J. Math. Kyoto Univ. (JMKYAZ)
20-3 (1980) 435-446

A simple expression of the characters of certain discrete series representations, II

By

Shunsuke MIKAMI

(Communicated by Prof. H. Yoshizawa, april 2, 1979)

Introduction

In the previous paper [5], we showed that Hirai's general formula of the characters is reduced to a simple one for certain discrete series unitary representations of $SO_0(p, q)$ ($p+q$: odd). In this paper, we study the similar problem for the connected simple Lie group G of type FI, which is the unique exceptional Lie group of "class II".

Let $\mathfrak{g}$ be the Lie algebra of G, and B a compact Cartan subgroup of G. We denote by $\mathfrak{b}_c^*$ the complexification of the dual space of $\mathfrak{b}$, the Lie algebra of B, and $\mathfrak{b}_B^*$ the lattice in $\mathfrak{b}_c^*$ consisting of such $\Lambda \in \mathfrak{b}_c^*$ that the mapping $\xi_\Lambda : B \ni \exp X \longmapsto e^{\Lambda(X)}$ ($X \in \mathfrak{b}$) defines a unitary character of B. Let G' be the totality of regular elements of G. Then by Harish-Chandra, it was shown that for each regular $\Lambda \in \mathfrak{b}_B^*$, there exists a discrete series representation of G whose character π_Λ is expressed on $B \cap G'$ as follows:

$$\pi_\Lambda = \varepsilon(\Lambda) \left(\sum_{w \in W_k} \operatorname{sgn}(w) \xi_{w\Lambda} \right) / \Delta^{\mathfrak{b}},$$

where $W_k (= W_G(\mathfrak{b}))$ denotes the little Weyl group and the number $\varepsilon(\Lambda) = \pm 1$ is determined by Λ and the positive system of the roots. In [1], Hirai gave a global formula of π_Λ' (the analytic function on G' corresponding to π_Λ) valid for any $\Lambda \in \mathfrak{b}_B^*$.

Since the root system of type F_4 belongs to "class II", Hirai's formula for type FI is very complicated for general Λ. In this note just as for the case of $SO_0(p, q)$ ($p+q$: odd), we study how the terms in the original formula are cancelling out each other when a regular element Λ in $\mathfrak{b}_B^*$ is dominant with respect to the positive system of Borel-de Siebenthal. We consider π_Λ' only on the connected component A of the identity of the split Cartan subgroup H of G. It is because we avoid a secondary complication and deal with the fundamental phenomena. Though the root system of type F_4 is not so simple, the factor group of W_k modulo the subgroup W_0 in §1 of W_k is of order 2. This contributes to the simplification of the formula. Our main result is the theorem at the end of this paper.

(We follow principally the definitions and the notations in [1] and [5]. So readers can find, if necessary, their exact meanings in these previous papers.)

§ 1. The root system of type F_4

Let $\mathfrak{g}$ be a normal real form of the simple complex Lie algebra $\mathfrak{g}_c$ of type F_4 and θ be a Cartan involution of $\mathfrak{g}$. Put $\mathfrak{k}=\{X\in\mathfrak{g}\ ;\ \theta X=X\}$ and $\mathfrak{p}=\{X\in\mathfrak{g}\ ;\ \theta X=-X\}$. Then there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$. Let $\Sigma(\mathfrak{h})$ be the root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$. In general, for a Lie algebra $\mathfrak{u}$, we denote by $\mathfrak{u}_c$ its complexification. As is well-known, there is an orthogonal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{h}_c^*$ for which

$$\Sigma(\mathfrak{h})=\left\{\begin{array}{l}\pm e_i(1\leqq i\leqq 4),\ \pm e_i\pm e_j\ \ (1\leqq i<j\leqq 4),\\ (\pm e_1\pm e_2\pm e_3\pm e_4)/2\end{array}\right\}.$$

We give the lexicographic order $\boldsymbol{P}$ with respect to (e_1, e_2, e_3, e_4) in $\Sigma(\mathfrak{h})$. Then the standard maximal orthogonal ordered sets with respect to $\boldsymbol{P}$ (cf. §1 in [5]) are the following ones:

$$\begin{aligned} E_0&=(e_1+e_2, e_1-e_2, e_3+e_4, e_3-e_4),\\ E_1&=(e_1+e_2, e_1-e_2, e_3, e_4),\\ E_2&=(e_1, e_2, e_3, e_4).\end{aligned}$$

For $0\leqq i\leqq 2$, let ν_{E_i} be an automorphism of $\mathfrak{g}_c$ defined in §1 in [5] as a product of appropriate Cayley transformations of $\mathfrak{g}_c$. Put $\mathfrak{b}=\nu_{E_0}(\mathfrak{h}_c)\cap\mathfrak{g}$, then $\mathfrak{b}$ is a compact Cartan subalgebra of $\mathfrak{g}$. For a root $\alpha\in\Sigma(\mathfrak{h})$, we define the element $\tilde{\alpha}$ in $\mathfrak{b}_c^*$ as $\tilde{\alpha}(X)=\alpha(\nu_{E_0}^{-1}(X))$ for $X\in\mathfrak{b}_c$. Apparently, $\tilde{\alpha}$ belongs to $\Sigma(\mathfrak{b})$, the root system of $(\mathfrak{g}_c, \mathfrak{b}_c)$. Then by taking the root vectors for $e_1\pm e_2$, $e_3\pm e_4$, e_1, e_2, e_3, and e_4 properly as in [5], there exist elements k_i $(i=1, 2)$ in K such that

$$\nu_{E_0|\mathfrak{h}_c}=\mathrm{Ad}(k_i)\nu_{E_i|\mathfrak{h}_c}\qquad \text{for } i=1,\ 2.$$

Here K denotes the subgroup of the adjoint group Int $(\mathfrak{g})$ corresponding to $\mathfrak{k}$. Therefore the root $\tilde{e}_i$ is compact if and only if i is even and the long roots $\tilde{e}_i\pm\tilde{e}_j$ are compact if and only if the parities of both i and j

are the same.

For convenience, we call a root of the form $(\Sigma\,\varepsilon_i\tilde{e}_i)/2$, $\varepsilon_i=\pm 1$, a root of ε-type. Put

$$\Sigma_1=\{\text{roots of } \varepsilon\text{-type such that } \varepsilon_1\varepsilon_3>0\} \quad \text{and}$$
$$\Sigma_2=\{\text{roots of } \varepsilon\text{-type such that } \varepsilon_1\varepsilon_3<0\}.$$

Then we have the following lemma.

Lemma 1. *Put Σ_k (resp. Σ_n) be the totality of compact (resp. non-compact) roots of $\Sigma(\mathfrak{b})$. Then one of the following two cases occurs.*

Case 1) $\Sigma_1\subset\Sigma_k$ *and* $\Sigma_2\subset\Sigma_n$;
Case 2) $\Sigma_1\subset\Sigma_n$ *and* $\Sigma_2\subset\Sigma_k$.

Proof. At first, we assume that a root $\alpha_0=(\Sigma\,\varepsilon_i\tilde{e}_i)/2$ in Σ_1 is comact. Let us denote the reflection with respect to a root α by s_α. When α is compact, the root $s_\alpha\beta$ is compact or not according as β is compact or not. Put $f_i=\nu_{E_0}e_i=\tilde{e}_i$ for $1\leqq i\leqq 4$. Then $s_{f_2}\alpha_0$ and $s_{f_4}\alpha_0$ are compact. This means that the root of the form $(\varepsilon_1 f_1+\varepsilon_3 f_3\pm f_2\pm f_4)/2$ belong to Σ_k. Furthermore, the roots $f_1\pm f_3$ are compact and $s_{f_1+f_3}s_{f_1-f_3}$ equals to $s_{f_1}s_{f_3}$, therefore $(-\varepsilon_1 f_1-\varepsilon_3 f_3\pm f_2\pm f_4)/2$ belong to Σ_k. Thus $\Sigma_1\subset\Sigma_k$. But the root $\beta_0=(\varepsilon_1 f_1-\varepsilon_3 f_3+\varepsilon_2 f_2+\varepsilon_4 f_4)/2$ is non-compact, because $\beta_0=\alpha_0-\varepsilon_3 f_3$, and the sum of a compact root and a non-compact one is non-compact. Analogously as above we can show that $\Sigma_2\subset\Sigma_n$. Thus the case 1 occurs.

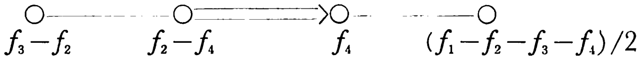
Conversly, suppose any root in Σ_1 is not compact. Then we can prove similarly that the case 2 occurs. Q. E. D.

In the case 1, we replace the order $\boldsymbol{P}$ with the lexicographic order $\boldsymbol{P}'$ corresponding to $(e_1', e_2', e_3', e_4')=(-e_1, e_2, e_3, e_4)$. With respect to this new order $\boldsymbol{P}'$, the standard maximal orthogonal ordred sets are as follows:

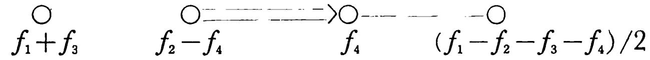
$$\begin{aligned} E_0' &= (e_1'+e_2',\ e_1'-e_2',\ e_3'+e_4',\ e_3'-e_4') \\ &= (-e_1+e_2,\ -e_1-e_2,\ e_3+e_4,\ e_3-e_4), \\ E_1' &= (e_1'+e_2',\ e_1'-e_2',\ e_3',\ e_4'), \\ &= (-e_1+e_2,\ -e_1-e_2,\ e_3,\ e_4), \\ E_2' &= (e_1',\ e_2',\ e_3',\ e_4') \\ &= (-e_1,\ e_2,\ e_3,\ e_4). \end{aligned}$$

Any two roots in $E_0^*=\{e_1\pm e_2,\ e_3\pm e_4\}$ are mutually strongly orthogonal. The root vectors corresponding to the roots in E_0^* which define the automorphism ν_{E_0} also satisfy the condition 5. 1 in [1] for E_0'. So the two Caylay transforms ν_{E_0} and $\nu_{E_0'}$ coincide. Since we are in Case 1 for (e_1, e_2, e_3, e_4), the roots of the form $(\varepsilon_1\tilde{e}_1'+\varepsilon_2\tilde{e}_2'+\varepsilon_3\tilde{e}_3'+\varepsilon_4\tilde{e}_4')/2$ is compact if and only if $\varepsilon_1\varepsilon_3<0$. This means that we are in Case 2 for (e_1', e_2', e_3', e_4'). Thus in the following, we may assume without loss of generality that all the roots in Σ_1 are non-compact.

We call a positive system in $\Sigma(\mathfrak{b})$ of Borel-de Siebenthal when the corresponding simple roots are compact except only one simple root γ and the multiplicity of γ in the highest root is equal to 1 or 2. There exists a positive system P_0 in $\Sigma(\mathfrak{b})$ such that the corresponding Dynkin diagram is expressed as follows:



Then P_0 is a positive system of Borel-de Siebenthal. The subset $P_0\cap\Sigma_k$ is a positive system of Σ_k and the corresponding Dynkin diagram is as follows:



Thus Σ_k is a root system of type $A_1\times C_3$. Put $w_1=s_{f_1+f_3}$, and the Weyl group of Σ_k, identified with W_k, is the direct product of the two subgroups $\{1, w_1\}$ and W_0. Here W_0 is the subgroup corresponding to the sub-root-system of type C_3.

Lemma 2. *We have the following characterization of* W_0:

$$W_0=\{w\in W(\Sigma(\mathfrak{b}))\ ;\ w(\Sigma_n\cap P_0)=\Sigma_n\cap P_0\}.$$

Here $W(\Sigma(\mathfrak{b}))$ *denotes the Weyl group of* $\Sigma(\mathfrak{b})$.

Proof. For an element w from the right hand side, let $w=s_{\alpha_1}\cdots\cdots s_{\alpha_p}$ be a reduced expression of it by means of the simple reflections with respect to P_0. Put $\beta_i=s_{\alpha_p}\cdots\cdots s_{\alpha_{i+1}}\alpha_i$ for $i=1, 2, \dots, p$. Then each root β_i is positive and $w\beta_i$ is negative. Assume that there exists such an index i_0 that α_{i_0} is non-compact and for any $i>i_0$, α_i is compact. Then the root $w\beta_{i_0}$ is negative and non-compact, which contradicts that w maps $\Sigma_n\cap P_0$ onto itself. Therefore each α_i is compact simple, so w belongs to W_0.

Conversely, an element $w\in W_0$ has a reduced expression $w=s_{\gamma_1}\cdots\cdots s_{\gamma_q}$ in W_0, where each γ_i is simple root in Σ_k. Note that γ_i's are also simple in $\Sigma(\mathfrak{b})$ with respect to P_0 and that the length q of w is equal to the number of positive roots mapped into negative ones by w. Then we see that no non-compact positive root is mapped into negative one by w. Hence $w(\Sigma_n \cap P_0) = \Sigma_n \cap P_0$. Q. E. D.

Let $(\mathfrak{b}_B^*)^+$ be the totality of such $\Lambda \in \mathfrak{b}_B^*$ that $(\Lambda, \alpha) \geqq 0$ for any $\alpha \in P_0$. Here $(\ ,\ )$ denotes the inner product in $\mathfrak{b}_c^*$ induced by the Killing form of $\mathfrak{g}_c$. Then, for a regular $\Lambda \in (\mathfrak{b}_B^*)^+$, the next corollary follows from the above lemma immediately.

Corollary. For any $s \in W_0$ and $\alpha \in \Sigma_n \cap P_0$, $\mathrm{sgn}(s^{-1}\Lambda, \alpha) = 1$.

§ 2. Hirai's character formula for general Λ

In this section, we restate Hirai's character formula. For $\alpha \in \Sigma(\mathfrak{h})$, define the character ξ_α of H as

$$\xi_\alpha(h) X_\alpha = Ad(h) X_\alpha,$$

where X_α is a root vector corresponding to α. Put

$$A(P) = \{h \in A\ ;\ \xi_\alpha(h) > 1 \text{ for any } \alpha > 0 \text{ with respect to } \boldsymbol{P}\}, \text{ and}$$
$$W(E_i) = \{u \in W(\Sigma(\mathfrak{h}))\ ;\ uE_i \in M^{or}(\boldsymbol{P})\}.$$

Here $W(\Sigma(\mathfrak{h}))$ denotes the Weyl group of $\Sigma(\mathfrak{h})$ and the readers can find the definition of $M^{or}(\boldsymbol{P})$ in §1 in [5]. Let $\Lambda \in \mathfrak{b}_B^*$. We define the functions of $h = \exp X \in A(P)$ $(X \in \mathfrak{h})$ as follows:

$$Y(h\ ;E_j, u, s^{-1}\Lambda) = \mathrm{sgn}_{P(E_j)}(s^{-1}\Lambda) \prod_{\alpha \in E_j^*} \exp\{-u\alpha(X)\,|(s^{-1}\Lambda, \tilde{\alpha})|/|\alpha|^2\},$$
$$Z(h\ ;E_j, \Lambda) = \sum_{s \in W_k} \sum_{u \in W(E_j)} \mathrm{sgn}(s) Y(h\ ;E_j, u, s^{-1}\Lambda),$$

where $\mathrm{sgn}_{P(E_j)}(s^{-1}\Lambda) = \mathrm{sgn}\{\prod_{\alpha \in P(E_j)} (s^{-1}\Lambda, \tilde{\alpha})\}$. (For $P(E_j)$, see §1 in [5].)
Then Hirai's character formula tells that for $h \in A(P)$,

$$\pi'_\Lambda(h) = \varepsilon(\Lambda)\Big(\sum_{j=0}^{2}(-1)^j Z(h\ ;E_j, \Lambda)\Big)/\Delta^{\mathfrak{h}}(h),$$

where

$$\varepsilon(\Lambda) = \mathrm{sgn}\{\prod_{\substack{\alpha \in \Sigma(\mathfrak{h}) \\ \alpha > 0}} (\Lambda, \tilde{\alpha})\},$$
$$\Delta^{\mathfrak{h}}(h) = \xi_\rho(h) \prod_{\substack{\alpha \in \Sigma(\mathfrak{h}) \\ \alpha > 0}} (1 - \xi_{-\alpha}(h)) \quad \text{and}$$
$$\rho = \Big(\sum_{\substack{\alpha \in \Sigma(\mathfrak{h}) \\ \alpha > 0}} \alpha\Big)/2.$$

In the following, we consider the character π'_Λ only in the case where $\Lambda\in(\mathfrak{b}_B^*)^+$. (So $\varepsilon(\Lambda)=-1$.)

§ 3. The summation of sgn(s) $Y(h\,;E_j,\ u,\ s^{-1}\Lambda)$'s over W_k

In this section, we reduce the summation of sgn$(s)\,Y(h\,;\,E_j,\ u,\ s^{-1}\Lambda)$'s over $s\in W_k$ into that over $s\in W_0$. Note that $W_k=W_0\cup W_0\cdot w_1$.

First of all, let $s\in W_0\subset W_k$. Since the roots $f_1\pm f_2$, $f_3\pm f_4$, f_1 and f_3 are all positive and non-compact, so are the roots $s(f_1\pm f_2)$, $s(f_3\pm f_4)$, sf_1 and sf_3.
Therefore we get that $\mathrm{sgn}_{P(E_j)}(s^{-1}\Lambda)=\mathrm{sgn}\{(s^{-1}\Lambda,\ f_2)(s^{-1}\Lambda,\ f_4)\}$ and

$$\begin{aligned}&\{|(s^{-1}\Lambda,\ f_1+f_2)|+|(s^{-1}\Lambda,\ f_1-f_2)|\}/2\\ &=\{(s^{-1}\Lambda,\ f_1+f_2)+(s^{-1}\Lambda,\ f_1-f_2)\}/2=(s^{-1}\Lambda,\ f_1),\\ &\{|(s^{-1}\Lambda,\ f_1+f_2)|-|(s^{-1}\Lambda,\ f_1-f_2)|\}/2=(s^{-1}\Lambda,\ f_2).\end{aligned}$$

On the other hand, for $u\in W(\Sigma(\mathfrak{h}))$,

$$\begin{aligned}uE_0&=(ue_1+ue_2,\ ue_1-ue_2,\ ue_3+ue_4,\ ue_3-ue_4),\\ uE_1&=(ue_1+ue_2,\ ue_1-ue_2,\ ue_3,\ ue_4),\\ uE_2&=(ue_1,\ ue_2,\ ue_3,\ ue_4).\end{aligned}$$

Therefore for $j=0$, 1, 2, we can rewrite Y's in the following form convenient for later caluculation;

$$\begin{aligned}&\mathrm{sgn}(s)\mathrm{sgn}\{(s^{-1}\Lambda,\ f_2)(s^{-1}\Lambda,\ f_4)\}\,Y(h\,;E_j,\ u,\ s^{-1}\Lambda)\\ &=\begin{cases}\mathrm{sgn}(s)\prod_{i=1}^{4}\exp\{-ue_i(X)(s^{-1}\Lambda,\ f_i)\} \quad \text{for } j=0, & (3.1)\\ \mathrm{sgn}(s)\prod_{i=1}^{3}\exp\{-ue_i(X)(s^{-1}\Lambda,\ f_i)\}\exp\{-ue_4(X)\,|(s^{-1}\Lambda,\ f_4)|\}\\ \qquad\qquad \text{for } j=1, & (3.2)\\ \mathrm{sgn}(s)\prod_{i=1}^{2}\exp\{-ue_{2i-1}(X)(s^{-1}\Lambda,\ f_{2i-1})\}\cdot\exp\{-ue_{2i}(X)\,|(s^{-1}\Lambda,\ f_{2i})|\}\\ \qquad\qquad \text{for } j=2. & (3.3)\end{cases}\end{aligned}$$

Next, let $s\in W_0\cdot w_1\subset W_k$. Put $s_1=w_1\cdot s$ $(=sw_1)$. Since w_1 maps the six roots $f_1\pm f_2$, $f_3\pm f_4$, f_1 and f_3 into negative non-comact ones, we get

$$\begin{aligned}&\mathrm{sgn}_{P(E_j)}(s^{-1}\Lambda)=\mathrm{sgn}\{(s_1^{-1}\Lambda,\ f_2)(s_1^{-1}\Lambda,\ f_4)\},\\ &|(s^{-1}\Lambda,\ f_1\pm f_2)|=(s_1^{-1}\Lambda,\ f_3\mp f_2)\quad (O.S.),\\ &|(s^{-1}\Lambda,\ f_3\pm f_4)|=(s_1^{-1}\Lambda,\ f_1\mp f_4)\quad (O.S.),\\ &|(s^{-1}\Lambda,\ f_1)|=(s_1^{-1}\Lambda,\ f_3)\ \text{and}\ |(s^{-1}\Lambda,\ f_3)|=(s_1^{-1}\Lambda,\ f_1).\end{aligned}$$

(Here $(O.S.)$ means that the orders of the signs correspond respectively.)
Therefore for E_0, it holds that

$$\operatorname{sgn}(s)Y(h\,;E_0,\ u,\ s^{-1}\Lambda)=-\operatorname{sgn}(s_1)\operatorname{sgn}\{(s_1^{-1}\Lambda,\ f_2)(s_1^{-1}\Lambda,\ f_4)\}\times$$
$$\exp\{-ue_1(X)(s_1^{-1}\Lambda,\ f_3)\}\exp\{-ue_3(X)(s_1^{-1}\Lambda,\ f_1)\}\prod_{i=1}^{2}\exp\{ue_{2i}(X)(s_1^{-1}\Lambda,\ f_{2i})\}.$$

Put $s'=s_1s_{f_1-f_3}s_{f_2}s_{f_4}$, then $s'\in W_0$ and $\operatorname{sgn}(s_1)=-\operatorname{sgn}(s')$.
Hence we get

$$\operatorname{sgn}(s)Y(h\,;E_0,\ u,\ s^{-1}\Lambda)=\operatorname{sgn}(s')\operatorname{sgn}\{(s'^{-1}\Lambda,f_2)(s'^{-1}\Lambda,\ f_4)\}\times$$
$$\prod_{i=1}^{4}\exp\{-ue_i(X)(s'^{-1}\Lambda,\ f_i)\}. \tag{3.4}$$

For E_1, put $s'=s_1s_{f_1-f_3}s_{f_2}$ and for E_2, put $s'=s_1s_{f_1-f_3}$. Then for each case, $s'\in W_0$ and

$$\operatorname{sgn}(s)\operatorname{sgn}\{(s^{-1}\Lambda,\ f_2)(s^{-1}\Lambda,\ f_4)\}=\operatorname{sgn}(s')\operatorname{sgn}\{(s'^{-1}\Lambda,\ f_2)(s'^{-1}\Lambda,\ f_4)\}$$

and the function $\operatorname{sgn}(s)Y(h\,;E_j,u,s^{-1}\Lambda)$ is expressed in the form in (3.2) or (3.3) where s is replaced by s', according as $j=1$ or 2. Thus we get that for $j=0,\ 1,\ 2$,

$$\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_j,u,\ s^{-1}\Lambda)=2\sum_{s\in W_0}\operatorname{sgn}(s)Y(h\,;E_j,\ u,\ s^{-1}\Lambda). \tag{3.5}$$

§ 4. The division of $W(E_i)$

From now on, for each $i=0,\ 1,\ 2$, we identify $u\in W(E_i)$ with the corresponing ordered set uE_i.

4.1 The set $W(E_0)$ is composed of the following three subsets:

$$W(E_0)^{(0)}=\left\{\begin{array}{l}u_j^{(0)}=(e_1+e_j,\ e_1-e_j,\ e_i+e_k,\ e_i-e_k)\ ;\\ \{i,\ j,\ k\}=\{2,\ 3,\ 4\},\ i<k\end{array}\right\},$$
$$W(E_0)^{(1)}=\left\{\begin{array}{l}u_j^{(1)}=(e_1+e_j,\ e_i+e_k,\ e_1-e_j,\ e_i-e_k)\ ;\\ \{i,\ j,\ k\}=\{2,\ 3,\ 4\},\ i<k\end{array}\right\},$$
$$W(E_0)^{(2)}=\left\{\begin{array}{l}u_j^{(2)}=(e_1+e_j,\ e_i-e_k,\ e_1-e_j,\ e_i+e_k)\ ;\\ \{i,\ j,\ k\}=\{2,\ 3,\ 4\},\ i<k\end{array}\right\}.$$

Lemma 3. *Let u and u' be in $W(E_0)$. If $u^{-1}u'$ is equal to either $s_{e_1-e_3}$ or $s_{e_2-e_4}$, then the following equality holds:*

$$\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_0,\ u,\ s^{-1}\Lambda)=-\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_0,\ u',\ s^{-1}\Lambda).$$

Proof. Put $\hat{s}=u^{-1}u'$, then $\hat{s}\in W(\Sigma(\mathfrak{h}))$ and under the identifications s_{e_i} with s_{f_i} and $s_{e_i-e_j}$ with $s_{f_i-f_j}$, $W(\Sigma(\mathfrak{h}))$ is isomorphic to $W(\Sigma(\mathfrak{b}))$ and $\hat{s}$ can be considered as an element in $W_k\subseteq W(\Sigma(\mathfrak{b}))$. Therefore,

$$\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_0,\ u\hat{s},\ s^{-1}\Lambda)$$
$$=\sum_{s\in W_k}\operatorname{sgn}(s)\operatorname{sgn}\{(s^{-1}\Lambda,\ f_2)(s^{-1}\Lambda,\ f_4)\}\prod_{i=1}^{4}\exp\{-u\hat{s}(e_i)(X)(s^{-1}\Lambda,\ f_i)\}$$
$$=\sum_{s\in W_k}\operatorname{sgn}(s)\operatorname{sgn}\{(s^{-1}\Lambda,\ f_2)(s^{-1}\Lambda,\ f_4)\}\prod_{i=1}^{4}\exp\{-ue_i(X)(s^{-1}\Lambda,\ \hat{s}f_i)\}.$$

Set $s'=s\hat{s}$, then $\operatorname{sgn}(s')=-\operatorname{sgn}(s)$ and

$$\operatorname{sgn}\{(s^{-1}\Lambda,\ f_2)(s^{-1}\Lambda,\ f_4)\}=\operatorname{sgn}\{(s'^{-1}\Lambda,\ f_2)(s'^{-1}\Lambda,\ f_4)\}.$$

When s varies on the whole W_k, so does s'. Hence we get

$$\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_0,\ u\hat{s},\ s^{-1}\Lambda)$$
$$=-\sum_{s'\in W_k}\operatorname{sgn}(s')\operatorname{sgn}\{(s'^{-1}\Lambda,\ f_2)(s'^{-1}\Lambda,\ f_4)\}\prod_{i=1}^{4}\exp\{-ue_i(X)((s\hat{s})^{-1}\Lambda,\ f_i)\}$$
$$=-\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_0,\ u,\ s^{-1}\Lambda).\qquad \text{Q.E.D.}$$

Through direct calculations, it can be shown easily that for $\hat{s}_0=\hat{s}_1=s_{e_2-e_4}$ and $\hat{s}_2=s_{e_1-e_3}$, we have $u_3^{(p)}=u_4^{(p)}\,\hat{s}_p$ for $p=0,\ 1,\ 2$. So we can apply Lemma 3 to the above three pairs $(u_3^{(p)},\ u_4^{(p)})$ $(p=0,\ 1,\ 2)$. Therefore we have

$$Z(h\,;E_0,\ \Lambda)=\sum_{u\in W(E_0)}\sum_{s\in W_k}\operatorname{sgn}(s)Y(h\,;E_0,\ u,\ s^{-1}\Lambda)$$
$$=2\sum_{p=0}^{2}\sum_{s\in W_0}\operatorname{sgn}(s)Y(h\,;E_0,\ u_2^{(p)},\ s^{-1}\Lambda).\tag{4.1}$$

Note that $u_2^{(0)}=1$, $u_2^{(1)}=s_\xi$ and $u_2^{(2)}=s_{e_4}s_\xi$, where $\xi=(e_1-e_2-e_3-e_4)/2$.

4.2. Next, we consider the set $W(E_1)$. It is also composed of the following three subsets:

$$W(E_1)^{(0)}=\left\{\begin{array}{l}(e_i+e_j,\ e_i-e_j,\ e_k,\ e_m)\ ;\\ \{i,\ j,\ k,\ m\}=\{1,2,3,4\}\quad i<j \text{ and } k<m\end{array}\right\},$$
$$W(E_1)^{(1)}=\left\{\begin{array}{l}u_{j,\pm}^{(1)}=(e_1\pm e_j,\ e_i\pm e_k,\ \{e_1\mp e_j+(e_i\mp e_k)\}/2,\\ \qquad\{e_1\mp e_j-(e_i\mp e_k)\}/2)\quad (O.S.)\ ;\\ \{i,\ j,\ k\}=\{2,\ 3,\ 4\}\ \text{ and }\ i<k\end{array}\right\},$$
$$W(E_1)^{(2)}=\left\{\begin{array}{l}u_{j,\pm}^{(2)}=(e_1\pm e_j,\ e_i\mp e_k,\ \{e_1\mp e_j+(e_i\pm e_k)\}/2,\\ \qquad\{e_1\mp e_j-(e_i\pm e_k)\}/2)\quad (O.S.);\\ \{i,\ j,\ k\}=\{2,\ 3,\ 4\}\ \text{ and }\ i<k\end{array}\right\}.$$

Put $s_0=s_{e_1-e_3}s_{e_2-e_4}$, then $s_0E_1=(e_3+e_4,\ e_3-e_4,\ e_1,\ e_2)$. Applying Lemma 4 in [5] to $W(E_1)^{(0)}$, we can divide $W(E_1)^{(0)}\setminus\{1,\ s_0\}$ into two subsets W_1

and W_2 such that $W_1 = W_2 \cdot s_{e_1 - e_3}$. As to $W(E_1)^{(1)}$ and $W(E_1)^{(2)}$, there exist the following correspondences in $W(E_1)^{(1)} \backslash \{u^{(1)}_{2,\pm}\}$ and $W(E_1)^{(2)} \backslash \{u^{(2)}_{2,\pm}\}$:

$$u^{(1)}_{3,\pm} = u^{(1)}_{4,\mp} \cdot s_{e_1 - e_3} \ (O.\ S.) \text{ and } u^{(2)}_{3,\pm} = u^{(2)}_{4,\pm} \cdot s_{e_1 - e_3} \ (O.\ S.).$$

Furthermore we have the next lemma concerning to $W(E_1)$, similar to Lemma 3.

Lemma 4. *For any two elements u and u′ in $W(E_1)$ such that $u^{-1}u' = s_{e_1 - e_3}$, the following equality holds:*

$$\sum_{s \in W_k} \operatorname{sgn}(s) Y(h\,; E_1, u, s^{-1}\Lambda) = -\sum_{s \in W_k} \operatorname{sgn}(s) Y(h\,; E_1, u', s^{-1}\Lambda).$$

From this lemma, we see that, in the summation over $u \in W(E_1)$ in $Z(h\,; E_1, \Lambda)$ in §2, there rest only the following u's: $\{1, s_0, u^{(1)}_{2,\pm}, u^{(2)}_{2,\pm}\}$. On the other hand, as elements in the Weyl group $W(\Sigma(\mathfrak{h}))$, we have

$$u^{(1)}_{2,+} = s_\xi = u^{(1)}_2,$$
$$u^{(1)}_{2,-} = s_{e_2} s_{e_4} s_\xi = u^{(1)}_2 \cdot s_0,$$
$$u^{(2)}_{2,+} = s_{e_4} s_\xi = u^{(2)}_2,$$
$$u^{(2)}_{2,-} = s_{e_2} s_\xi = u^{(2)}_2 \cdot s_0.$$

For brevity, we write $u^{(p)}$ for $u^{(p)}_2$ in the following. Then we get

$$Z(h\,; E_1, \Lambda) =$$
$$2\sum_{p=0}^{2} \sum_{s \in W_0} \operatorname{sgn}(s) \{Y(h\,; E_1, u^{(p)}, s^{-1}\Lambda) + Y(h\,; E_1, u^{(p)} s_0, s^{-1}\Lambda)\}. \tag{4.2}$$

Moreover for any pair (u, u') in $W(E_1)$ such that $u' = us_0$, we get that

$$\operatorname{sgn}(s) Y(h\,; E_1, u', s^{-1}\Lambda) = \operatorname{sgn}(s) \operatorname{sgn}\{(s^{-1}\Lambda, f_2)(s^{-1}\Lambda, f_4)\}$$
$$\times \prod_{i=1}^{3} \exp\{-(us_0)(e_i)(X)(s^{-1}\Lambda, f_i)\} \exp\{-(us_0)(e_4)(X)\,|\,(s^{-1}\Lambda, f_4)\,|\}$$
$$= \operatorname{sgn}(s') \operatorname{sgn}\{(s'^{-1}\Lambda, f_2)(s'^{-1}\Lambda, f_4)\} \prod_{i=1, \neq 2}^{4} \exp\{-ue_i(X)(s'^{-1}\Lambda, f_i)\}$$
$$\times \exp\{-ue_2(X)\,|\,(s'^{-1}\Lambda, f_2)\,|\}, \tag{4.3}$$

where $s' = s \cdot s_0$. Note that $s' \in W_0$ and as s varies on the whole W_0, so does s'.

4.3. Lastly, since the set $W(E_2)$ consists of the three elements $u^{(p)}$ $(p = 0, 1, 2)$, we see from (3.5) that

$$Z(h\,; E_2, \Lambda) = 2\sum_{p=0}^{2} \sum_{s \in W_0} \operatorname{sgn}(s) Y(h\,; E_2, u^{(p)}, s^{-1}\Lambda). \tag{4.4}$$

§ 5. The alternating sum of $Z(h\,; E_i, \Lambda)$'s

In this section, we calculate the alternating sum of $Z(h\,; E_i, \Lambda)$'s. First, we divide W_0 into the following four subsets:

$$\begin{aligned} R_1 &= \{s \in W_0\,; sf_2 \in P_0 \text{ and } sf_4 \in P_0\}, \\ R_2 &= \{s \in W_0\,; sf_2 \in P_0 \text{ and } -sf_4 \in P_0\}, \\ R_3 &= \{s \in W_0\,; -sf_2 \in P_0 \text{ and } sf_4 \in P_0\}, \\ R_4 &= \{s \in W_0\,; -sf_2 \in P_0 \text{ and } -sf_4 \in P_0\}. \end{aligned}$$

For each j, the signs of $(s^{-1}\Lambda, f_2)$ and $(s^{-1}\Lambda, f_4)$ are uniquely determined for all $s \in R_j$.

5. 1. For $s \in R_1$, paying attention to the explicit expressions in (3. 1)$\sim$(3. 4) and (4. 3), we see immediately that for $p=0, 1, 2$,

$$\begin{aligned} &\operatorname{sgn}(s) Y(h\,; E_0, u^{(p)}, s^{-1}\Lambda) = \operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}s_0, s^{-1}\Lambda) \\ &= \operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}, s^{-1}\Lambda) = \operatorname{sgn}(s) Y(h\,; E_2, u^{(p)}, s^{-1}\Lambda). \end{aligned}$$

Similarly for $s \in R_2$ and $p=0, 1, 2$,

$$\begin{aligned} &\operatorname{sgn}(s) Y(h\,; E_0, u^{(p)}, s^{-1}\Lambda) = \operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}s_0, s^{-1}\Lambda), \\ &\operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}, s^{-1}\Lambda) = \operatorname{sgn}(s) Y(h\,; E_2, u^{(p)}, s^{-1}\Lambda). \end{aligned}$$

Moreover for $s \in R_3$ and $p=0, 1, 2$,

$$\begin{aligned} &\operatorname{sgn}(s) Y(h\,; E_0, u^{(p)}, s^{-1}\Lambda) = \operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}, s^{-1}\Lambda), \\ &\operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}s_0, s^{-1}\Lambda) = \operatorname{sgn}(s) Y(h\,; E_2, u^{(p)}, s^{-1}\Lambda). \end{aligned}$$

5. 2. Put $X(h\,; u^{(p)}, E_i, j)$ as follows:

$$X(h\,; u^{(p)}, E_i, j) = \begin{cases} \sum_{s \in R_j} \operatorname{sgn}(s) Y(h\,; E_i, u^{(p)}, s^{-1}\Lambda), & \text{for } i=0, 2, \\ \sum_{s \in R_j} \{\operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}, s^{-1}\Lambda) \\ \quad + \operatorname{sgn}(s) Y(h\,; E_1, u^{(p)}s_0, s^{-1}\Lambda)\}, & \text{for } i=1. \end{cases}$$

Then for each $j=1, 2, 3$, we see from 5. 1 that

$$\sum_{i=0}^{2} (-1)^i X(h\,; u^{(p)}, E_i, j) = 0. \tag{5. 1}$$

5. 3. Finally, we study the case of $s \in R_4$. For E_0, using the concrete expression in (3. 1), we have

$$\operatorname{sgn}(s) Y(h\,; E_0, u^{(p)}, s^{-1}\Lambda) = \operatorname{sgn}(s) \prod_{i=1}^{4} \exp\{-u^{(p)}e_i(X)(s^{-1}\Lambda, f_i)\} \tag{5. 2}$$

For E_1 and $u=u^{(p)}$,

$$-\mathrm{sgn}(s)Y(h;E_1,u^{(p)},s^{-1}\Lambda)$$
$$=-\mathrm{sgn}(s)\prod_{i=1}^{3}\exp\{-u^{(p)}e_i(X)(s^{-1}\Lambda,f_i)\}\exp\{u^{(p)}e_4(X)(s^{-1}\Lambda,f_4)\}.$$

Put $s'=s\cdot s_{f_4}$. Then the above formula turns to be the same as (5. 2) where s is replaced by s', and s' varies on the whole R_2 as s varies on R_4. Similarly, for $u=u^{(p)}s_0$,

$$-\mathrm{sgn}(s)Y(h;E_1,u^{(p)}s_0,s^{-1}\Lambda)=\mathrm{sgn}(s'')\prod_{i=1}^{4}\exp\{-u^{(p)}e_i(X)(s''^{-1}\Lambda,f_i)\},$$

where $s''=s\cdot s_{f_2}$ and varies on the whole R_3 as s varies on R_4.

For E_2,

$$\mathrm{sgn}(s)Y(h;E_2,u^{(p)},s^{-1}\Lambda)$$
$$=\mathrm{sgn}(s)\prod_{i=1}^{2}\exp\{-u^{(p)}e_{2i-1}(X)(s^{-1}\Lambda,f_{2i-1})\}\exp\{u^{(p)}e_{2i}(X)(s^{-1}\Lambda,f_{2i})\}.$$

Put $s'''=s\cdot s_{f_2}s_{f_4}$, then

$$\mathrm{sgn}(s)Y(h;E_2,u^{(p)},s^{-1}\Lambda)=\mathrm{sgn}(s''')\prod_{i=1}^{4}\exp\{-u^{(p)}e_i(X)(s'''^{-1}\Lambda,f_i)\},$$

and s''' varies on the whole R_4.

Using these results, we get for $p=0$, 1, 2,

$$\sum_{i=0}^{2}(-1)^i X(h;u^{(p)},E_i,4)=\sum_{s\in W_0}\mathrm{sgn}(s)\prod_{i=1}^{4}\exp\{-u^{(p)}e_i(X)(s^{-1}\Lambda,f_i)\}. \tag{5. 3}$$

5. 4. From (4. 1), (4. 2), (4. 4), (5. 1) and (5. 3), we see that for $h=\exp X\in A(P)$,

$$\sum_{i=0}^{2}(-1)^i Z(h;E_i,\Lambda)$$
$$=2\sum_{p=0}^{2}\sum_{s\in W_0}\mathrm{sgn}(s)\prod_{i=1}^{4}\exp\{-u^{(p)}e_i(X)(s^{-1}\Lambda,f_i)\}.$$

Furthermore both $u^{(1)}$ and $u^{(2)}$ belongs to W_0 and $u^{(1)}=u^{(2)}s_\xi s_{e_4}s_\xi$, so the sub-summation over $s\in W_0$ and $p=1$, 2 is equal to zero. Thus we get the following theorem.

Theorem. *Let $\Lambda\in\mathfrak{b}_B^*$ be regular and dominant with respect to the posisive system of Borel-de Siebenthal: $\Lambda\in(\mathfrak{b}_B^*)^+$. Then for $h=\exp X\in A(P)$ $(X\in\mathfrak{h})$, the function $\kappa^{\mathfrak{h}}(h)=\varepsilon(\Lambda)\pi'_\Lambda(h)\Delta^{\mathfrak{h}}(h)$ corresponding to the character π_Λ has the following simple expression :*

$$\kappa^{\mathfrak{h}}(h)=2\sum_{s\in W_0}\operatorname{sgn}(s)\prod_{i=1}^{4}\exp\{-e_i(X)(s^{-1}\Lambda, f_i)\}.$$

Here $\{e_i\}$ *is the orthogonal basis of the root system* $\Sigma(\mathfrak{h})$ *and* $f_i=\nu_{E_0}e_i\in\Sigma(\mathfrak{b})$, *and* W_0 *denotes the subgroup of the little Weyl group* W_k *corresponding to the sub-root-system of type* C_3 *of* Σ_k.

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE,
KYOTO UNIVERSITY

References

[1] Hirai, T.; The characters of the discrete series for semisimple Lie groups. to appear
[2] Hirai, T.; Explicit form of the characters of discrete series representations of semisimple Lie groups. In: Proceeding of Symposia in Pure Mathematics 26, 281-287 Amer. Math. Soc. 1973.
[3] Hirai, T.; Character formulas for the discrete series semisimple Lie groups. Lecture Notes in Mathematics 739, 315-340, Springer (1979).
[4] Borel, A. and de Siebenthal, J.; Les sous groupes fermés de rang maximum des groupes de Lie clos. Comment. Math. Helv. **23**, 200-221 (1949).
[5] Mikami, S.; A simple expression of the characters of certain discrete series representations. J. Math. Kyoto Univ. **19**-3, 435-470 (1979).
[6] Sugiura, M,; Conjugacy classes of Cartan subalgebras in real semisimple Lie algebra. J. Math. Soc. Japan **11**, 474-534 (1959).
[7] Vargas, J. A.; A character formula for the discrete series of a semisimple Lie group. Preprint.

CURRENT ADDRESS
DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
FUKUI UNIVERSITY
FUKUI 910, JAPAN