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Notes on the Riemann-Roch theorem on open Riemann surfaces

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Introduction

Riemann-Roch theorem, one of the most important theorems in the classical theory of Riemann surfaces, was at first extended to open Riemann surfaces by Y. Kusunoki [4], and afterwards generalized along his method by H. Mizumoto [7], M. Yoshida [12] and M. Shiba [9]. Comparing these generalizations, however, they can be classified superficially into two types, namely the generalization by Mizumoto and those of Yoshida and Shiba have somewhat different forms, where Shiba's result is clearly an extension of Yoshida's one. Whereas the relationship between the Mizumoto's result and Yoshida's one was not known, and so we intend in this paper to discuss about this relationship.

In this paper, we recall in § 1 the notion of Yamaguchi's regular operators and some related results (Yamaguchi [11]), and next in § 2 and § 3, we consider the convergence of the sequence of the certain harmonic functions by using the regular operator's method (Cf. Theorem 1). Finally, in § 4, by applying the results in § 2 and § 3, we show that the Yoshida's theorem can be regarded as an extension of the Mizumoto's one (Cf. Theorem 2). As for the notations and the terminologies concerning the differentials in this paper, we shall use those in Ahlfors and Sario [1] without repetitions, though we restrict ourselves to real differentials.

§ 1. Regular operator

Let R be an open Riemann surface, W an end towards the Alexandroff's ideal boundary Δ of R (namely, the complement of W is the closure of a regular region of R) and $\{R_n\}$ a regular exhaustion of R. Denote $W\cup\partial W$ by $\overline{W}$ and set

$HD(R)$=a Banach space of harmonic Dirichlet functions on R with respect to the norm $|||u|||=\|du\|+|u(a_0)|$, where $u\in HD(R)$, $\|du\|$ the Dirichlet norm on R of du and a_0 is a fixed point on R,

$D_0(R)$=the set of all Dirichlet potentials on R,

X=a subspace of $HD(R)$,

$C^\omega(\partial W)=\{f: f$ is a real analytic function on the relative boundary ∂W of $W\}$,

$H(\overline{W})=\{$restriction to $\overline{W}$ of a harmonic function on an open set containing $\overline{W}\}$.

Definition 1 (Yamaguchi [11]). We say a linear operator $L=L_W: C^{\omega}(\partial W) \to H(\overline{W})$ is regular (with respect to W), if it satisfies the following conditions:

(i) $Lf=f$ on ∂W,

(ii) $\|dLf\|_W<\infty$, where $\|dLf\|_W$ denotes the Dirichlet integral over W,

(iii) $\langle dLf, dLg\rangle_W=\int_{\partial W} f(dLg)^*$ for any $f, g\in C^{\omega}(\partial W)$, where $\langle dLf, dLg\rangle_W$ means the mixed Dirichlet integral over W.

Hereafter, we shall use frequently the following results (Yamaguchi [11]).

Proposition 1. (i) *If $u=L_W u$ on W for $u\in HD(R)$, u must reduce to a constant. In addition, if $L_W 1=1$ on W, the constant must reduce to zero.*

(ii) *Denote by $\{L\}$ the set of all regular operators with respect to W and $\{X\}$ the set of all subspaces of $HD(R)$, then there exists an one to one correspondence between $\{L\}$ and $\{X\}$ such that for any $u\in H(\overline{W})$, the following conditions* (1) *and* (2) *are equivalent to each other:*

(1) *$u=Lf$ on ∂W,*

(2) *$u=f$ on ∂W, $u=v+g_0$ on W for some $v\in X$ and $g_0\in D_0(R)$ and the set $\{h: h\in HD(R)$ and $\lim_{n\to\infty}\int_{\partial R_n} hdu^*=0\}$ coincides with X for each $\{R_n\}\}$.*

Hereafter, we denote by L^X the regular operator associated with the space X.

Proposition 2. (i) *$(L^X)1=1$ if $X\ni 1$, and $(L^X)1\neq 1$ if $X\not\ni 1$.*

(ii) *If $X\ni 1$, $(dL^X f)^*\in\{\omega|_W\in dX^{*\perp}+\Gamma_{e0}\cap\Gamma^1\}$ where $\omega|_W$ denotes the restriction of ω to W and $dX^{*\perp}$ the orthogonal complement of $dX^*=\{du^*: u\in X\}$ in Γ_h.*

(iii) *The closure of the linear space $\{u_f: (L^X)f=u_f+g_0$ on W where $g_0\in D_0(R)$ and $u_f\in X\}$ coincides with X.*

Proposition 3. *Suppose $L_W=L$ is a regular operator associated with X and s a harnonic function on $\overline{W}$ except for isolated singularities not accumulating to ∂W.*

(i) *If $L1=1$ and $\int_{\partial W} ds^*=0$, there exists a harmonic function on R except for the singularities of s such that* (a) *$p-s=L(p-s)$ on W,* (b) *p is independent of W,* (c) *p is unique save for an additive constant.*

(ii) *If $L1\neq 1$, for any s there exists uniquely a harmonic function p on R except for the singularities of s satisfying the above conditions* (a) *and* (b).

Proposition 4. (i) *Let $\{X_n\}$ be a sequence of subspaces of $HD(R)$ such that $\bigcap_{n=1}^{\infty}$ closure $\{\bigcap_{k=n}^{\infty} X_k\}=$ closure $\{\bigcap_{n=1}^{\infty}\sum_{k=n}^{\infty} X_k\}$, which we denote by X. Then, for any f and W, we have $\lim_{n\to\infty}|||(L^{X_n})f-(L^X)f|||_W=0$, where $|||v|||_W=\|dv\|_W+|v(a_0)|$.*

(ii) *Let $\{\Omega_n\}$ be a sequence of regions such that $\overline{\Omega}_n\subset\Omega_{n+1}$ and $\bigcup_{n=1}^{\infty}\Omega_n=R$. Suppose that X_n (resp. X) is a subspace of $HD(\Omega_n)$ (resp. $HD(R)$) which satisfies the following conditions* (a) *and* (b):

(a) *for each* $u\in X$, *there exists a sequence* $\{u_n\}$ *with* $u_n\in X_n$ *such that*

$$|||u_n-u|||_{\Omega_n}=\|du_n-du\|_{\Omega_n}+|u_n(a_0)+u(a_0)|\longrightarrow 0 \quad \text{as } n\to\infty,$$

(b) *if* $\{u_n: u_n\in X_n\}$ *is a sequence suct that* $\sup_n |||u_n|||_{\Omega_n}<\infty$, *the limit of each locally uniformly convergent subsequence* $\{u_{n_k}\}$ *belongs to* X.

Then, for any f *and* W, *we have* $|||(L^X)f-(L^{X_n})f|||_{W_n}\to 0$ *as* $n\to\infty$, *where* $W_n=W\cap\Omega_n$.

§ 2. Convergence theorems of X principal functions

In this paper, we denote by $\{\Omega_n\}$ a sequence of regions on R such that

(i) $\bar{\Omega}_n\subset\Omega_{n+1}$, $\bigcup_{n=1}^{\infty}\Omega_n=R$ and each component of $R-\Omega_n$ is non compact,

(ii) $\partial\Omega_n$ consists of a finite number of Jordan curves for each n,

(iii) $\partial\Omega_n$ is homologous to $\partial\Omega_m$ for $m>n$.

Definition 2. (Matsui [5]). Suppose, for each n, $X_n(\Omega_n)$ (resp. $X_n(R)$) is a subspace of $HD(\Omega_n)$ (resp. $HD(R)$). We say that a sequence $\{X_n(\Omega_n)\}_{n=1}^{\infty}$ (resp. $\{X_n(R)\}_{n=1}^{*}$) converges to a subspace $X(R)$ of $HD(R)$ if the following conditions are fulfilled:

(i) for each $u\in X(R)$ there exists a sequence $\{u_n\}$ with $u_n\in X_n(\Omega_n)$ (resp. $u_n\in X_n(R)$) such that $|||u_n-u|||_{\Omega_n}\to 0$ (resp. $|||u_n-u|||\to 0$) as $n\to\infty$, where a_0 is a fixed point on R,

(ii) if $\{u_n\}$ with $u_n\in X_n(\Omega_n)$ (resp. $u_n\in X_n(R)$) is a sequence such that $\sup_n |||u_n|||_{\Omega_n}<\infty$ (resp. $\sup_n |||u_n|||<\infty$), the limit of each locally uniformly convergent subsequence $\{u_{n_k}\}$ belongs to $X(R)$.

In this case, we write simply $X_n(\Omega_n)\Rightarrow X(R)$ (resp. $X_n(R)\Rightarrow X(R)$).

Let W be an end towards Δ, P_k, $k=1, 2, \cdots, K$ a finite number of points on R and V a regular region such that $\cup P_k\subset V\subset R-W$ and $R-\bar{W}-\bar{V}$ is connected, and we set

$X_n=X_n(\Omega_n)$ (resp. $X=X(R)$)=a subspace of $HD(\Omega_n)$ (resp. $HD(R)$),

L=the regular operator such that, for any $f\in C^{\omega}(\partial V\cup\partial W)$, $Lf=(L^X)f$ on W and Lf=Dirichlet solution H_f^V (which we denote by H^Vf) on V,

L_n=the regular operator such that, for any $f\in C^{\omega}(\partial V\cup\partial W)$, $Lf=(L^X)f$ on W_n and $L_nf=H^Vf$ on V, where $W_n=W\cap\Omega_n$,

s=a function on $\bar{W}\cup\bar{V}$ such that $s|_W=0$ and $s|_{\bar{V}}\in H(\bar{V}-P_i)$.

p (resp. p_n)=the solution on R (resp. on Ω_n) of the equation $p-s=L(p-s)$ on $W\cup V$ (resp. $p-s=L_n(p-s)$ on $W_n\cup V$).

Hereafter, *we call the above function* p *a* X *principal function on* R *with the singularities* s.

Lemma 2.1. *Suppose* $X_n(\Omega_n)\Rightarrow X(R)$, *then we have the following:*

(i) *if* $\int_{\partial V}ds^*=0$, $X\ni 1$ *and* $X_n\ni 1$ *for each* n, *there exists, under the suitable* *choice of additive constants, a sequence* $\{n_k\}$ *of integers such that* $n_k\to\infty$ *and* $p_{n_k}-p\to 0$ *as* $k\to\infty$, *locally uniformly on* R,

(ii) *if* $X\not\ni 1$ *and* $X_n\not\ni 1$ *for each* n, *there exists, for any* s, *a sequence* $\{n_k\}$ *of integers such that* $n_k\to\infty$ *and* $p_{n_k}-p\to 0$ *as* $k\to\infty$, *locally uniformly on* R,

(iii) *if* $\int_{\partial V} ds^*=0$, $X\ni 1$ *and* $X_n\not\ni 1$ *for each* n, *then there exists a sequence* $\{n_k\}$ *of integers such that* $n_k\to\infty$ *and* $d(p_{n_k}-p)\to 0$, *locally uniformly on* R.

Proof. At first we prove the case (i). We extend s to $R-V-W$ so that we obtain $\hat{s}\in C^2(R-V-W)$. Because of $\int_{\partial V} ds^*=0$ and the fact $R-\bar{V}-\overline{W}$ is connected, by Lemma 2 in Yamaguchi [11] we can extend ds^* to a closed differential σ so that $\sigma\in\Gamma^1(R-V-W)$ and $\sigma=ds^*$ on $W\cup(V-\cup P_i)$, hence $\sigma^*+d\hat{s}\in\Gamma(R)$. Therefore, $\sigma^*+d\hat{s}$ has a decomposition of the form

$$\sigma^*+d\hat{s}=\omega_{cn}+df_{on}^*=\omega_{hn}+df_{on}^*+dg_{on}=\omega_n+\tau_n+df_{on}^*+dg_{on}\,,$$

$$=\omega_c+df_o^*=\omega_h+df_o^*+dg_0=\omega+\tau+df_o^*+dg_o\,,$$

where $\omega_{cn}\in\Gamma_c(\Omega_n)$, $\omega_{hn}\in\Gamma_h(\Omega_n)$, $df_{on}\in\Gamma_{eo}(\Omega_n)$, $dg_{on}\in\Gamma_{eo}(\Omega_n)$, $\omega_n\in dX_n$, $\tau_n\in(dX_n)^\perp$, $\omega_c\in\Gamma_c(R)$, $\omega_h\in\Gamma_h(R)$, $df_o\in\Gamma_{eo}(R)$, $dg_o\in\Gamma_{eo}(R)$, $\omega\in dX$ and $\tau\in(dX)^\perp$. Here we note that df_{on} and dg_{on} (resp. df_o and dg_o) are harmonic on W_n (resp. W). Now we set $dp_n=ds-du_n-dg_{on}$ and $dp=ds-du-dg_o$ where $du_n=\omega_n$ and $du=\omega$, then from Theorem 3 in Yamaguchi [11] p_n (resp. p) is the solution of the equation $p-s=L_n(p-s)$ on $W_n\cup V$ (resp. $p-s=L(p-s)$ on $W\cup V$). On the other hand, we have from above decomposition forms

$$\sup_n\{\|dp-dp_n\|_{\Omega_n}\}<\infty \quad\text{and}\quad \sup_n\{\|\omega_n\|+\|\tau_n\|+\|df_{on}\|+\|dg_{on}\|\}<\infty\,.$$

Therefore, from the fact df_{on} and dg_{on} are harmonic on W_n and Lemma 3.2 in Matsui [5], there exists a sequence $\{n_k\}$ of integers such that $n_k\to\infty$, $dp_{n_k}\to dp'$ as $k\to\infty$, locally uniformly on R and moreover, $dp'=du'+dF_o=\tau_x+dG_o^*$, where $du\in dX$, $\tau_x\in(dX)^\perp$, $dF_o\in\Gamma_{eo}\cap\Gamma^1$ and $dG_o\in\Gamma_{eo}\cap\Gamma^1$. Therefore, from Propositions 1 and 2 we have $dp-dp'=dh\in\Gamma_{he}(R)$ and $h=Lh$ on W, and so we have $h=$ constant. Next, we prove the case (ii). At first, we notice that the linear space $X_n+\{\text{constant}\}$ (resp. $X+\{\text{constant}\}$) is a closed space in $HD(\Omega_n)$ (resp. $HD(R)$) (Yamaguchi [11]) and L_n (resp. L) induces the space X_n (resp. X) on Ω_n (resp. R). Now we set

$\tilde{L}f=L(f-c_f)+c_f$, $\tilde{L}_nf=L_n(f-c_{fn})+c_{fn}$,

$\tilde{X}_n=$the space induced by the operator $\tilde{L}_n$ on Ω_n,

$\tilde{X}=$the space induced by the operator $\tilde{L}$ on R,

$\tilde{s}=s-\left(\int_{\partial V} ds^*\Big/\int_{\partial W} dL1^*\right)L1$, $\tilde{s}_n=s-\left(\int_{\partial V} ds^*\Big/\int_{\partial W} dL_n1^*\right)L_n1$,

$\tilde{p}_n=$the solution of the equation: $p-\tilde{s}_n=\tilde{L}_n(p-\tilde{s}_n)$ on $W_n\cup V$,

$\tilde{p}=$the solution of the equation: $p-\tilde{s}=\tilde{L}(p-\tilde{s})$ on $W\cup V$,

where $f\in C^\omega(\partial W\cup\partial V)$,

$$c_f=\left(\int_{\partial(W\cup V)}dLf^*\Big/\int_{\partial(W\cup V)}dL1^*\right) \text{ and } c_{fn}=\left(\int_{\partial(W\cup V)}dL_nf^*\Big/\int_{\partial(W\cup V)}dL_n1^*\right).$$

Then we have $\tilde{L}_n1=1$, $\tilde{L}1=1$ and $\tilde{s}_n-\tilde{s}\to 0$ uniformly on $V\cup W$ and moreover, from the Proposition 2, $\tilde{X}_n=X_n+\{\text{constant}\}$ and $\tilde{X}=X+\{\text{constant}\}$. Consequently, there exists a sequence $\{n_k\}=\{k\}$ of integers such that $\tilde{p}_k-\tilde{p}\to 0$ as $k\to 0$, locally uniformly on R. According to Theorem 3 in Yamaguchi [11], we have

$$p=\tilde{p}-\left\{\int_{\partial(W\cup V)}dL(p-s)^*\Big/\int_{\partial W}dL1^*\right\},\quad p_k=\tilde{p}_k-\left\{\int_{\partial(W\cup V)}dL_k(\tilde{p}_k-\tilde{s}_k)^*\Big/\int_{\partial W}dL_k1^*\right\}.$$

But from Lemma 1 in [11], we get $\|dL_k\tilde{p}_k\|_{W_k}\leqq\sup_k\|d\Phi_k\|_{W_k}<\infty$, where ϕ_k denotes $H^{W_k}_{p_k}$, hence we have $\sup_k|c_{fk}|<\infty$. Consequently, there exists a sequence $\{k_\mu\}=\{\mu\}$ of integers such that $\tilde{p}_\mu-\tilde{p}\to 0$ as $\mu\to\infty$. Since $d\tilde{p}_\mu-d\tilde{p}=dp_\mu-dp$, the case (iii) is evident.

Lemma 2.2. *Let $\{X_n(R)\}$ be a sequence of subspaces of $HD(R)$ such that $\bigcap_{n=1}^{\infty}$ closure $\{\sum_{k=n}^{\infty}X_k(R)\}=$closure $\{\sum_{n=1}^{\infty}\bigcap_{k=n}^{\infty}X_k(R)\}$, which we denote by $X(R)$. Then, we have $X_n(R)\Rightarrow X(R)$.*

Proof. Since $\bigcap_{k=n}^{\infty}X_k(R)\subset\bigcap_{k=n+1}^{\infty}X_k(R)$, there exists, for each $u\in X(R)$, a sequence $\{u_n\}$ with $u_n\in\bigcap_{k=n}^{\infty}X_k(R)\subset X_n(R)$ such that $|||u-u_n|||\to 0$ as $n\to\infty$. Next, since $dX^\perp=\bigcap_{n=1}^{\infty}$ closure $\{\sum_{k=n}^{\infty}dX_k^\perp\}=$closure $\{\sum_{n=1}^{\infty}\bigcap_{k=n}^{\infty}dX_k^\perp\}$, there exists, for each $\omega\in dX^\perp$, a sequence $\{\omega_n\}$ with $\omega_n\in\bigcap_{k=n}^{\infty}dX_k^\perp\subset dX_n^\perp$ such that $\|\omega-\omega_n\|\to 0$ as $n\to\infty$. Therefore, for each $\omega\in dX$ and the limit u of a locally uniformly convergent subsequence $\{u_{n_k}\}$ such that $u_{n_k}\in X_{n_k}(R)$ and $\sup_k\|u_{n_k}\|<K$, we have $|\langle\omega, du\rangle|=\varepsilon K+|\langle du,\omega\rangle_D|\leqq\lim_{k\to\infty}|\langle du_{n_k},\omega\rangle_D|+2K\varepsilon\leqq\lim_{k\to\infty}|\langle du_{n_k},\omega_{n_k}\rangle+3K\varepsilon=3K\varepsilon$, where D denotes a regular region such that $\|\omega\|_{R-D}<\varepsilon$. Hence $du\in dX$. If $X\ni 1$, $u\in X$, and if $X\not\ni 1$, then $1\notin\bigcap_{k=n}^{\infty}X_k$ for each n, and so we have $|u(a)-v_n(a)|\leqq K_a\|du-dv_n\|\to 0$ as $n\to\infty$, where K_a is a positive constant and $\{v_n\}$ a sequence with $v_n\in\bigcap_{k=n}^{\infty}X_k$ such that $\|du-dv_n\|\to 0$ as $n\to\infty$ (Cf. Lemma 3 in [11]). Consequently, we have $|||u-v_k|||\to 0$ as $n\to\infty$, hence $u\in X$.

Theorem 1. *Let $\{X_n(\Omega_n)\}_{n=1}^{\infty}$ with $X_n(\Omega_n)\subset HD(\Omega_n)$ be a sequence of subspaces such that $X_n(\Omega_n)\Rightarrow X(R)$, where $X(R)$ is a subspace of $HD(R)$.*

(i) *If $X(R)\ni 1$, $X_n(\Omega_n)\ni 1$ for each n and $\int_{\partial V}ds^*=0$, there exists, under suitable choice of additive constants, a sequence $\{n_k\}=\{k\}$ of integers such that $|||p_k-p|||_{\Omega_k}\to 0$ as $k\to\infty$.*

(ii) *If $X(R)\not\ni 1$, $X_n(\Omega_n)\not\ni 1$ for each n, there exists, for any s, a sequence $\{n^k\}=\{k\}$ such that $|||p-p_k|||_{\Omega_k}\to 0$ as $k\to\infty$.*

(iii) *If* $X(R)\ni 1$, $X_n(\Omega_n)\not\ni 1$ *for each* n *and* $\int_{\partial V} ds^*=0$, *then there exists a sequence* $\{n_k\}=\{k\}$ *of integers such that* $\|d(p-p_k)\|_{\Omega_k}\to 0$ *as* $k\to\infty$. *Analogously, suppose* $\{X_n(R)\}$ *is a sequence of subspaces of* $HD(R)$ *such that* $\bigcap_{n=1}^{\infty}$ *closure* $\{\sum_{k=n}^{\infty} X_k(R)\}$ $=$*closure* $\{\sum_{n=1}^{\infty}\bigcap_{k=n}^{\infty} X_k(R)\}$, *which we denote by* $X(R)$. *Then, we have the same conclusions as above* (i), (ii) *and* (iii) *except for setting* R *in place of* Ω_n.

We denote simply $\lim_{n\to\infty}\int_{\partial R_n}\omega$ by $\int_{\Delta}\omega$ for a differential ω if it exists, where $\{R_n\}$ is a regular exhaustion.

Proof. Supoose $\{p_k\}$ is the sequence of $X_k(\Omega_k)$ principal functions in Lemma 2.1 and Δ_k the ideal boundary of Ω_k. From Proposition 1, we have

$$\int_{\Delta_k} p_k\,dp_k^*=\int_{\Delta_k}(u_k+f_{ok})dL_k p_k^*=\int_{\Delta_k} f_{ok}\,dL_k p_k^*=0\,,$$

$$\int_{\Delta_k} p\,dp^*=\int_{\Delta_k}(u+f_o)dLp^* \longrightarrow \int_{\Delta} f_o\,dLp^*=0 \qquad \text{as } k\to\infty\,,$$

where $u_k\in X_k(\Omega_k)$, $f_{ok}\in D_o(\Omega_k)$, $u\in X(R)$ and $f_o\in D_o(R)$. On the other hand, for $k<r$ we have from the Proposition 4

$$\begin{aligned}\int_{\Delta_r} p_r\,dp^*&=\varepsilon_r+\int_{\Delta_k} p_r\,dp^*+\int_{\Delta_r} p_r\,dL_r p^*-\int_{\Delta_k} p_r\,dL_r p^*\\ &=\varepsilon_r+\langle dp_r,\, d(Lp-L_r p)\rangle_{\Omega_k-V}+\int_{\partial V} p_r\,d(Lp-L_r p)^*\,,\end{aligned}$$

where $\varepsilon_r\to 0$ as $r\to\infty$. But from Lemma 1 in [11], we have $\sup_r \|dp_r\|_{\Omega_r-V}<\infty$, hence $\lim_{r\to\infty}\int_{\Delta_r} p_r\,dp^*=0=\lim_{r\to\infty}\int_{\Delta_r} p\,dp_r^*$ (Cf. Proposition 4). Therefore, we get $\|\|p_k-p\|\|_{\Omega_k}\to 0$ as $k\to\infty$. The last part in in this Lemma is evident (Cf. Lemma 2.2).

§ 3. Regular operators and subspaces

3.A. $\boldsymbol{HF_o(\alpha, R)}$ **and** $\boldsymbol{\Gamma_{ho}(\beta, R)}$**.** Let R_s^* be the Kerékjártó-Stoïlow's compactification of R, $\Delta_s=R_s^*-R$ and $P(\Delta_s)=\alpha\cup\beta$ a partition of Δ_s such that α is closed and $\beta=\Delta_s-\alpha$ is relative open. We set

$D(R)$=the Banach space of Dirichlet functions with respect to $\|\|*\|\|$,

$F_0^2(\alpha, R)=\{f: f\in C^2(R)\cap D(R)$ and the support of f is disjoint with a neighbourhood of $\alpha\}$,

$HF_o(\alpha, R)=\{$closure of $F_o(\alpha, R)$ in $D(R)\}\cap HD(R)$,

$\Gamma_{heo}(\alpha, R)=\{df: f\in HF_o(\alpha, R)\}$, $HM(R)=\{u: du\in\Gamma_{hm}(R)\}$,

$\Gamma_{ho}(\beta, R)$=the orthogonal complement of $\Gamma_{heo}(\alpha, R)^*$ in $\Gamma_h(R)$.

In case where α and β are both closed, we call G an end towards α if $R_s^*-\bar{D}=G^*\cup\tilde{G}^*$, $G^*\supset\alpha$ and $G^*\cap\beta=\emptyset$, where D is a regular region and $G^*=G\cup$

(closure of G in $R_s^*)\frown\varDelta_s$.

3.B. Regular operators on a finite surface. Let Ω be a finite surface and $\partial\Omega=\alpha\cup\beta\cup\gamma$, $\alpha\frown\beta=\beta\frown\gamma=\gamma\frown\alpha=\emptyset$ where α, β and γ consist of contours. We set

$HD(\bar{\Omega})=\{u\in HD(\Omega): u$ is harmonic on $\partial\Omega\}$,
$\Gamma_h(\bar{\Omega})=\{\omega\in\Gamma_h(\Omega): \omega$ is harmonic on $\partial\Omega\}$,
$HM(\beta, \Omega)=\{u\in HD(\Omega): u$ is constant on each component of $\beta\}$.

Lemma 3.1. (Matsui [6]). (i) $HF_0(\alpha, \Omega)=$ *closure of* $\{HD(\bar{\Omega})\frown HF_0(\alpha, \Omega)\}$ *in* $D(\Omega)$,

(ii) $\Gamma_{h0}(\beta\cup\gamma, \Omega)=$ *closure of* $\{\Gamma_{h0}(\beta\cup\gamma, \Omega)\frown\Gamma_h(\bar{\Omega})\}$ *in* $\Gamma_h(\Omega)$,

(iii) $\Gamma_{h0}(\beta\cup\gamma, \Omega)\frown\Gamma_{hse}(\Omega)=$ *closure of* $\{\Gamma_{h0}(\beta\cup\gamma, \Omega)\frown\Gamma_{hse}(\Omega)\frown\Gamma_h(\bar{\Omega})\}$ *in* $\Gamma_h(\Omega)$.

Remark. It is evident that $HM(\beta, \Omega)=HM(\Omega)+HF_0(\alpha, \Omega)$ for a finite surface Ω.

Lemma 3.2. $HM(\beta, \Omega)\frown HF_0(\alpha, \Omega)=$ *closure* $\{HM(\beta, \Omega)\frown HF_0(\alpha, \Omega)\frown HD(\bar{\Omega})\}$.

Proof. We have only to prove the relation: closure $\{HM(\beta, \Omega)\frown HF_0(\alpha, \Omega)\frown HD(\bar{\Omega})\}\supset HM(\beta, \Omega)\frown HF_0(\alpha, \Omega)$. For each $u\in HM(\beta, \Omega)\frown HF_0(\alpha, \Omega)$, we set $\phi(P)=u(P)$ for $P\in\Omega$ and $\phi(P)=-u(j_\alpha P)$ for $P\in\Omega_\alpha-\Omega$, where $\hat{\Omega}_\alpha$ is the double of Ω with respect to α, j_α being the involutory mapping of $\hat{\Omega}_\alpha$, then we have $\phi\in HM(\beta\cup j_\alpha\beta, \hat{\Omega}_\alpha)$, and so from Lemma 3.1, there exists a sequence $\{\phi_n\}$ with $\phi_n\in HM(\beta\cup j_\alpha\beta, \hat{\Omega}_\alpha)\frown HD(\bar{\hat{\Omega}}_\alpha)$ such that $\|\phi_n-\phi\|_{\hat{\Omega}_\alpha}\to 0$ as $n\to\infty$. Therefore, by setting $f_n(p)=\frac{1}{2}\{\phi_n(p)-\phi_n(j_\alpha p)\}$ and $f(p)=\frac{1}{2}\{\phi(p)-\phi(j_\alpha p)\}$, we have $f_n|_\Omega\in HF_0(\alpha, \Omega)\frown HM(\beta, \Omega)\frown HD(\bar{\Omega})$, $f|_\Omega=u$ and $\|f_n-f\|_\Omega\to 0$ as $n\to\infty$. q.e.d.

Suppose W_α, W_β and W_γ are ends towards α, β and γ, respectively, where $\overline{W}_\alpha\frown\overline{W}_\beta=\overline{W}_\beta\frown\overline{W}_\gamma=\overline{W}_\alpha\frown\overline{W}_\gamma=0$.

Lemma 3.3. *Let L be the regular operator associated with $HM(\beta, \Omega)\frown HF_0(\alpha, \Omega)$. Then, $L=H^{W_\alpha}$ an W_α, $L=(Q)L_1$ on W_β and $L=L_0$ on W_γ, where $(Q)L_1$ (resp. L_0) denotes the Sario's $(Q)L_1$ (resp. L_0) principal operator for W_β (resp. W_γ) and $H^{W_\alpha}=$the Dirichlet operator $C^\omega(\partial W\frown\Omega)\to H(\overline{W}_\alpha)$ such that $H^{W_\alpha}f=0$ on α and $H^{W_\alpha}f=f$ on $\partial W_\alpha\frown\Omega$.*

Proof. At first, we denote by Y the space associated with L, then we can prove easily $Y\subset HF_0(\alpha, \Omega)\frown HM(\beta, \Omega)$ by Proposition 1, 2 and Lemma 3.1. Conversely, for each $u\in HF_0(\alpha, \Omega)\frown HM(\beta, \Omega)\frown HD(\bar{\Omega})$ and any $f\in C^\omega(\partial W)$ we have $\int_\gamma u(dLf)^*=0$, $\int_\beta u(dLf)^*=0$ and $\int_\alpha u(dLf)^*=0$ (Cf. Proposition 1), and so from Lemma 3.2 we have $HF_0(\alpha, \Omega)\frown HM(\beta, \Omega)\subset Y$. q.e.d.

3.C. Regular operators on a bordered surface. Let Ω be a bordered surface whose border $\partial\Omega$ consists of a finite number of contours. We set $\partial\Omega=\alpha\cup\beta$, $\alpha\cap\beta=\emptyset$ where α and β consists of contours, $\gamma=\Delta_s^\Omega-\partial\Omega$, Δ_s^Ω being the ideal boundary of Ω, and set

$$HM(\beta, \Omega)=\{u\in HD(\Omega): u=\text{constant on each component of } \beta\}.$$

Remark. $HM(\beta, \Omega)=HM(\Omega)+HF_o(\beta, \Omega)$.

Suppose $\{G_n\}$ is an exhaustion of Ω such that $\partial G_n\supset\partial\Omega$ for each n and further, $\{\hat{G}_n\}$ is a regular exhaustion of $\hat{\Omega}$ where $\hat{G}_n$ (resp. $\hat{\Omega}$) is the double of G_n (resp. Ω) with respect to α. Then, from Lemma 3.3 and Proposition 4, we have

Lemma 3.4. $HM(\beta, G_n)\cap HF_o(\alpha, G_n)\Rightarrow HM(\beta, \Omega)\cap HF_o(\alpha, \Omega)$.

Next, we consider the case $\partial\Omega=\alpha'$ and $\beta'\cup\gamma'=\Delta_s^\Omega-\alpha'$ where γ' and β' are disjoint and both closed. Let $\{\Omega_n\}$ be an exhaustion of Ω such that each component of $\partial\Omega_n$ is dividing for each n and $\Omega-\Omega_n\cup\partial\Omega_n$ is an end towards β' on Ω for each n. Denoting $\partial\Omega_n-\alpha'$ by β_n, we set

$$HM(\beta', \Omega)=\{u: \text{there exists a sequence } \{u_n\} \text{ with } u_n\in HM(\beta_n, \Omega_n) \text{ such that } |||u-u_n|||_{\Omega_n}\longrightarrow 0 \quad \text{as } n\to\infty\}.$$

Lemma 3.5. (i) $HM(\beta', \Omega)=closure\{HM(\Omega)+HF_o(\beta', \Omega)\}$,
(ii) $HM(\beta_n, \Omega_n)\cap HF_o(\alpha', \Omega_n)\Rightarrow HM(\beta', \Omega)\cap HF_o(\alpha', \Omega)$.

Proof. (i) Since $HM(\beta', \Omega)\ni 1$, we have only to prove $dHM(\beta', \Omega)=$closure $\{\Gamma_{hm}(\Omega)+\Gamma_{heo}(\beta', \Omega)\}$ by Lemma 3 in Yamaguchi [11]. But, it is evident that $dHM(\beta', \Omega)\supset$closure$\{\Gamma_{hm}(\Omega)+\Gamma_{heo}(\beta', \Omega)\}$. By the definition of $HM(\beta', \Omega)$, we have $dHM(\beta', \Omega)\subset\{\Gamma_{ho}(\gamma'\cup\alpha', \Omega)\cap\Gamma_{hse}(\Omega)\}^{*\perp}=$closure$\{\Gamma_{hm}(\Omega)+\Gamma_{heo}(\beta', \Omega)\}$.

(ii) By the analogous method as in Lemma 3.3, we can prove the fact that, for each $v\in HM(\beta', \Omega)\cap HF_o(\alpha', \Omega)$, there exists a sequence $\{v_n\}$ with $v_n\in HM(\beta_n, \Omega_n)\cap HF_o(\alpha', \Omega_n)$ such that $|||v_n-v|||_{\Omega_n}\to 0$ as $n\to\infty$. Next, suppose $\{u_n\}$ with $u_n\in HM(\beta_n, \Omega_n)\cap HF_o(\alpha', \Omega_n)$ be the sequence such that $\sup_n |||u_n|||_{\Omega_n}<K$. Then, for each sequence $\{n_k\}=\{k\}$ of integers such that $u_k\to u$ as $k\to\infty$ locally uniformly on Ω, we have $u=0$ on α' since $u_n=0$ on α' for all n, and so $u\in HF_o(\alpha', \Omega)$. For $\varepsilon>0$ and $\omega\in\Gamma_{ho}(\alpha'\cup\beta', \Omega)\cap\Gamma_{hse}(\Omega)$, there exists a region D such that $\partial D\supset\alpha'$, $\|\omega\|_{\Omega-D}<\varepsilon$ and $\Omega-D\cup\partial D$ is an end towards $\beta'\cup\gamma'$. Consequently, we have $|\langle\omega, du^*\rangle|<K\varepsilon+|\langle\omega, du^*\rangle_D|=K\varepsilon+\lim_{k\to\infty}|\langle\omega, du_k^*\rangle_D|<3K\varepsilon+\lim_{k\to\infty}|\langle\omega, du_k^*\rangle_{\Omega_k}|=3K\varepsilon$, hence we have $u\in HM(\beta', \Omega)$.

3.D. Regular operators on an open Riemann surface (1). Let $P(\Delta_s)=\alpha\cup\beta\cup\gamma$ be a regular partition of Δ_s and W_α, W_β and W_γ ends towards α, β and γ, respectively. Suppose $\{\tilde{\Omega}_n\}$ is an exhaustion of R such that, for each n, each component of $\partial\tilde{\Omega}_n$ is a dividing Jordan curve and $R-\tilde{\Omega}_n\cup\partial\tilde{\Omega}_n$ is an end towards β. We set

$HM(\beta, R)=\{u$: there exists a sequence $\{u_n\}$ with $u_n\in HM(\partial\tilde{\Omega}_n, \tilde{\Omega}_n)$ such that $|||u-u_n|||_{\tilde{\Omega}_n}\to 0$ as $n\to\infty\}$.

Lemma 3.6. *$HM(\beta, R)=closure\{HM(R)+HF_0(\beta, R)\}$.*

Proof. Omitted.

Next, we consider an exhaustion $\{\Omega_n^\#\}$ of R such that $R-\Omega_n^\#\cup\alpha_n$ is, for each n, an end towards α, where $\alpha_n=\partial\Omega_n^\#$.

Lemma 3.7. *$X_n=HM(\beta, \Omega_n^\#)\cap HF_0(\alpha_n, \Omega_n^\#)\Rightarrow HM(\beta, R)\cap HF_0(\alpha, R)=X$.*

Proof. For each $u\in X$, we set $f_n=u$ on $\Omega_n^\#-W_n$ and $f_n=H_u^{W_n}$ on W_n, where W_α is an end towards α and $W_n=W_\alpha\cap\Omega_n^\#$. Then, f_n has a decomposition of the form: $f_n=u_n+f_{on}$, where $u_n\in HD(\Omega_n^\#)$ and $f_{on}\in D_0(\Omega_n^\#)$. Obviously, $u_n\in X_n$ (Cf. Lemma. 3.1). Since $|||f_n-u|||_{\Omega_n^\#}\to 0$ as $n\to\infty$, we can get a sequence $\{u_n'\}$ with $u_n'=u_n+c_n$, c_n being a constant for each n, such that $|||u-u_n|||_{\Omega_n^\#}\to 0$ as $n\to\infty$. Because $\|df_{on}\|_{\Omega_n^\#}\to 0$ as $n\to\infty$, $f_{on}|_{W_n}=H_{u-u_n'}^{W_n}+H_{c_n}^{W_n}$ and $\sup_n\|dH_{c_n}^{W_n}\|_{W_n}<\infty$, we have $c_n\to 0$ as $n\to\infty$, where H^{W_n} denotes the Dirichlet operator for W_n, hence $|||u_n-u|||_{\Omega_n^\#}\to 0$ as $n\to\infty$. Next, let $\{v_n\}$ with $v_n\in X_n$ be a sequence such that $\sup_n|||v_n|||_{\Omega_n^\#}<\infty$. For the limit v of a locally uniformly convergent subsequence $\{v_{n_k}\}$, we have, by the analogous method as in Lemma 3.5, $v\in HM(\beta, R)$. On the other hand, it holds $v_k=H_{v_k}^{W_k}$ on W_k and $v_k\to H_v^{W_\alpha}$ as $k\to\infty$, and so $v=H_v^{W_\alpha}$ $=\int v\,d\mu_a$ where μ_a is the harmonic measure of W_α with respect to a (Cf. p. 28 in C. Constantinescu und A. Cornea [2]). Therefore, applying Theorem 2.4 in Fuji-i-e [3] and Lemma 3 in Ohtsuka [8] to u, we can get easily $u\in HF_0(\alpha, R)$.

Corollary. *Let L be the regular operator associated with $HM(\beta, R)\cap HF_0(\alpha, R)$. Then, $L=(Q)L_1$ on W_β, $L=L_0$ on W_γ and $L=H^{W_\alpha}$, where $(Q)L_1$ (resp. L_0) is the Sario's $(Q)L_1$ (resp. L_0) principal operator and H^{W_α} the Dirichlet operator on W_α such that $(H^{W_\alpha}f)(a)=H_f^{W_\alpha}(a)=\int fd\mu_a$, μ_a being the harmonic measure of W_α with respect to a.*

3.E. Regular operators on an open surface R (2). Let $P(\Delta_s)=\alpha\cup\beta\cup\gamma$ be a partition of Δ_s such that α and $\alpha\cup\beta$ are closed and $\alpha\cap\beta=\beta\cap\gamma=\gamma\cap\alpha=\emptyset$. Suppose $\{R_n\}$ is a regular canonical exhaustion of R and $R-R_n\cup\partial R_n=\bigcup_k D_{nk}$, where D_{nk} denotes a component for each pair (n, k). Denote $G\cup$ (closure of G in $R_s^*)\cap\Delta_s$ by G^* where G is a region on R, and we set

$$\alpha_n^*=\Delta_s\cap[\bigcup_k\{D_{nk}^*:\alpha\cap D_{nk}^*\neq 0\}],\qquad \gamma_n^*=\Delta_s\cap[\bigcup_k\{D_{nk}^*:\Delta_s\cap D_{nk}^*\subset\gamma\}],$$

$$\beta_n^*=\Delta_s-\alpha_n^*-\gamma_n^*.$$

Then, $\alpha_n^*\cup\beta_n^*\cup\gamma_n^*$ is a regular partition of Δ_s and $\alpha_n^*\downarrow\alpha$, $\gamma_n^*\uparrow\gamma$ as $n\to\infty$ (Cf. [10]).

Denoting $HM(\alpha_n^*, R)=HM(\alpha_n^*)$ and $HF_o(\beta_n^*, R)=HF_o(\beta_n^*)$, we get the followings:

Lemma 3.8. (i) *Let λ and μ be closed and relatively open sets on Δ_s, then we have $HM(\lambda\cup\mu)=HM(\lambda)\cap HM(\mu)$,*

(ii) *$HF_o(\alpha_n^*)\cap HM(\beta_m^*)\supset HF_o(\alpha_n^*)\cap HM(\beta_n^*)$ for $m>n$,*

(iii) *Denote $HF_o(\alpha_n^*)\cap HM(\beta_n^*)$ by X_n, then closure$\{\sum_{n=1}^{\infty}\bigcap_{k=n}^{\infty}X_k\}=\bigcap_{n=1}^{\infty}$closure $\{\sum_{k=n}^{\infty}X_k\}$, which we denote by X. From Lemma 2.2 we have $X_k\Rightarrow X$.*

Proof. (i) Omitted.

(ii) For each m $(m>n)$, it holds $\beta_m=(\beta_m^*\cap\alpha_n^*)\cup(\beta_m^*\cap\beta_n^*)$, and so $HF_o(\alpha_n^*)\cap HM(\beta_m^*)=HF_o(\alpha_n^*)\cap HM(\beta_m^*\cap\alpha_n^*)\cap HM(\beta_m^*\cap\beta_n^*)$. But, $HF_o(\alpha_n^*)\subset HM(\alpha_n^*)\subset HM(\alpha_n^*\cap\beta_m^*)$, we have $HF_o(\alpha_n^*)\cap HM(\beta_m^*)=HF_o(\alpha_n^*)\cap HM(\beta_m^*\cap\beta_n^*)\supset HF_o(\alpha_n^*)\cap HM(\beta_n^*)$.

(iii) It is evident that closure$\{\sum_{n=1}^{\infty}\bigcap_{k=n}^{\infty}X_k\}\subset\bigcap_{n=1}^{\infty}$closure$\{\sum_{k=n}^{\infty}X_k\}$. On the other hand, from $HF_o(\alpha_k^*)\supset HF_o(\alpha_n^*)$ for $k>n$, we have the relations: closure$\{\sum_{n=1}^{\infty}\bigcap_{k=n}^{\infty}X_k\}$ $=$closure$\{\sum_{n=1}^{\infty}HF_o(\alpha_n^*\cap\bigcap_{k=n}^{\infty}HM(\beta_k^*)\}\supset$closure$\{\sum_{n=1}^{\infty}HF_o(\alpha_n^*)\cap HM(\beta_n^*)\}\supset\bigcap_{n=1}^{\infty}$closure$\{\sum_{k=n}^{\infty}X_k\}$.

Note. X is independent on the choice of $\{R_n\}$.

From the definition of $\Gamma_{ho}(*, R)$ we have $\Gamma_{ho}(\gamma_n^*, R)\subset\Gamma_{ho}(\gamma, R)$ since $\gamma_n^*\supset\gamma$. Hence we can get

Corollary. $HM(\alpha_n^*\cup\beta_n^*)\Rightarrow$ *closure*$\{\sum_{n=1}^{\infty}HM(\alpha_n^*\cup\beta_n^*)\}$.

We denote the closure of $\{\sum_{n=1}^{\infty}HM(\alpha_n^*\cup\beta_n^*)\}$ by $HM(\alpha\cup\beta, R)=HM(\alpha\cup\beta)$.

Next, let P_i, $i=1, 2, \cdots, K$ be points of R and V a regular canonical region containing $\cup P_i$, and we set

$$s\in H(\bar{V}-\bigcup_{i=1}^{K}P_i),$$

$$X=\text{closure}\{\sum_{n=1}^{\infty}HF_o(\alpha_n^*, R)\cap HM(\beta_n^*, R)\}.$$

Further, let α_n'', β_n'' and γ_n'' be sets of Jordan curves on ∂R_n which are the derivations of α_n^*, β_n^* and γ_n^*, respectively. From Lemmatta 3.4, 3.5, 3.7 and 3.8 we can construct the another regular canonical exhaustion $\{G_n\}$ such that $X_n'=HF_o(\alpha_n', G_n)\cap HM(\beta_n', G_n)\Rightarrow X$, where α_n' and β_n' are the Jordan curves on ∂G_n derivated by α_n^* and β_n^*, respectively. Therefore, by Theorem 1 we have

Lemma 3.9. (i) *Let p (resp. p_n) be the X (resp. X_n') principal function on R (resp. G_n) with singularity s such that $\int_{\partial V}ds^*=0$, then there exists a sequence $\{k_n\}$ of integers such that $\|dp_{k_n}-dp\|_{G_{k_n}}\rightarrow 0$ as $n\rightarrow\infty$.*

(ii) *Let q (resp. q_n) be the $Z=HM(\alpha\cup\beta, R)$ (resp. $Z_n=HM(\alpha_n'\cup\beta_n', G_n)$) prin-*

cipal function on R (resp. G_n) with s such that $\int_{\partial V} ds^* = 0$, then there exists a sequence $\{n_k\}$ of integers such that $\|q_{n_k} - q\|_{G_{n_k}} \to 0$ as $k \to \infty$.

§ 4. Notes on the Riemann-Roch theorem

Let $P(\Delta_s) = \alpha \cup \beta \cup \gamma$ be the partition of Δ_s as in § 3. E. We set

$\tilde{\Lambda}_{he}(R) = \{du \in \Gamma_{he}(R)$: there exists a sequence $\{du_n \in \Gamma_{he}(R_n)\}$ such that (i) $u_n = u$ on α_n'' and $\partial u_n / \partial n = 0$ on γ_n'', (ii) $u_n =$constant on each component l of β_n'' such that $\int_l du^* = 0$, (iii) $\|u_n - u\|_{R_n} \to 0$ as $n \to \infty\}$.

Remark. From Propositions 1 and 4, $\tilde{\Lambda}_{he}(R)$ is independent on the choice of canonical exhaustion $\{R_n\}$.

Further, let D be a regular region and $R - D = \cup \Omega$ where Ω is a component. Divide $\partial\Omega$ into disjoint subarcs C_k ($k = 1, 2, \cdots, \nu$, $\partial\Omega = \sum_{k=1}^{\nu} C_k$) and let Q_k be a point on C_k. Suppose $\omega_P^{\Omega} = \omega(P, C_k, \Omega)$ is the generalized harmonic measure of C_k with respect to Ω (Cf. Mizumoto [7]). We set

$\Lambda_{he}(R) = \{du \in \Gamma_{he}(R)$: there exists a regular region D such that, for each component Ω of $R - \bar{D}$ with the condition: (closure of Ω in $R^*) \cap \alpha = 0$, we have $u(P) = \int u(Q) d\omega_P^{\Omega} = \lim_{\nu \to \infty} \sum_{k=1}^{\nu} u(Q_k) \omega(P, C_k, \Omega)$ for $P \in \Omega\}$.

Lemma 4.1. $\tilde{\Lambda}_{he}(R) \subset \Lambda_{he}(R)$.

Proof. At first, we set $\beta_n^{\Omega} = \Omega^* \cap \beta_n^*$ where $\Omega^* = \Omega \cup$(closure of Ω in $R^*) \cap \Delta_s$, $X_{n\Omega} = HF_0(\partial\Omega, \Omega) \cap HM(\beta_n^{\Omega}, \Omega)$ and $X_{\Omega} =$closure$\{\sum_{n=1}^{\infty} X_{n\Omega}\}$. Further, let W_{Ω} be an end towards $\Delta_s^{\Omega} = \Omega^* - \Omega$ and $L_{X\Omega}$: $C^{\omega}(\partial W_{\Omega}) \to H(W_{\Omega} \cup \partial W_{\Omega})$ the regular operator associated with X_{Ω}. For each $du \in \tilde{\Lambda}_{he}(R)$ we associated a function $v = \int u d\omega_P'$ on W_{Ω}, where $\omega_P' = \omega_P^{W\Omega}$. Then, by use of Lemma 1.1. in [7], we have $L_{X\Omega}(u - v) = u - v$ on W_{Ω}, hence from Proposition 1 we have $du \in \Lambda_{he}(R)$. q. e. d.

Lemma 4.2. *Suppose R satisfies the condition: $\Lambda_{he}(R) = \{0\}$, then a differential of X principal function with s on R is also a differential of Z principal function with s on R.*

Proof. Let p (resp. q) be the X (resp. Z) principal function on R with s, then from Lemmatta 3.9 and 4.1 we have $dp - dq \in \tilde{\Lambda}_{he}(R) = \{0\}$, and so $dp = dq$.

Corollary. *Suppose R satisfies the condition: $\Lambda_{he}(R) = \{0\}$, then $dX = dZ$.*

Proof. Cf. Lemma 4.2 and Corollary of Theorem 2 in Yoshida [12].

Let Y be a subspace of $HD(R)$. Now we generalize the definition of Y principal function on R (Cf. Yoshida [12]). Suppose V is a parametric disc and c a simple arc on V such that $\partial c=\zeta_1-\zeta_2$. We set $s=0$ on W and $s=\arg(z-\zeta_1)-\arg(z-\zeta_2)$ on V, where z is a local parameter of V and W an end towards Δ_s. Then $\int_{\partial V} ds^*=0$, hence there exists a harmonic differential ω on $R-\{\zeta_1, \zeta_2\}$ which has the following properties: (i) $\|\omega-ds\|_V<\infty$, (ii) ω is the differential of a harmonic function p in $R-c$ such that $p-s=L(p-s)$ on $W\cup(V-c)$ where $Lf=L^Y f$ on W and $Lf=H_f^V$ on $V-c$ for each $f\in C^\omega(\partial W\cup\partial V)$.

We call also the harmonic function p in above (ii) *a Y principal function with s on R.*

Remark. For thus generalized Y principal function, all of the Theorems and Lemmatta in §2, §3 and §4 are also true.

Let $\delta=\delta_P/\delta_Q$ be a finite divisor such that δ_P and δ_Q are integral divisor. We set

$\Lambda_1(Y, \delta)=\{\psi$: (i) ψ is a meromorphic differential on R such that $\mathrm{Re}(\psi)$ is a finite sum of differentials of Y principal functions on R, (ii) divisor of ψ (which we denote by $(\psi))>\delta$ and $\sum \mathrm{Res}(\psi)=0\}$

$S_1(Y, \delta)=\{f$: (i) f is a meromorphic function on R such that $\mathrm{Re}(df)$ is a finite sum of differentials of Y principal functions on R, (ii) $(f)>\delta\}$.

From Lemma 4.2 and its Corollary, we can get

Lemma 4.3. *Suppose R satisfies the condition: $\Lambda_{he}(R)=\{0\}$, then we have*

$S_1(X, 1/\delta)=S_1(Z, 1/\delta)$, $\Lambda_1(X, 1/\delta)=\Lambda_1(Z, 1/\delta)$ *and* $\Lambda_1(X, 1/\delta_Q)=\Lambda_1(Z, 1/\delta_Q)$.

Further, we consider the following linear spaces (over the real number field). Let D be a fixed regular canonical region such that $D\supset\delta$. We set

$\Lambda_2(Y, \delta, D)=\{\psi$: (i) ψ is a meromorphic differential such that $\mathrm{Re}(\psi)|_{R-D}$ is an exact differential du, (ii) du is a finite sum of differentials of Y principal functions on R, (iii) $(\psi)>\delta$ and $\sum \mathrm{Res}(\psi)=0\}$.

$S_2(Y, 1/\delta, D)=\{f$: (i) f is a multivalued meromorphic function such that $du=\mathrm{Re}(df)$ is exact on R and $f|_D$ is single valued, (ii) du is a finite sum of differentials of Y principal functions on R, (iii) $(f)>\delta\}$.

Analogously as in Lemma 4.3, we can get by Lemma 4.2 and its Corollary

Lemma 4.4. *Suppose R satisfies the condition: $\Lambda_{he}(R)=\{0\}$, then we have*

$$S_2(X, 1/\delta, D)=S_2(Z, 1/\delta, D),\ \Lambda_2(X, \delta, D)=\Lambda_2(Z, \delta, D) \text{ and}$$

$$\Lambda_2(X, 1/\delta_Q, D)=\Lambda_2(Z, 1/\delta_Q, D).$$

Now we consider the relationship between the Riemann-Roch theorems by Mizumoto and by Yoshida. First, we have

Theorem A. (Cf. Mizumoto [7]). *Suppose R satisfies the condition*: $\Lambda_{he}(R)=\{0\}$, *then we have*

$$\dim S_2(X, 1/\delta, D)=2(d+1-g)+\dim \Lambda_2(X, \delta, D),$$

where g is the genus of D and $d=\deg \delta$.

Next, Yoshida proved the following:

Theorem B. (Cf. Yoshida [12]). *Suppose $Y\supset HM(R)$, then we have*

$$\dim S_1(Y, 1/\delta)=2\{\deg \delta_P+1-\min(1, \deg \delta_Q)\}-\dim\frac{\Lambda_1(Y, 1/\delta_Q)}{\Lambda_1(Y, \delta)}.$$

As we see in the above Theorem A and Theorem B, the formulations of the Riemann-Roch theorems by Mizumoto and by Yoshida are different each other, and so, in order to compare these two theorem, it is necessary to express them in the analogous form. Therefore, we modify Theorem A (resp. Theorem B) and reformulate it in terms of $S_1(X, 1/\delta)$ and $\Lambda_1(X, \delta)$ (resp. $S_2(Y, 1/\delta, D)$ and $\Lambda_2(Y, \delta, D)$) as follows:

Theorem A′. *Suppose R satisfies the condition*: $\Lambda_{he}(R)=\{0\}$, *then we have*

$$\dim S_1(X, 1/\delta)=2\{\deg \delta_P+1-\min(1, \deg \delta_Q)\}-\dim\frac{\Lambda_1(X, 1/\delta_Q)}{\Lambda_1(X, \delta)}.$$

Theorem B′. *Suppose $Y\supset HM(R)$, then we have*

$$\dim S_2(Y, 1/\delta, D)=2(d+1-g)+\dim (\Lambda_2(Y, \delta, D),$$

where g is the genus of D and $d=\deg \delta$.

Proof. Theorem A′ and Theorem B′ can be proved by the same method as in Kusunoki [4] or Yoshida [12], and so omitted.

Now, to compare Theorem A (resp. Theorem B) with Theorem B′ (resp. Theorem A′), we consider a Riemann surface satisfying the condition $\Lambda_{he}(R)=\{0\}$. Then, by setting $Y=Z$ in Theorem B, we have Theorem A′ from Lemma 4.3, and moreover, if we set $Y=Z$ in Theorem B′, it reduces to Theorem A from Lemma 4.4. Therefore, we have the following:

Theorem2. *Concerning the Riemann-Roch theorem on open Riemann surfaces, the theorem by Yoshida can be regarded as an extension of that by Mizumoto.*

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