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On some elements in MSp_{4*} which are related to the image of J-homomorphism

By

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§ 1. Introduction and statement of results

Let MSp be the Thom spectrum of symplectic vector bundles and S^0 be the sphere spectrum. Consider the cofibering,

$$S^0 \xrightarrow{i} MSp \xrightarrow{q} MSp/S^0,$$

where i is the inclusion of the bottom sphere and q is the canonical projection. Associated to it we have an exact sequence of homotopy groups,

$$\cdots \xrightarrow{\partial} \pi_l(S^0) \xrightarrow{i_*} \pi_l(MSp) \xrightarrow{q_*} \pi_l(MSp/S^0) \xrightarrow{\partial} \pi_{l-1}(S^0) \longrightarrow \cdots$$

In [7] Ray proved that $i_*(\mathrm{Im}\, J_l)=0$ for $l \geqq 2$, where J_l: $\pi_l(SO)\to\pi_l(S^0)$ is the J-homomorphism. Let j_k be a generator of $\mathrm{Im}\, J_{4k-1}$. Then clearly there exist elements $y_k \in \pi_{4k}(MSp/S^0)$ and $x_k \in \pi_{4k}(MSp)$ such that $\partial y_k = j_k$ and $q_*(x_k)=m(2k)y_k$, where $m(2k)$ is the order of j_k[1] [5].

In this paper we shall study the properties of such elements x_k and y_k.

Throughout this paper the coefficients of homology and cohomology groups are always integers, Z.

Let h: $\pi_*(MSp)\to H_*(MSp)$ and h': $\pi_*(MSp/S^0)\to H_*(MSp/S^0)$ be the Hurewicz homomorphisms of MSp and MSp/S^0. Let BSp be the classifying space of Sp-bundles. Recall that

$$H_*(MSp)=Z[b_1, b_2,\cdots],$$

$$H_*(BSp) = Z[\beta_1, \beta_2,\cdots].$$

For each sequence $R=(r_1, r_2,\cdots)$ with almost all $r_i=0$, we denote $\beta_1^{r_1}\beta_2^{r_2}\cdots$ by $\beta^R \in H_{4|R|}(BSp)$ and its dual by $p^R \in H^{4|R|}(BSp)$, similarly we use the notations $b^R \in H_{4|R|}(MSp)$ and $S^R \in H^{4|R|}(MSp)$, where $|R|=\sum ir_i$. On the other hand, as is well known, $H^*(BSp)$ is a polynomial ring of symplectic Pontrjagin classes $\{p_k\}$, so each p^R is uniquely expressed by polynomials of $\{p_k\}$. Define an integer μ^R by the equation,

$$p^R = \mu^R p_k + \text{other terms},$$

here $|R| = k$.

We denote the KO-orientation class of MSp by $\tau\colon MSp \to KO$, where KO is the representative spectrum of real K-theory.

Our results are as follows.

Theorem 1. *For any y_k such that $\partial y_k = j_k$,*

i) *if k is odd then*

$$h'(y_k) \equiv -(2k-1)! \sum_{|R|=k} \mu^R b^R \quad \text{mod.}\ \mathrm{Im}\ h_{4k},$$

ii) *if k is even then*

$$4h'(y_k) \equiv -(2k-1)!2 \sum_{|R|=k} \mu^R b^R \quad \text{mod.}\ \mathrm{Im}\ h_{4k},$$

here we identify $H_*(MSp)$ and $H_*(MSp/S^0)$ for $*>0$.

Theorem 2. *For any y_k such that $\partial y_k = j_k$, the order of $h'(y_k)$ in $H_{4k}(MSp)/\mathrm{Im}\ h_{4k}$ is $m(2k)$.*

Theorem 3. *$\tau_*(x_k)$ can not be divided by any proper divisor of $m(2k)$, especially $x_k \not\equiv 0$ mod. $2\pi_{4k}(MSp)$.*

Corollary 4. *Let $\mu_j \in \pi_{8j+1}(S^0)$ be μ-series defined in [2], then*

$$i_*(\mu_j) = \alpha x_2^j \neq 0,$$

where $j \geqq 0$ and $\alpha \in \pi_1(MSp) = Z/2Z$ is a unique non zero element.

Combining Theorem 1 and 2, we can determine the Hurewicz image of x_k under some choice of x_k, for example,

$$h(x_1) = -24b_1$$

$$h(x_2) = 1440b_2 - 72b_1^2$$

$$h(x_3) = -5!7\cdot 8\cdot 9(3b_3 - 3b_2 b_1 + b_1^3).$$

We also find the relations among $\{x_k\}$ under some condition in Theorem 5 of §4, for example, in $\pi_{4*}(MSp)/8\pi_{4*}(MSp)$

$$x_1^2 \equiv 4x_2, \quad x_1 x_2 \equiv x_3, \quad x_1 x_3 \equiv 4x_4, \quad x_2^2 \equiv x_4.$$

§2. Proof of Theorem 1

Clearly y_k is determined up to $\mathrm{Im}\ q_*$. In order to study $h'(y_k)$ it is sufficient to calculate for a special y_k. Let ξ and ξ' be generators of $\pi_{4k}(BSp)$ and $\pi_{4k}(BSO)$ respectively, where BSO is the classifying space of SO-bundles. As is well known, $r_*(\xi) = \xi'$ for k=odd and $=4\xi'$ for k=even, where $r\colon BSp \to BSO$ is the realification map. Recall that the mapping cone of $j_k = J(\xi')$ equals to the Thom complex of ξ' as stable complexes. Let $T(\xi)$ be the Thom map of ξ.

Suppose $k=\text{odd}$, then it is easy to see that there exists an element $y_k \in \pi_{4k}(MSp/S^0)$ such that the following diagram commutes up to homotopy;

$$\begin{array}{ccccccc}
S^0 & \longrightarrow & S^0 \cup e^{4k} & \longrightarrow & S^{4k} & \xrightarrow{j_k} & S^1 \\
\| & & \big\downarrow{\scriptstyle T(\xi)}^{j_k} & & \big\downarrow{\scriptstyle y_k} & & \| \\
S^0 & \xrightarrow{i} & MSp & \xrightarrow{q} & MSp/S^0 & \longrightarrow & S^1 ,
\end{array}$$

here holyzontal lines are cofibrations. Let $z_k \in H_{4k}(S^0 \underset{j_k}{\cup} e^{4k})$ be a generator, then $h'(y_k)=T(\xi)_*(z_k)$, where we identify $H_{4k}(MSp)$ and $H_{4k}(MSp/S^0)$. Consider the following commutative diagram;

$$\begin{array}{ccc}
H_{4k}(S^0 \underset{j_k}{\cup} e^{4k}) & \xrightarrow{T(\xi)_*} & H_{4k}(MSp) \\
\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \cong} \\
H_{4k}(S^{4k}) & \xrightarrow{\quad \xi_* \quad} & H_{4k}(BSp) ,
\end{array}$$

here vertical arrows indicate the Thom isomorphisms. So for our purpose it is sufficient to determine $\xi_*(g)$, for a generator g of $H_{4k}(S^{4k})$. As is well known, $p_i(\xi)=-(2k-1)!\bar{g}$ for $i=k$ and $=0$ otherwise, where $\bar{g} \in H^{4k}(S^{4k})$ is the dual of g.

Put $\xi_*(g)=\sum_{|R|=k} \lambda^R \beta^R$. Using duality of $H_*(BSp)$ and $H^*(BSp)$ we have

$$\lambda^R=\langle p^R(\xi),\, g\rangle=\langle \mu^R p_k(\xi),\, g\rangle=-(2k-1)!\mu^R.$$

Thus we have

$$h'(y_k)=-(2k-1)! \sum_{|R|=k} \mu^R b^R.$$

Therefore for any y_k such that $\partial y_k=j_k$,

$$h'(y_k) \equiv -(2k-1)! \sum_{|R|=k} \mu^R b^R \quad \text{mod.}\ \mathrm{Im}\, h_{4k}.$$

This completes the proof of i). The proof of ii) is similar, so we omit it.

Remark. For $k=\text{even}$ the author does not know how to determine $h'(y_k)$.

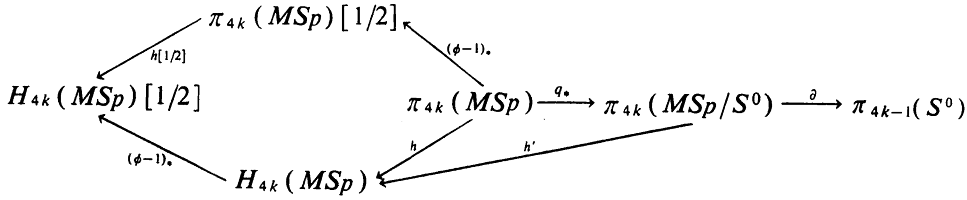
§3. Interpretations of Adams *e*-invariant

In this section we shall prove Theorem 2, Theorem 3 and Corollary 4. Our proof is based on the results of [2].

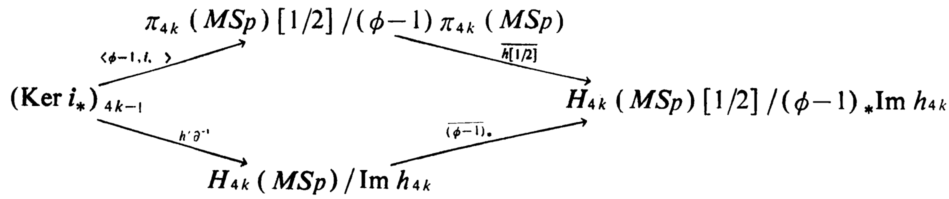
Proof of Theorem 2

Adams [2] and Mahowald [5] proved that any element of $\mathrm{Im}\, J$ is detected by e'_R-invariant. This means that any element of $\mathrm{Im}\, J_{4k-1}$ is detected by the functional ψ^2-1 operation, where $\psi^2\colon KO^*(\ )\to KO^*(\ )\,[1/2]$ is the real stable Adams operation. From Theorem 2.6 in [4] it is easily seen that there is a operation $\phi\colon MSp^*(\ ) \to MSp^*(\ )\,[1/2]$ such that $\tau\phi=\psi^2\tau$. Therefore any element of $\mathrm{Im}\, J_{4k-1}$ is detected by the functional $\phi-1$ operation, that is, the Toda bracket $\langle \phi-1,\, i,\, \gamma\rangle \not\equiv 0$ in $\pi_{4k}(MSp)\,[1/2]/(\phi-1)\pi_{4k}(MSp)$ for any $\gamma \in \mathrm{Im}\, J_{4k-1}$.

Consider the following commutative diagram;



From the above commutative diagram and definition of Toda bracket we have a following commutative diagram;



where $\bar{f}$ means a homomorphism induced by f.

Recall that torsion in $\pi_*(MSp)$ is only two-primary, so $h[1/2]$ is monic. Therefore the composition

$$\overline{h[1/2]}\circ\langle\phi-1,\ i,\ \rangle|_{\mathrm{Im}\, J_{4k-1}} \text{ is monic.}$$

This completes the proof of Theorem 2.

Remark. The invariant $h'\partial^{-1}\colon (\ker i_*)_{4k-1}\to H_{4k}(MSp)/\mathrm{Im}\, h_{4k}$ is closely related to $MSp-e$-invariant e_{MSp}:

$$(\ker i_*)_{4k-1} \longrightarrow \mathrm{Ext}^{1,4k}_{MSp_*MSp}(MSp_*, MSp_*).$$

That is, there is a commutative diagram;

$$\begin{array}{ccc} (\ker i_*)_{4k-1} & \xrightarrow{e_{MSp}} & \mathrm{Ext}^{1,4k}_{MSp_*MSp}(MSp_*, MSp_*) \\ & {\scriptstyle h'\partial^{-1}}\searrow \quad \swarrow{\scriptstyle f} & \\ & H_{4k}(MSp/S^0)/\mathrm{Im}\, h_{4k} = H_{4k}(MSp)/\mathrm{Im}\, h_{4k} & \end{array}$$

here the homomorphism f is given by the following composition (See 15 in Part III of [3]),

$$\begin{array}{l} \mathrm{Ext}^{1,4k}_{MSp_*MSp}(MSp_*, MSp_*) \\ \quad \big\downarrow \text{inclusion} \\ \pi_{4k}(MSp\wedge MSp/S^0)/\mathrm{Im}\,(\pi_{4k}(MSp) \xrightarrow{q_*} \pi_{4k}(MSp/S^0) \xrightarrow{h^{MSp}} \pi_{4k}(MSp\wedge MSp/S^0)) \\ \quad \big\downarrow H\text{-orientation of } MSp \\ H_{4k}(MSp/S^0)/\mathrm{Im}\, h_{4k}. \end{array}$$

Proof of Theorem 3

By definition of x_k and Toda bracket, we easily see that

$$x_k \in \langle i, j_k, m(2k)\rangle .$$

Then $\tau_*(x_k) \in \tau \circ \langle i, j_k, m(2k)\rangle \subset \langle \tau \circ i, j_k, m(2k)\rangle$. Note that

$$(\psi^2 - 1)\langle \tau \circ i, j_k, m(2k)\rangle = \langle \psi^2 - 1, \tau \circ i, j_k\rangle m(2k)$$

as cosets in $KO_{4k}[1/2]/m(2k)(\psi^2-1)KO_{4k}$. If for some proper divisor d of $m(2k)$, $x_k = dx'$ in $\pi_{4k}(MSp)$, then

$$d(\psi^2-1)\tau_*(x') \in \langle \psi^2 - 1, \tau \circ i, j_k\rangle m(2k).$$

So we have that

$$(\psi^2-1)\tau_*(x') \in \langle \psi^2 - 1, \tau \circ i, j_k\rangle m(2k)/d.$$

But this contradicts with a fact that the order of $e'_R(j_k) = \langle \psi^2 - 1, \tau \circ i, j_k\rangle$ in $KO_{4k}[1/2]/(\psi^2-1)KO_{4k}$ is $m(2k)$.
This completes the proof.

Proof of Corollary 4

Our proof is based on the following facts;
i) $m(4) = 2^4 \cdot 3 \cdot 5$. ii) $x_2 \in \langle i, j_2, m(4)\rangle \subset \langle i, 8\sigma, 2\rangle$, where $\sigma = 3 \cdot 5 \cdot j_2$. iii) Indeterminacy of $\langle i, 8\sigma, 2\rangle = 2\pi_8(MSp)$, because $\pi_8(MSp)$ is torsion free (See, for example, [8]). iv) $\mu_{j+1} \in \langle 8\sigma, 2, \mu_j\rangle$ (See [2]). v) $\mu_0 = \eta$ and $i_*\eta = \alpha$, where η is a unique non zero element of $\pi_1(S^0)$.

Using the above facts and by induction,

$$\begin{aligned} i_*\mu_{j+1} \in i\langle 8\sigma, 2, \mu_j\rangle &= \langle i, 8\sigma, 2\rangle \mu_j \\ &= \langle i, 8\sigma, 2\rangle \cdot i_*\mu_j = x_2 \cdot \alpha x_2^j = \alpha x_2^{j+1}. \end{aligned}$$

So we have that $i_*\mu_j = \alpha x_2^j$ for $j \geqq 0$.
On the other hand

$$\begin{aligned} \tau_*(\alpha x_2^j) &= (\tau_*\alpha) \cdot (\tau_* x_2^j) \\ &= e \cdot y^j \quad \text{(By Theorem 3)} \\ &\neq 0, \end{aligned}$$

where $e \in \pi_1(KO)$ and $y \in \pi_8(KO)$ are generators. This completes the proof of Corollary 4.

§4. Relations in $\{x_k\}$

In this section we shall prove the following;

Theorem 5. *If* $i_*\pi_{4l}(S^0) = 0$, $i_*\pi_{4k}(S^0) = 0$ *and* $i_*\pi_{4(k+l)}(S^0) = 0$ *and if for any* $\gamma \in \pi_{4(k+l)-1}(S^0)$ *such that* $i_*\gamma = 0$, $8\gamma = 0$ *and* $e'_R(\gamma) = 0$, $\langle i, \gamma, 8\rangle \ni 0$, *then*

$$x_k x_l - a_{k,l} x_{k+l} \equiv 0 \quad \text{mod.}\ 8\pi_{4(k+l)}(MSp),$$

where $a_{k,l}=4$ *for* $k\equiv l\equiv 1$ mod. 2, *and* $=1$ *otherwise.*

Proof. Take $\sigma_n \in \operatorname{Im} J_{4n-1}$ so that $e'_R(\sigma_n)=(-1)^n 1/8$. Then $i\sigma_n=8\sigma_n=0$. It is clear from [7] and [9] that

$$0\in i\langle\sigma_k, 8, \sigma_l\rangle \quad \text{and} \quad 0\in\langle\sigma_k, 8, \sigma_l\rangle 8.$$

From Proposition 1.5 of [9], there exist elements $\lambda\in\langle i, \sigma_k, 8\rangle$, $\mu\in\langle\sigma_k, 8, \sigma_l\rangle$ and $\nu\in\langle 8, \sigma_l, 8\rangle$ such that $\lambda\sigma_l=i\mu=8\mu=\sigma_k\nu=0$ and

$$\langle\lambda, \sigma_l, 8\rangle+\langle i, \mu, 8\rangle-\langle i, \sigma_k, \nu\rangle\ni 0.$$

We can take x_k as λ and $\nu=8\delta$ for some $\delta\in\pi_{4l}(S^0)$. From Theorem 11.1 of [2] we can take $-a_{k,l}\sigma_{k+l}+\gamma$ as μ where $\gamma\in\pi_{4(k+l)-1}(S^0)$ such that $8\gamma=0$, $i\gamma=0$ and $e'_R(\gamma)=0$. It is easy that $\langle x_k, \sigma_l, 8\rangle\ni x_k x_l$. From the second assumption

$$\langle i, -a_{k,l}\sigma_{k+l}+\gamma, 8\rangle\supset-\langle i, a_{k,l}\sigma_{k+l}, 8\rangle\ni -a_{k,l}x_{k+l}.$$

From the first assumption

$$\langle i, \sigma_k, 8\delta\rangle\supset\langle i, \sigma_k, 8\rangle i_*\delta=0.$$

Therefore the set $x_k x_l - a_{k,l}x_{k+l}+$(Indeterminacy of brackets) contains zero. So from the first assumption we have

$$x_k x_l - a_{k,l} x_{k+l} \equiv 0 \quad \text{mod.}\ 8\pi_{4(k+l)}(MSp).$$

This completes the proof.

Remark. The first assumption is correct for amall k and l, because $\pi_{4n}(MSp)$ is torsion free for small n (See, for example, [8]). Using the detail study of the spectral sequence such that $E^2=H_*(MSp)\otimes\pi_*(S^0)\Rightarrow\pi_*(MSp)$, we can check the second assumption for small k and l, especially for $k+l\leqq 5$ this holds [6].

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