\det N_0(x ; \xi)| \geq \delta > 0 \quad \text{for} \quad (x, \xi) \in R^l \times R^l.
\]
Next, consider the equation (2.12) (mod. \(S^\circ\)), that is,
\[(2.20) \quad N_0(x ; \xi) = N_0(x ; D)M_0(x ; D) + (N_0(x ; \xi)M_1(x ; \xi) + N_{-1}(x ; \xi)M_0(x ; \xi))
\[(\mod. S^\circ) .
\]
It follows from (2.20) that
\[(2.21) \quad \sum_{\alpha=1}^{m} N_0^{(\alpha)}(x ; \xi)M_{\alpha}(x ; \xi) = (N_0(x ; \xi)M_1(x ; \xi) + N_{-1}(x ; \xi)M_0(x ; \xi))
\]
\[= \sum_{\alpha=1}^{m} (\varpi_0^{(\alpha)}(x ; \xi)N_{\alpha}(x ; \xi)) + (\varpi_0(x ; \xi)N_{-1}(x ; \xi) + \varpi_0(x ; \xi)N_0(x ; \xi)).\]
We put \(N_{-1}(x ; \xi)N_0^{(\alpha)}(x ; \xi) = \tilde{N}_{-1}(x, \xi) = (\tilde{n}_{ij}(x, \xi)), \) then we have
\[(2.22) \quad \tilde{N}_{-1}(x ; \xi)\varpi_0(x ; \xi) - \varpi_0(x ; \xi)\tilde{N}_{-1}(x ; \xi)
\[= \varpi_0(x, \xi) - \left\{N_0(x ; \xi)M_1(x ; \xi)N_0^{-1}(x ; \xi)
\right\}
\[= \tilde{n}_{ij}(x ; \xi) - \sum_{\alpha=1}^{m} \left(\varpi_0^{(\alpha)}(x ; \xi)N_{\alpha}(x ; \xi) - N_0^{(\alpha)}(x ; \xi)M_{\alpha}(x ; \xi)N_0^{-1}(x ; \xi)\right)\}
\]
We put \(R_{ij}(x, \xi) = (r_{ij}(x, \xi))\) where
\[(2.23) \quad R_{ij}(x, \xi) = N_0(x ; \xi)M_1(x ; \xi)N_0^{-1}(x ; \xi)
\[= \sum_{\alpha=1}^{m} \left(\varpi_0^{(\alpha)}(x ; \xi)N_{\alpha}(x ; \xi) - N_0^{(\alpha)}(x ; \xi)M_{\alpha}(x ; \xi)N_0^{-1}(x ; \xi)\right)\}
\]
Then we choose \(\varpi_0(x ; \xi)\) such that
\[(2.24) \quad \varpi_0(x ; \xi) = \text{diagonal of } R_{ij}(x, \xi).
\]
Define
\[(2.25) \quad \tilde{n}_{ij}(x ; \xi) = \begin{cases}
(\lambda_i(x, \xi) - \lambda_j(x, \xi))^{-1}r_{ij}(x ; \xi) & (i \neq j) \\
0 & (i = j).
\end{cases}
\]
Then $\varrho(x; \xi)$ and $N_{-1}(x; \xi) = \tilde{N}_{-1}(x; \xi)N_0(x; \xi)$ satisfy (2.21). This completes the proof.

3. Condition (C); $\varrho^*(x, D) = \varrho(x, D) \pmod{S^\circ}$.

In this section under the condition (A) we analyse the condition (B). We start from the following

**Proposition 3.1.** Let $P(x, D)$ be a scalar pseudodifferential operator on $\mathbb{R}^d$ with symbol $\sigma(P) = p(x, \xi) = p_2(x, \xi) + p_1(x, \xi) + \cdots$ ($p_j$ is homogeneous of degree $j$ in $\xi$). Denote by $P^*(x, D)$ formally adjoint operator to $P(x, D)$. Then we have

\[(3.1) \quad P^*(x, D) = P(x, D) \pmod{S^\circ}\]

if and only if

\[(3.2) \quad p_2(x, \xi) \text{ and } p_1(x, \xi) \text{ are real-valued functions}\]

where $p_1(x, \xi)$ is the subprincipal symbol of $P(x, D)$, i.e.

\[(3.3) \quad p_1(x, \xi) = \frac{1}{2\pi} \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{\alpha_1}}{\alpha!} \frac{\partial^{\alpha}}{\partial x_1 \cdots \partial \xi_\alpha} p_2(x, \xi).\]

**Proof.** By well known formula for pseudodifferential operators we have

\[(3.4) \quad \sigma(P^*) = \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{\alpha_1}}{\alpha!} p_2^{(\alpha)}(x, \xi)\]

\[= \bar{p}_2(x, \xi) + \left( \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{\alpha_1}}{\alpha!} \bar{p}_2^{(\alpha)}(x, \xi) \right) + \cdots.\]

Thus $P^*(x, D) = P(x, D) \pmod{S^\circ}$ holds if and only if

\[(3.5) \quad \begin{cases} p_2(x, \xi) = p_2(x, \xi) \\ p_1(x, \xi) + \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{\alpha_1}}{\alpha!} \bar{p}_2^{(\alpha)}(x, \xi) = p_1(x, \xi), \end{cases}\]

that is,

\[(3.6) \quad \begin{cases} \text{Im } p_2(x, \xi) = 0, \\ \text{Im } p_1(x, \xi) - \text{Im} \left( p_1(x, \xi) - \frac{1}{2\pi} \sum_{\alpha \in \mathbb{N}^d} \bar{p}_2^{(\alpha)}(x, \xi) \right) = 0. \end{cases} \quad \text{(Q. E. D.)}\]

Now we back to section 2 and analyse the condition (B). We calculate the subprincipal symbol of $\varrho(x, D)$.

**Lemma 3.2.** We have

\[(3.7) \quad \varrho_1(x, \xi) = \varrho_0(x, \xi) - \frac{1}{2} \sum_{\alpha \in \mathbb{N}^d} \varrho_0^{(\alpha)}(x, \xi)\]

\[= \text{diagonal of } \left[ \tilde{N}_\alpha(x, \xi)M_\alpha(x, \xi)N_0^{-1}(x, \xi) \right]\]

\[- \frac{1}{2} \sum_{\alpha \in \mathbb{N}^d} \left( \varrho_0(x, \xi)N_0^{(\alpha)}(x, \xi)N_0^{-1(\alpha)}(x, \xi) \right)\]
Non-kowalewskian equation

\(-N_0 \omega_0(x, \xi) M_4(x, \xi) N_0^{-1}(\omega_0(x, \xi))\)
\(-N_0(\omega_0(x, \xi)) N_0^{-1}(\omega_0(x, \xi)) M_4(x, \xi) N_0^{-1}(\omega_0(x, \xi))\)
\(-2(\omega_0^2(x, \xi) N_0(x, \xi) - \omega_0(x, \xi) N_0(x, \xi))\)

where \(M_4(x, \xi)\) is the subprincipal symbol of \(M(x, D)\) defined by (1.13).

**Proof.** Using the identities

\[N_0(x, \xi) M_4(x, \xi) N_0(x, \xi)^{-1} = \mathcal{D}_2(x, \xi),\]

and

\[\mathcal{D}_2(x, \xi) = \text{diagonal of } R_4(x, \xi),\]

where

\[R_4(x, \xi) = N_0(x, \xi) M_4(x, \xi) N_0^{-1}(x, \xi) + \sum_{|\alpha|=1} (N_0^{\alpha}(x, \xi) M_4(x, \xi) - \mathcal{D}_2^{\alpha}(x, \xi) N_0(x, \xi) N_0^{-1}(x, \xi),\]

we have the above result after elementary but tedious calculus. (Q.E.D.)

From the above lemma we have

**Lemma 3.3.** Under the condition (A),

\[(3.8) \quad \text{Im } \mathcal{D}_2(x, \xi) = 0\]

holds if and only if

\[(3.9) \quad I_j(x, \xi) (\text{Im } M_4(x, \xi)) r_j(x, \xi) = \frac{1}{2i} \sum_{|\alpha|=1} \left[ I_j^{\alpha}(x, \xi) (\lambda_j(x, \xi) I - M_4(x, \xi)) r_j^{\alpha}(x, \xi) \right.\]
\[-I_j^{\alpha}(x, \xi) (\lambda_j(x, \xi) I - M_4(x, \xi)) r_j^{\alpha}(x, \xi)\]
\[-(\lambda_j(x, \xi), I_j(x, \xi)) r_j(x, \xi) = 0, \quad j = 1, \ldots, m.\]

**Remark 3.4.** The condition (3.9) is invariant for the choice of the null vectors satisfying \(I_j(x, \xi) r_k(x, \xi) = \delta_{jk}\) except the last term \((\lambda_j(x, \xi), I_j(x, \xi)) r_j(x, \xi)\).

We replace the null vectors \(I_j(x, \xi), r_j(x, \xi)\) by

\[(3.10) \quad N_0(x, \xi) = \begin{cases} \text{exp}(\varphi_j(x, \xi)) l_j(x, \xi), & r_j(x, \xi) = \text{exp}(-\varphi_j(x, \xi)) r_j(x, \xi), \\
\{\lambda_j(x, \xi), I_j(x, \xi)\} r_j(x, \xi) = \delta_{jk}, \end{cases}\]

then we have

\[(3.11) \quad \{\lambda_j(x, \xi), I_j(x, \xi)\} r_j(x, \xi) \]
\[= \{\lambda_j(x, \xi), \varphi_j(x, \xi)\} + \{\lambda_j(x, \xi), I_j(x, \xi)\} r_j(x, \xi)\]
\[= H_j \varphi_j(x, \xi) + \{\lambda_j(x, \xi), I_j(x, \xi)\} r_j(x, \xi).\]

Thus we have proved
Lemma 3.5. Under the condition (A)

\[(3.8) \quad \text{Im} \mathcal{Q}_i(x, \xi) = 0\]

holds if and only if there exists a $C^\infty$ real-valued solution homogeneous of degree 0 in $\xi$ for the equation

\[(3.12) \quad H_{ij}(x, \xi) = h_j(x, \xi), \quad j = 1, \ldots, m,\]

where $h_j(x, \xi)$ is defined by (1.10).

Boundedness of a solution of (3.12) needs for $N(x, D)$ to satisfy the condition (2.13).

As a conclusion of this section we have proved the following

Proposition 3.6. The conditions (A) and (B) imply the condition (C); $\mathcal{Q}_+(x, D) = 0 (\text{mod. } S^0)$ and the condition $|\det N(x, \xi)| \geq \delta > 0$ for $(x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^4$.


In this section we derive an energy inequality for solutions of the equation (1.4), \((b=2)\). Let

\[(4.1) \quad P(x; D, D_t) = D^n + a_1(x, D)D^{n-1} + \cdots + a_m(x, D), \quad D = D_x,\]

be an operator satisfying the conditions (A) and (B). Consider the equation

\[(4.2) \quad P(x; D, D_t)u(x, t) = f(x, t), \quad (x, t) \in \Omega.\]

As in section 2, we reduce (4.2) to a system

\[(4.3) \quad L(x; D, D_t)U(x, t) = D_tU(x, t) - M(x, D)U(x, t) = F(x, t)\]

where

\[(4.4) \quad U(x, t) = \langle (A^2 + 1)^{-1}u(x, t), (A^2 + 1)^{-2}D_tu(x, t), \ldots, D^{n-1}u(x, t) \rangle,\]

and

\[(4.5) \quad M(x, D) = \begin{pmatrix} 0 & & A^2 \cdot & \cdot & \cdot & 0 \\ & \ddots & & \ddots & \ddots & \ddots \\ & & \ddots & & \ddots & \ddots \\ & & & \ddots & & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{pmatrix} - a_m(x, D')|D|^2 \cdot \ddots \cdot a_1(x, D')|D|^2 \quad (\text{mod. } S^0).\]

At first we derive an energy inequality for solutions of (4.3). Let $U(x, t) = U(t)$ be a solution of (4.3) with
Non-kowalewskian equation

\[ F(x, t) = F(t) \in C^0([0, T]; \Pi H^\alpha(R^d)) \]
such that
\[ U(t) \in C^0([0, T]; \Pi H^\alpha(R^d)) \cap C^1([0, T]; \Pi H^{\alpha+1}(R^d)) \]

In section 2 we have diagonalized (4.3) as follows:
\[ (D_x - \mathcal{D}(x, D))N(x, D)U(x, t) = B(x, D)U(x, t) + N(x, D)F(x, t). \]
Here \( \mathcal{D} \) is a diagonal pseudodifferential operator of order 2, \( B \) and \( N \) pseudodifferential operators of order 0 such that \( |\text{det} \sigma(N(x, \xi))| \geq 0 \).

If \( U(t) \in C^0([0, T]; \Pi H^\alpha) \cap C^1([0, T]; \Pi H^{\alpha+1}) \), then \( NU(t) \in C^0([0, T]; \Pi H^\alpha) \cap C^1([0, T]; \Pi H^{\alpha+1}) \). We set
\[ V(x, t) = N(x, D)U(x, t). \]

It follows from (4.8) that
\[ \frac{d}{dt} \| V(t) \|^2 = 2 \text{Re}\left( \frac{d}{dt} V(t), V(t) \right) = 2 \text{Re}(i\mathcal{D} V(t), V(t)) + 2 \text{Re}(iB U(t) + iN F(t), V(t)). \]

By virtue of the condition (C) we have
\[ |\text{Re}(i\mathcal{D} V(t), V(t))| \leq \text{const} \| V(t) \|^2. \]
Thus we have
\[ \frac{d}{dt} \| V(t) \|^2 \leq \text{const} (\| V(t) \|^2 + \| U(t) \|^2 + \| NF(t) \|^2). \]

We set
\[ [U(t)]^2 = \| NU(t) \|^2 + \beta \|(A^2 + 1)^{-1} U(t) \|^2, \quad (\beta > 0 \text{ sufficiently large}). \]
Then \( [U(t)] \) defines an equivalent \( L^2 \)-norm to \( \| U(t) \| \), uniformly in \( t \in [0, T] \).

Operate \( (A^2 + 1)^{-1} \) to (4.3) we have
\[ \frac{d}{dt} (A^2 + 1)^{-1} U(t) = i(A^2 + 1)^{-1} MU(t) + i(A^2 + 1)^{-1} F(t). \]
It follows from (4.14) that
\[ \frac{d}{dt} \| (A^2 + 1)^{-1} U(t) \|^2 \leq \text{const} \| U(t) \|^2 + \| (A^2 + 1)^{-1} F(t) \|^2. \]
From (4.12) and (4.15) it follows that
\[ \frac{d}{dt} [U(t)]^2 \leq \gamma [U(t)]^2 + [F(t)]^2 \quad (\gamma > 0). \]
This implies that
\[ [U(t)]^2 \leq C(T) \left\{ [U(t_0)]^2 + \left| \int_{t_0}^t [F(s)]^2 ds \right| \right\}. \]
Thus we have proved the following

**Proposition 4.1.** Assume that the conditions (A) and (B) hold for (4.3). For $F(t) \in C([0, T]; \overline{H^0})$ and solutions $U(t) \in C([0, T]; \overline{H^0})$ of (4.3) the energy inequality

$$
\|U(t)\|^2 \leq C(T) \left\{ \|U(t_0)\|^2 + \left| \int_{t_0}^t \|F(s)\|^2 \, ds \right| \right\}
$$

holds where $C(T)$ is a positive constant independent of $U(t)$ and $F(t)$.

In view of (4.2), (4.3) and (4.4) we have the following

**Proposition 4.2.** Assume that the conditions (A) and (B) hold for (4.1). For $f(t) \in C([0, T]; H^0)$ and solutions $u(t) \in C([0, T]; H^{1_0}) \cap C^{n-1}([0, T]; H^0)$ of (4.2) the energy inequality

$$
\|u(t)\|^2 \leq C(T) \left\{ \|u(t_0)\|^2 + \left| \int_{t_0}^t \|f(s)\|^2 \, ds \right| \right\}
$$

holds where

$$
\|u(t)\|^2 = \sum_{j=1}^\infty \|D_t^{j-1}u(t)\|_{L^{m-j}}^2.
$$

5. **Proof of Theorem 1.1.**

As in section 4 we define an inner product $(\cdot, \cdot)_H$ and a norm $\|\cdot\|_H$ equivalent to the usual $L^2(\mathbb{R}^3)$-inner product and $L^2(\mathbb{R}^3)$-norm as follows:

$$
(U(t), V(t))_H = (N(x, D)U(t), N(x, D)V(t)) + c_0((A^2+1)^{-1}U(t), (A^2+1)^{-1}V(t))
$$

for large positive $c_0$ (fixed),

$$
\|U(t)\|_H = \sqrt{(U(t), U(t))_H} \quad \text{for } U(t), V(t) \in C([0, T]; \overline{H^0}).
$$

By virtue of (2.13) there exist positive constants $c_1(T)$, $c_2(T)$ such that

$$
c_1(T) \|U(t)\|_H \leq \|U(t)\|_H \leq c_2(T) \|U(t)\|_H \quad \text{for } t \in [0, T].
$$

We define the Hilbert space $H = \overline{H^0(\mathbb{R}^3)}$ with norm $\|\cdot\|_H$. We have reduced (2.4) to a system (2.7). We take for the domain of definition $D(M)$ of $M(x, D)$ the Sobolev space $\overline{H^0(\mathbb{R}^3)}$.

**Lemma 5.1.** Assume that the conditions (A) and (B) hold. Then there exist a constant $\beta$ and a positive constant $\delta_0$ such that

$$
\|(\lambda I - iM(x, D))U\|_H \geq \beta \|U\|_H + \delta_0 \|U\|_H
$$

holds for real $\lambda$ such that $\lambda > \beta$ and $U(x) \in \overline{H^0}$ which shows that $(\lambda I - iM)$ is one-to-one from $\overline{H^0}$ to $\overline{H^0}$ and the image $(\lambda I - iM)\overline{H^0}$ is closed in $\overline{H^0}$. 

Proof. For \( U(x) \in \mathbb{H}^s(R^n) \) and real \( \lambda \) we have

\[
\| (\lambda - iM)U \|_{\mathbb{K}}^2 = \lambda^2 \| U \|_{\mathbb{K}}^2 - 2\lambda \text{Re} (iMU, U)_{\mathbb{K}} + \| MU \|_{\mathbb{K}}^2,
\]

and

\[
2 \text{Re} (iMU, U)_{\mathbb{K}} = i \{(MU, U)_{\mathbb{K}} - (U, MU)_{\mathbb{K}} \}
\]

\[
= i \{(NU, NU) - (NU, NMU) \}
\]

\[
+ i \{(\partial NU, NU) - (NU, \partial NU) \}
\]

\[
+ i \{(MU, NU) - (NU, BU) \}
\]

\[
+ i \{(NU, BU) - (NU, BU) \}
\]

\[
= i \{(\partial NU, NU) - (NU, \partial NU) \}. \tag{5.6}
\]

By virtue of the condition (C) we have

\[
| (\partial NU, NU) - (NU, \partial NU) | = |((\partial - \partial^*)(NU, NU))| \leq r_1 \| NU \|^2 \quad (r_1 > 0). \tag{5.7}
\]

Thus (5.6) and (5.7) imply that

\[
|2 \text{Re} (iMU, U)_{\mathbb{K}}| \leq r \| U \|_{\mathbb{K}}^2 \quad (r > 0). \tag{5.8}
\]

Therefore for large \( \lambda \) we have

\[
\| (\lambda - iM)U \|_{\mathbb{K}}^2 \geq (|\lambda| - \beta_1)^2 \| U \|_{\mathbb{K}}^2 + \| MU \|_{\mathbb{K}}^2. \tag{5.9}
\]

By the definition and the condition (A) we have

\[
\| MU \|_{\mathbb{K}}^2 \geq \| NNU \|^2 + c_5 \| (A^4 + 1)MU \|^2 \geq \| NU \|^2 - c_4 \| U \|^2 \geq \delta_4 \| NU \|^2 - c_4 \| U \|^2 \tag{5.10}
\]

and

\[
\| NU \|^2 \geq c_4 \| (A^4 + 1)NU \|^2 \geq c_4 \| N(A^4 + 1)U \|^2 - c_4 \| U \|^2 \tag{5.11}
\]

where \( N' \) is a pseudodifferential operator of order 0 such that

\[
|\det \sigma (N')(x, \xi)| \geq \delta_2 > 0. \tag{5.12}
\]

From (5.11) and (5.12) it follows that

\[
\| NU \|^2 \geq \delta_6 \| (A^4 + 1)U \|^2 - c_4 \| U \|^2 \geq \delta_6 \| U \|^2 - c_4 \| U \|^2 \tag{5.13}
\]

and

\[
\| MU \|_{\mathbb{K}}^2 \geq \delta_6 \| U \|^2 - c_4 \| U \|^2. \tag{5.14}
\]

(5.4) follows from (5.9) and (5.14). (Q. E. D.)
**Lemma 5.2.** The formally adjoint operator \( L^*(x; D, D_t) = D_t - M^*(x, D) \) satisfies the conditions (A) and (B) for some diagonalizer \( \tilde{N}(x, D) \). More precisely we have

\[
\tilde{N}(x, D)L^*(x; D, D_t) = (D_t - M^*(x, D))\tilde{N}(x, D) \quad \text{(mod. } S^o)\]

and

\[
|\det \sigma(\tilde{N})(x, \xi)| \geq \delta^* > 0 \quad \text{for } (x, \xi) \in \mathbb{R}^1 \times \mathbb{R}^1.
\]

From Lemmas 5.1 and 5.2 we have

**Lemma 5.3.** For \( V(x) \in \tilde{H}^2(\mathbb{R}^1) \) and real \( \lambda(\lambda \geq \beta') \) we have

\[
\|(\lambda I - iM^*)V\|^2_\delta \geq (|\lambda| - \beta')^2 \|V\|^2_\delta + \delta'_0 \|V\|^2_\delta \quad (\delta'_0 > 0)
\]

where

\[
\|V\|^2_\delta = \|\tilde{N}(x, D)V\|^2 + c\|(M^* + 1)^{-1}V\|^2 \quad (c: \text{large positive constant})
\]

which is an equivalent norm to \( \tilde{H}^* \)-norm \( \| \cdot \| \).

**Lemma 5.4.** The image \( (\lambda I - iM)\mathbb{H} \) is dense in \( \mathbb{H}^0 \) for large \( |\lambda|, \lambda \in \mathbb{R} \).

**Proof.** Suppose that the image is not dense in \( \mathbb{H}^0 \). Then there exists a \( V(x) \in \mathbb{H}^0, V \neq 0 \) such that

\[
(\lambda I - iM)V = 0 \quad \text{for all } U \in \mathbb{H}^2,
\]
a fortiori for all \( U \in \mathbb{H}^0 \). This implies that

\[
(\lambda I + iM^*)V = 0.
\]

It follows from (5.20) that \( M^*V \in \mathbb{H}^0 \). Denote by \( \phi(\xi) \) a \( C^\infty(\mathbb{R}^1) \) function such that

\[
\phi(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq 1 \\ 0 & \text{for } |\xi| \geq 2 
\end{cases}
\]

and

\[
0 \leq \phi(\xi) \leq 1.
\]

Define \( \phi_n(\xi) = \phi\left(\frac{\xi}{n}\right) \), \( \phi_n^{(\alpha)}(\xi) = \left(\frac{\partial}{\partial \xi}\right)^\alpha \phi_n(\xi) \) and

\[
\phi_n(D)f(x) = (2\pi)^{-1} \int e^{ix\xi} \phi_n(\xi)f(\xi) d\xi.
\]

It follows from (5.21) that \( \phi_n(D)V(x) \) and \( \phi_n(D)M^*V \) belong to \( H^0 \). Applying the inequality (5.17) we have

\[
0 = \|\phi_n(D)(\lambda I + iM^*)V\|^2_\delta
\]

\[
= \|\lambda I + iM^*\phi_n(D)V - i[M^*, \phi_n(D)]V\|^2_\delta.
\]
Expanding the commutator we have

\begin{equation}
[M^*, \psi_n(D)] V(x) = \sum_{i \geq 1, i \leq 2} \frac{(-1)^i}{\nu!} \delta(D)^{x^{(2)}} - \sum_{i \geq 1, i \leq 3} \frac{(-1)^i}{\nu!} \delta(D)^{x^{(3)}} - c[M^*, \psi_n(D)] V^3.
\end{equation}

The order of $D^2M^*$ is 2, thus we have

\begin{equation}
\|R^2(V)\| \leq \text{const} n^{-2} \|V\|.
\end{equation}

From (5.23) and (5.26) it follows that

\begin{equation}
0 \geq (|\lambda| - \beta^2) \|\phi_n(D) V\|_2 + \delta' \|\phi_n(D) V\|_3 - \text{const} \sum_{i \geq 1, i \leq 2} \|\phi_n^{(i)}(D) V\|_3 - \text{const} n^{-2} \|V\|^2.
\end{equation}

More generally we have

\begin{equation}
0 \geq (|\lambda| - \beta^2) \|\phi_n^{(i)}(D) V\|_2 + \delta' \|\phi_n^{(i)}(D) V\|_3 - \text{const} \sum_{i \geq 1, i \leq 2} \|\phi_n^{(i)}(D) V\|_3 - \text{const} n^{-2} \|V\|^2.
\end{equation}

For large positive $R$, it follows from (5.27) and (5.28) that

\begin{equation}
(\text{5.29})_1 \quad 0 \geq (|\lambda| - \beta^2) \sum_{i \geq 1, i \leq 2} R \|\phi_n^{(i)}(D) V\|_2 + \delta' \sum_{i \geq 1, i \leq 2} R \|\phi_n^{(i)}(D) V\|_3 - \text{const} n^{-2} R \|V\|^2.
\end{equation}

Summing up these inequalities we have

\begin{equation}
0 \geq (|\lambda| - \beta^2) \sum_{i \geq 1, i \leq 2} R \|\phi_n^{(i)}(D) V\|_2 + \delta' \sum_{i \geq 1, i \leq 2} R \|\phi_n^{(i)}(D) V\|_3 - \text{const} n^{-2} R^2 \|V\|^2.
\end{equation}
We choose a constant $R$ such that
\[
\delta' R - \text{const} > 0, \\
\delta' R^2 - \text{const} R - \text{const} > 0.
\]
Since $\|\phi_n(D)V\| \to \|V\|$ as $n \to \infty$, there exists a positive constant $n_0$ such that
\[
\|\phi_n(D)V\|^2 \geq \frac{1}{2} \|V\|^2 \quad \text{for } n \geq n_0.
\]
Thus for $n \geq n_0$, we have
\[
0 \geq \frac{1}{2} (|\lambda| - \beta)^2 \|V\|^2 - \text{const}(1 + R + R^2) n^{-2} \|V\|^2 \\
+ (|\lambda| - \beta)^2 \sum_{k=1}^n R^{(k)} \|\phi_k(D)V\|^2.
\]
If $\|V\| \neq 0$ and $|\lambda|$ is large, then the second member of (5.31) is positive which is contradiction. (Q. E. D.)

From Lemmas 5.1 and 5.4 we have the following fundamental

**Proposition 5.5.** Under the conditions (A) and (B) there exists a constant $\beta$ such that for any real $\lambda$ with $|\lambda| > \beta$ the operator $(\lambda I - iM)$ defines a one-to-one mapping of $\overline{\Pi} H^2$ onto $\overline{\Pi} H^0$, i.e., the resolvent $(\lambda I - iM)^{-1}$ exists for any $|\lambda| > \beta$, $\lambda \in \mathbb{R}^1$ and the inequality
\[
\| (\lambda I - iM)^{-n} \|_{L(\overline{\Pi} H^0, \overline{\Pi} H^0)} \leq \frac{c}{(|\lambda| - \beta)^n} \quad (n=1, 2, \ldots)
\]
holds where $c$ is a positive constant independent of $\lambda$ and $n$.

**Corollary of Proposition 5.5.** If $U(x) \in \overline{\Pi} H^0(\mathbb{R}^1)$ such that $MU(x) \in \overline{\Pi} H^0$, then $U(x) \in \overline{\Pi} H^0(\mathbb{R}^1)$, i.e.,
\[
D(M) = \overline{\Pi} H^2 = \{U(x) \in \overline{\Pi} H^0; MU(x) \in \overline{\Pi} H^0\}.
\]
Proposition 5.5 implies immediately the existence of a unique solution of the Cauchy problem (1.4) by applying the Hille-Yosida theorem. (Q. E. D.)

### 6. Examples.

In this section we give some examples of operators satisfying the conditions (A) and (B) (or (B')). At first consider the first order operators in $t$.

**Example 6.1.** (Takeuchi [5], [6])
\[
P(x; D_x, D_t) = D_t + D_x^2 + a(x)D_x, \quad x \in \mathbb{R}^1, \quad t \in [0, T].
\]
In this case we can choose $\varphi(x, \xi)$ in condition (B') such that

$$\varphi(x, \xi) = \varphi(x) = -\frac{1}{2} \int_0^x \Im(a(y))dy,$$

$$N(x, \xi) = N(x) = e^{\varphi(x)} \quad \text{(in (3.10)).}$$

If we assume the following condition (B'):

$$\int_0^x \Im(a(y))dy$$

is a bounded function,

then the Theorem 1.1 holds for this operator ($m=1$). The following equality holds:

$$\exp \left( -\frac{1}{2} \int_0^x \Im(a(y))dy \right) (D_t + D_x^2 + a(x)D_x)$$

$$\equiv (D_t + D_x^2 + \Re a(x)D_x) \exp \left( -\frac{1}{2} \int_0^x \Im(a(y))dy \right) \quad \text{(mod. S').}$$

Example 6.2.

$$P(x; D_x, D_t) = D_t + a(x)D_x^2 + b(x)D_x, \quad x \in \mathbb{R}^1, \quad t \in [0, T].$$

In this case we assume the following conditions:

Condition (A): $a(x)$ is a real-valued function such that

$$M \geq |a(x)| \geq \delta > 0, \quad x \in \mathbb{R}^1.$$

Condition (B): $\int_0^x \frac{\Im(b(y))}{a(y)}dy$ is a bounded function.

(Under the condition (A) this is equivalent to the condition: $\int_0^x \Im(b(y))dy$ is bounded). Then Theorem 1.1 holds for this operator ($m=1$). We choose $\varphi(x, \xi)$ and $N(x, \xi)$ as follows:

$$\varphi(x, \xi) = \varphi(x) = -\frac{1}{2} \left\{ \int_0^x \frac{\Im(b(y))}{a(y)}dy + \log|a(x)| \right\},$$

$$N(x, \xi) = N(x) = e^{\varphi(x)} = \frac{1}{\sqrt{|a(x)|}} \exp \left( -\frac{1}{2} \int_0^x \frac{\Im(b(y))}{a(y)}dy \right).$$

The following equality holds:

$$N(x)(D_t + a(x)D_x^2 + \Re b(x)N_x)N(x) \quad \text{(mod. S').}$$

Example 6.3.

$$P(x; D_x, D_t) = D_t + |D_x|^2 + \sum_{j=1}^k b_j(x)D_j, \quad x \in \mathbb{R}^1, \quad t \in [0, T], \quad D_j = -i \frac{\partial}{\partial x_j}.$$

We choose $\varphi(x, \xi)$ in the condition (B') as follows:

$$\varphi(x, \xi) = -\frac{1}{2} \int_0^x \left\{ \sum_{j=1}^k b_j(x)D_j \right\} \left( x - t \frac{\xi_j}{|\xi|} \right) dt,$$

$$N(x, \xi) = \exp(\varphi(x, \xi)).$$
Condition (B'): \( \varphi(x, \xi) \) defined by (6.11) is bounded implies the conclusion of Theorem 1.1 \((m=1)\). The following equality holds:

\[
\exp(\varphi(x, D_x)) \left( D_x + |D_x|^2 + \sum_{j=1}^{i} b_j(x) D_j \right) \\
\equiv \left( D_x + |D_x|^2 + \sum_{j=1}^{i} \text{Re} b_j(x) D_j \right) \exp(\varphi(x, D_x)) \quad (\text{mod. } S^0).
\]

**Example 6.4.**

\[
\begin{align*}
P(x; D_2, D_1) &= D_1 + A(x, D_x), \\
A(x, D_x) &= \sum_{i,j=1}^{l} a_{ij}(x) D_i D_j + \sum_{j=1}^{i} b_j(x) D_j, \quad x \in \mathbb{R}^l, \quad t \in [0, T].
\end{align*}
\]

We assume the condition (A):

\[
\begin{align*}
\text{and } \left| \sum_{i,j=1}^{l} a_{ij}(x) \xi_i \xi_j \right| &\geq \delta |\xi|^2, \quad (\delta > 0).
\end{align*}
\]

Our procedure is interpreted as follows: At first we transform \( A(x, D) \) by \( N(x, D) = \exp(\varphi(x, D)) \) where \( \varphi(x, D) \) is still to be determined:

\[
\exp(\varphi(x, D)) A(x, D) \equiv \tilde{A}(x, D) \exp(\varphi(x, D)), \quad (\text{mod. } S^0).
\]

Here the symbol of \( \tilde{A}(x, D) \) has the following form:

\[
\begin{align*}
\sigma(\tilde{A}) &= a_2(x, \xi) + a_4(x, \xi) + i \{ a_3(x, \xi), \varphi(x, \xi) \}, \\
\begin{align*}
a_4(x, \xi) &= \sum_{i,j=1}^{l} a_{ij}(x) \xi_i \xi_j, \\
a_3(x, \xi) &= \sum_{i=1}^{l} b_i(x) \xi_i.
\end{align*}
\end{align*}
\]

Next we calculate the formal adjoint \( \tilde{A}^* \) to \( \tilde{A} \). For real-valued \( \varphi(x, \xi) \),

\[
\begin{align*}
\sigma(\tilde{A}^*) &= a_4(x, \xi) + a_2(x, \xi) + \frac{1}{2} \sum_{i,j=1}^{l} \frac{\partial a_{ij}(x)}{\partial x_i} \xi_j - i \{ a_3(x, \xi), \varphi(x, \xi) \}.
\end{align*}
\]

We decompose \( \tilde{A}(x, D) \) as follows:

\[
\tilde{A}(x, D) = \frac{\tilde{A} + \tilde{A}^*}{2} + i \frac{\tilde{A} - \tilde{A}^*}{2i},
\]

where

\[
\begin{align*}
\sigma\left( \frac{\tilde{A} + \tilde{A}^*}{2} \right) &= a_2(x, \xi) + \text{Re} a_4(x, \xi) + \frac{1}{2} \sum_{i,j=1}^{l} \frac{\partial a_{ij}(x)}{\partial x_i} \xi_j, \\
\sigma\left( \frac{\tilde{A} - \tilde{A}^*}{2i} \right) &= \{ a_3(x, \xi), \varphi(x, \xi) \} + \text{Im} a_4(x, \xi) + \sum_{i,j=1}^{l} \frac{\partial a_{ij}(x)}{\partial x_i} \xi_j.
\end{align*}
\]

Finally we choose \( \varphi(x, \xi) \) homogeneous of degree 0 such that condition (B) holds:

\[
\begin{align*}
\{ a_3(x, \xi), \varphi(x, \xi) \} + \sum_{j=1}^{l} \text{Im} b_j(x) \xi_j + \sum_{i,j=1}^{l} \frac{\partial a_{ij}(x)}{\partial x_i} \xi_j = 0.
\end{align*}
\]

Then the Theorem 1.1 \((m=1)\) and the following equality hold:
(6.24) \[ \exp(\phi(x, D))\left(D_i + \sum_{j=1}^{i} a_{ij}(x)D_j + \sum_{j=1}^{i} b_j(x)D_j\right) = \left(D_i + \sum_{j=1}^{i} a_{ij}(x)D_j + \sum_{j=1}^{i} \Re b_j(x)D_j\right)\exp(\phi(x, D)), \quad \text{(mod. } S^0) \].

Now we give an example of differential operators of order 2 in \( t \) (i.e. \( m=2 \)).

**Example 6.5.**

(6.25) \[ P(x ; D_x, D_t)=D_{\xi}^2 - |D_x|^2 + \sum_{j=1}^{i} b_j(x)D_j + \sum_{a_i=3} c_a(x)D_a^2, \]
\[ x \in \mathbb{R}^4, \quad t \in [0, T]. \]

(6.26) \[ P^n(\xi, \tau) = \tau^2 - |\xi|^4(\tau - |\xi|^4) + |\xi|^4. \]

It follows from (6.26) that the condition (A) is satisfied. In the notations of section 1, \( a\|_2(x, \xi) = 0 \), \( a\|_2(x, \xi) = -|\xi|^4 \), \( a\|_4(x, \xi) = \sum_{j=1}^{i} b_j(x)\xi_j \), \( a\|_4(x, \xi) = \sum_{a_i=3} c_a(x)\xi^a \), \( \lambda_- = |\xi|^2 \), \( \lambda_+ = -|\xi|^2 \). Consider the equation

\[ P(x ; D, D_t)u(x, t) = f(x, t). \]

Putting \( U(x, t) = ((A^2+1)u(x, t), D_t u(x, t)) \), we have

(6.27) \[ D_t U(x, t) = M(x, D)U(x, t) + F(x, t), \]
where

\[
\begin{bmatrix}
M(x, D) = M_4(D) + M_1(x, D) \\
\sigma(M_4) = \begin{bmatrix} 0 & |\xi|^2 \\
|\xi|^2 & 0 \end{bmatrix} \\
\sigma(M_1) = \begin{bmatrix} 0 & -\sum_{a_i=3} c_a(x) (\frac{\xi}{|\xi|})^a |\xi| & -\sum_{j=1}^{i} b_j(x) (\frac{\xi}{|\xi|}) |\xi| \end{bmatrix}
\end{bmatrix} \text{ (mod. } S^0),
\]

(6.28)

We take a diagonalizer \( N(x, D) \) as follows.

\[
\begin{bmatrix}
N(x, D) = N_4(x, D) + N_1(x, D) \\
\sigma(N_4) = \begin{bmatrix} \varphi_{41}(x, \xi) & \varphi_{42}(x, \xi) \\
-\varphi_{41}(x, \xi) & \varphi_{42}(x, \xi) \end{bmatrix} \\
\sigma(N_1) = \begin{bmatrix} -\frac{1}{4} (a^2(x, \xi)) - a(x, \xi) & \varphi_{41}(x, \xi) \\
\varphi_{42}(x, \xi) & -\frac{1}{4} (a^2(x, \xi)) + a(x, \xi) \end{bmatrix} \text{ (mod. } S^0),
\end{bmatrix}
\]

(6.29)

where real-valued functions \( \varphi_{ij}(x, \xi) \) are still to be determined,
Then we have

\begin{equation}
N(x, D)M(x, D) = \mathcal{D}(x, D)N(x, D) \quad \text{(mod. } S^0),
\end{equation}

where

\begin{equation}
\mathcal{D}(x, D) = \mathcal{D}_v(D) + \mathcal{D}_t(x, D) = \begin{pmatrix} \hat{\lambda}_i(x, D) & 0 \\ 0 & \hat{\lambda}_j(x, D) \end{pmatrix},
\end{equation}

\begin{equation}
\sigma(\hat{\lambda}_i) = |\xi|^2 - \frac{1}{2} \left( \sum_{|a|=3} c_a(x) \left( \frac{\xi}{|\xi|} \right)^a + \sum_{j=1}^i b_j(x) \frac{\xi_j}{|\xi|} \right)|\xi|^2 - \frac{2}{i^2} \sum_{j=1}^i \xi_j \frac{\partial \varphi_1}{\partial x_j},
\end{equation}

\begin{equation}
\sigma(\hat{\lambda}_j) = -|\xi|^2 + \frac{1}{2} \left( \sum_{|a|=3} c_a(x) \left( \frac{\xi}{|\xi|} \right)^a - \sum_{j=1}^i b_j(x) \frac{\xi_j}{|\xi|} \right)|\xi|^2 + \frac{2}{i^2} \sum_{j=1}^i \xi_j \frac{\partial \varphi_2}{\partial x_j}.
\end{equation}

We decompose \( \sigma(\hat{\lambda}_i) \) and \( \sigma(\hat{\lambda}_j) \) as follows:

\begin{equation}
\sigma(\hat{\lambda}_i) = \left[ |\xi|^2 - \frac{1}{2} \left( \sum_{|a|=3} \text{Re } c_a(x) \left( \frac{\xi}{|\xi|} \right)^a + \sum_{j=1}^i \text{Re } b_j(x) \frac{\xi_j}{|\xi|} \right) \right] |\xi|^2
+ 2i \left[ \sum_{j=1}^i \frac{\xi_j}{|\xi|} \frac{\partial \varphi_1}{\partial x_j} - \frac{1}{4} \left( \sum_{|a|=3} \text{Im } c_a(x) \left( \frac{\xi}{|\xi|} \right)^a + \sum_{j=1}^i \text{Im } b_j(x) \frac{\xi_j}{|\xi|} \right) \right] |\xi|,
\end{equation}

\begin{equation}
\sigma(\hat{\lambda}_j) = \left[ -|\xi|^2 + \frac{1}{2} \left( \sum_{|a|=3} \text{Re } c_a(x) \left( \frac{\xi}{|\xi|} \right)^a - \sum_{j=1}^i \text{Re } b_j(x) \frac{\xi_j}{|\xi|} \right) \right] |\xi|^2
- 2i \left[ \sum_{j=1}^i \frac{\xi_j}{|\xi|} \frac{\partial \varphi_2}{\partial x_j} - \frac{1}{4} \left( \sum_{|a|=3} \text{Im } c_a(x) \left( \frac{\xi}{|\xi|} \right)^a - \sum_{j=1}^i \text{Im } b_j(x) \frac{\xi_j}{|\xi|} \right) \right] |\xi|.
\end{equation}

We choose the functions \( \varphi_i(x, \xi) \) such that

\begin{equation}
\sum_{j=1}^i \frac{\xi_j}{|\xi|} \frac{\partial \varphi_1}{\partial x_j} - \frac{1}{4} \left( \sum_{|a|=3} \text{Im } c_a(x) \left( \frac{\xi}{|\xi|} \right)^a + \sum_{j=1}^i \text{Im } b_j(x) \frac{\xi_j}{|\xi|} \right) = 0,
\end{equation}

and

\begin{equation}
\sum_{j=1}^i \frac{\xi_j}{|\xi|} \frac{\partial \varphi_2}{\partial x_j} - \frac{1}{4} \left( \sum_{|a|=3} \text{Im } c_a(x) \left( \frac{\xi}{|\xi|} \right)^a - \sum_{j=1}^i \text{Im } b_j(x) \frac{\xi_j}{|\xi|} \right) = 0
\end{equation}

hold, that is,

\begin{equation}
\varphi_1(x, \xi) = \frac{1}{4} \int_0^{\frac{\xi}{|\xi|}} \left\{ \sum_{|a|=3} \text{Im } c_a \left( x - t \frac{\xi}{|\xi|} \right) \left( \frac{\xi}{|\xi|} \right)^a + \sum_{j=1}^i \text{Im } b_j \left( x - t \frac{\xi_j}{|\xi|} \right) \frac{\xi_j}{|\xi|} \right\} dt,
\end{equation}

and

\begin{equation}
\varphi_2(x, \xi) = \frac{1}{4} \int_0^{\frac{\xi}{|\xi|}} \left\{ \sum_{|a|=3} \text{Im } c_a \left( x - t \frac{\xi}{|\xi|} \right) \left( \frac{\xi}{|\xi|} \right)^a - \sum_{j=1}^i \text{Im } b_j \left( x - t \frac{\xi_j}{|\xi|} \right) \frac{\xi_j}{|\xi|} \right\} dt.
\end{equation}

The condition (B) is as follows: Functions defined by (6.36) and (6.37) are bounded. Then Theorem 1.1 \((m=2)\) holds.
Non-kowalewskian equation

References


Remarks added in proof.

Remark 1. Condition (A) implies that

\[ \sum_{k=1}^{l} \xi_k \frac{\partial \lambda_j}{\partial \xi_k}(x, \xi) = 2\lambda_j(x, \xi) \neq 0 \text{ on } T^*R^l \setminus 0 \]

by Euler’s identity, i.e., \( \nabla \xi \lambda_j(x, \xi) \neq 0 \) on \( T^*R^l \setminus 0 \). Thus any integral curve of the Hamilton field \( H_{\lambda_j} \) is regular and defined on \( R^l \) by virtue of the homogeneity of \( \lambda_j(x, \xi) \) (homogeneous of degree 2 in \( \xi \), \( 1 \leq j \leq m \)).

Remark 2. Condition (B) is stated more explicitly as follows. (cf. Duistermaat-Hörmander [1]).

(B-1) No complete integral curve of the Hamilton field \( H_{\lambda_j} \) is contained in a compact subset of \( T^*R^l \setminus 0 \), \( 1 \leq j \leq m \),

(B-2) for every compact subset \( K \) of \( T^*R^l \setminus 0 \) there exists a compact subset \( K' \) of \( T^*R^l \setminus 0 \) such that every compact interval on an integral curve (of the Hamilton field \( H_{\lambda_j} \)) with end points in \( K \) contained in \( K' \), \( 1 \leq j \leq m \).

From conditions (B-1) and (B-2) it follows that

(1.8) \[ H_{\lambda_j} \varphi_j(x, \xi) = h_j(x, \xi) \]

has a real-valued \( C^\infty(T^*R^l \setminus 0) \) solution \( \varphi_j(x, \xi) \) for any real-valued \( C^\infty(T^*R^l \setminus 0) \) function \( h_j(x, \xi) \), \( 1 \leq j \leq m \).

(B-3) For a function \( h_j(x, \xi) \) defined by (1.10) which is \( C^\infty(T^*R^l \setminus 0) \) real-valued, bounded on \( R^l \times S^{l-1} \) and homogeneous of degree 1 in \( \xi \), (1.8) has a real-valued, bounded \( C^\infty(T^*R^l \setminus 0) \) solution \( \varphi_j(x, \xi) \) homogeneous of degree 0 in \( \xi \), \( 1 \leq j \leq m \).
In the case where the operator $P(x; D_x, D_t)$ defined by (1.1) and (1.2) with $b=2$ has the principal part $P^0(x; D_x, D_t)$ defined by (1.3) with constant coefficients, conditions (B-1) and (B-2) are automatically satisfied.