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On the existence of meromorphic functions with certain lower order on non-hyperelliptic Riemann surfaces

By

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§ 1. Introduction.

On all compact Riemann surfaces of genus $g \geqq 2$ there exist meromorphic functions of order $n \geqq g+1$. It is also known that every non-hyperelliptic Riemann surface of genus $g \geqq 4$ admits infinitely many meromorphic functions of order g, and that there exists no meromorphic function of odd order $n \leqq g$ on hyperelliptic Riemann surfaces of genus g. But it seems to be unknown whether there are some integers $n \leqq g-1$ such that every non-hyperelliptic Riemann surface of sufficiently high genus g admits meromorphic functions of order n.

In this paper we shall prove that on every non-hyperelliptic Riemann surface of genus $g \geqq 4$ there exists at least one meromorphic function of order $g-1$. The main idea of the proof is to show the non-emptiness of the subsets $\mathring{W}^2_{g-1}$ of the Jacobi varieties (see below). The same idea will be applied to prove that every non-hyperelliptic Riemann surface of genus $g \geqq 4$ admits infinitely many meromorphic functions of order g.

In § 2, we shall recall some properties of the subvarieties W^ν_r of the Jacobi variety of a compact Riemann surface, and relate the subsets $\mathring{W}^\nu_r$ of nongap points of W^ν_r (Gunning [5]) with the existence of meromorphic functions of order r on the surface (cf. Martens [11]). We shall study in § 3 the characterization of hyperelliptic Riemann surfaces by the attainment of the maximal dimension of the subvarieties W^ν_r (see Theorem 3).

Next in § 4, we shall generalize a result of Andreotti-Mayer [2] about the subvarieties W^2_{g-1} for the trigonal Riemann surfaces in Theorem 3, and get some properties of the subvarieties W^ν_r for the trigonal Riemann surfaces similar to those for the hyperelliptic Riemann surfaces, especially the fact that the singular loci of some varieties W^ν_r for a trigonal Riemann surface are $W^{\nu+1}_r$. In § 5, we shall study the subvarieties W^2_ν of the Jacobi varieties for elliptic-hyperelliptic Riemann surfaces.

For $n=g-2$ we cannot make similar assertions to those for $n=g$, $g-1$ any more, since the elliptic-hyperelliptic Riemann surfaces of genus g admit no meromorphic function of odd order $n \leqq g-2$. There is another example of such

Riemann surfaces given by Meis [16], which are of genus $g=6$ and admit no meromorphic function of order $g-2=4$. We shall characterize such surfaces in terms of the subvarieties of the Jacobi varieties in Corollary 8 of § 4.

Thus it remains to be studied to classify the non-hyperelliptic Riemann surfaces of genus g by making use of the existence of meromorphic functions of order $n\leqq g-2$, and to characterize the classes in terms of the subvarieties of the Jacobi varieties if possible.

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§ 2. Preliminaries.

Let M denote a compact Riemann surface of genus $g>0$ and D denote a divisor on M. We denote $l(D)$ the dimension of the space of meromorphic functions on M whose divisors are the multiples of $-D$, $i(D)$ the dimension of the space of abelian differentials on M whose divisors are the multiples of D, and K the canonical divisor.

Let $J(M)$ be the Jacobi variety of M and φ be the Jacobi homomorphism. Let $W_r\subseteqq J(M)$ denote the sets $\{\varphi(D)\,|\,D$: a positive divisor of degree r on $M\}$, which are irreducible subvarieties of $J(M)$, and $W_r^\nu\subseteqq W_r$ denote the sets $\{\varphi(D)\,|\,D$: a positive divisors of degree r on M such that $l(D)\geqq\nu\}$. These subsets W_r^ν are analytic subvarieties of $J(M)$, but they may not be irreducible.

If S and T are subvarieties of $J(M)$, we can define the following subvarieties.

$$-S=\{-s\,|\,s\in S\},$$

$$S+T=\{s+t\,|\,s\in S,\ t\in T\}, \qquad S-T=S+(-T),$$

$$S\ominus T=\{u\in J(M)\,|\,T+u\subseteqq S\}=\bigcap_{t\in T}(S-t).$$

We shall recall some properties of the subvarieties W_r^ν of Jacobi variety $J(M)$.

Lemma 1. *$k-W_r^\nu=W_s^\mu$ $(s=2g-2-r,\ \mu=g-1-(r-\nu),\ r,\ s\geqq 0$ and $\nu,\ \mu\geqq 1)$, where $k=\varphi(K)$ is the canonical point of $J(M)$.*

Proof. This follows from the Riemann-Roch theorem and the fact that $i(D)=l(K-D)$.

Corollary 1. $k-W_{g-1}^\nu=W_{g-1}^\nu$.

Lemma 2.

1) $W_r\ominus W_s=W_{r-s}$ $\quad(r\geqq s)$,

2) $W_r^\nu=W_{r-\nu+1}\ominus(-W_{\nu-1})$ $\quad(1\leqq\nu\leqq r+1)$,

3) $W_r^\nu\ominus(-W_s)=W_{r+s}^{\nu+s}$ $\quad(1\leqq\nu\leqq r$ *and* $0\leqq s)$,

4) $W_r^\nu \ominus W_s = W_{r-s}^\nu \quad (1 \leqq \nu \leqq r$ *and* $0 \leqq s \leqq r-\nu+1)$.

Proof. The inclusion relation $W_{r-s} \subseteqq W_r \ominus W_s$ is obvious. Conversely for $u \in W_r \ominus W_s$, $u+W_s \subseteqq W_r$. Then there exists a non-negative divisor D such that $\varphi(D)=u$ and $D+D_s \sim D_r$ for some positive divisors D_s and D_r of degree s and r, where $\sim$ means linear equivalence. If $D+D_s=D_r$, then D is a non-negative divisor of degree $r-s$. If $D+D_s \neq D_r$, then $D \sim D_r - D_s \sim D_{r-s}$ (some non-negative divisor of degree $r-s$) and the proof of 1) is completed. To prove 2), we observe that $u-W_{\nu-1} \subseteqq W_{r-\nu+1}$ if and only if for every non-negative divisor D of degree $\nu-1$ there is a non-negative divisor D' of degree $r-\nu+1$ such that $\varphi(D+D')=u$, means $u \in W_r^\nu$ (cf. [13]).

The next observation is useful to prove 3) and 4): For any subsets A, B, C of $J(M)$,

$$(A \ominus B) \ominus C = A \ominus (B+C).$$

Using this observation, we have

$$\begin{aligned} W_r^\nu \ominus (-W_s) &= [W_{r-\nu+1} \ominus (-W_{\nu-1})] \ominus (-W_s) \\ &= W_{r-\nu+1} \ominus (-W_{s+\nu-1}) = W_{r+s}^{\nu+s} \\ W_r^\nu \ominus W_s &= [W_{r-\nu+1} \ominus (-W_{\nu-1})] \ominus W_s \\ &= W_{r-\nu+1} \ominus (W_s \ominus W_{\nu-1}) \\ &= [W_{r-\nu+1} \ominus W_s] \ominus (-W_{\nu-1}) \\ &= W_{r-s-\nu+1} \ominus (-W_{\nu-1}) = W_{r-s}^\nu . \end{aligned}$$

Proposition 1. $\dim W_r = r \quad (1 \leqq r \leqq g)$.

Proof. It follows from Abel's theorem that the Jacobi mapping $\varphi: M \rightarrow J(M)$ is a complex analytic homeomorphism between M and W_1, and from the Jacobi inversion theorem that $W_g = J(M)$ so that $\dim W_1 = 1$ and $\dim W_g = g$. Since $W_{r+1} = W_r + W_1$ so that $\dim W_{r+1} \leqq \dim W_r + 1$, we have $\dim W_r = r$ $(1 \leqq r \leqq g)$.

Proposition 2. $W_r^\nu = \emptyset$ $(1 \leqq \nu \leqq r \leqq 2g-2$, $\nu \leqq g+1$ *and* $2\nu > r+2)$

Proof. By Lemma 2, 3), we have $W_r^\nu = W_{r-\nu+1} \ominus (-W_{\nu-1})$. If $u \in W_r^\nu$, then $u - W_{\nu-1} \subseteqq W_{r-\nu+1}$. But since $g \geqq \nu-1 > r-\nu+1$, Proposition 1 asserts that $\nu-1 = \dim(u-W_{\nu-1}) \leqq \dim W_{r-\nu+1} = r-\nu+1$, which contradicts our hypothesis.

Proposition 3. *If* $W_r^\nu \neq \emptyset$, *then*

$$\dim W_{r+1}^{\nu+1} < \dim W_r^\nu \ (1 \leqq \nu \leqq r \leqq g-1) \text{ and}$$

$$\dim W_{r-1}^\nu < \dim W_r^\nu \ (1 \leqq \nu < r \leqq g).$$

Proof. If $W_{s+1}^{\nu+1} \neq \emptyset$, choose an irreducible component V of $W_{r+1}^{\nu+1}$ such that dim $V=\dim W_{r+1}^{\nu+1}$. Since $W_{r+1}^{\nu+1}=W_r^{\nu}\ominus(-W_1)$ by Lemma 2. 3), $V\subseteqq V-W_1\subseteqq W_{r+1}^{\nu+1}-W_1\subseteqq W_r^{\nu}$. As the image of an irreducible subvariety $V\times W_1$ under the addition map $J(M)\times J(M)\rightarrow J(M)$, $V-W_1$ is an irreducible subvariety of $J(M)$. If $\dim(V-W_1)$, then $V=V-W_1$ and by induction $V=V-W_g=J(M)$, which is impossible since $V(\subseteqq W_r\subseteqq W_{g-1})$ is a proper subvariety of $J(M)$. Thus we have $\dim W_{r+1}^{\nu+1}=\dim V<\dim(V-W_1)\leqq\dim W_r^{\nu}$. The second assertion can be proved similarly.

Proposition 4. *For $1\leqq\nu\leqq r\leqq g-1$, if $W_r^{\nu}\neq\emptyset$, then*

$$r\nu-(\nu-1)(g+\nu)\leqq\dim W_r^{\nu}\leqq r-2\nu+2\,.$$

Proof. Applying the first inequality of Proposition 3, it follows that

$$\dim W_r^{\nu}\leqq\dim W_{r-1}^{\nu-1}-1\leqq\cdots\leqq\dim W_{r-(\nu-1)}-(\nu-1)$$
$$=r-(\nu-1)-(\nu-1)=r-2\nu+2\,.$$

For the proof of the left hand side inequality, we refer to Martens [11] or Gunning [5, Th. 14(*b*)]. We here have to note that in the left hand side inequality $\dim W_r^{\nu}$ can be replaced by the lowest dimension of the components of W_r^{ν}.

According to Gunning [5], we call the subset $W_{r-1}^{\nu}+W_1\subseteqq W_r^{\nu}$ the gap variety of W_r^{ν}. Its complement is an open subset $\mathring{W}_r^{\nu}\subseteqq W_r^{\nu}$ which is called the subset of nongap points of W_r^{ν}.

Proposition 5. *Associated with any point $u\in\mathring{W}_r^{\nu}$ $(\nu\geqq2)$ there exists a meromorphic function f of order r on M whose polar divisor is precisely the divisor $-D$ of degree $-r$ such that $\varphi(D)=u$ and $l(D)=\nu$, and vice versa.*

Proof. Let D be a divisor of degree r such that $u=\varphi(D)$ and $D=P_1+\cdots+P_r$. Since $u\notin W_{r-1}^{\nu}+W_1$ so that $l(P_1)+\cdots+P_{i-1}+P_{i+1}+\cdots+P_r)\leqq\nu-1$ for each i $(1\leqq i\leqq r)$ and $l(D)\geqq\nu$, there exist meromorphic functions f_i such that the polar divisors of f_i are

$$(f_i)_\infty=D_i+P_i$$

where D_i are positive divisors such as D is multiple of D_i. A suitable linear combination of f_i will have precisely the divisor D as its polar divisor*). Conversely let f be a meromorphic function of order r having D as its polar divisor and $l(D)=\nu$, so that $\varphi(D)\in W_r^{\nu}$. If $\varphi(D)\in W_{r-1}^{\nu}+W_1$, then $l(D)\geqq\nu+1$ which is impossible.

Corollary 2. *There exists at least one meromorphic function of order r on M if and only if* $\bigcup_{\nu=2}^{[r/2]+1}\mathring{W}_r^{\nu}\neq\emptyset\qquad(r\geqq2)$.

Proposition 6. *If $W_r^{\nu}\neq\emptyset$, $J(M)$, then*

* If $u\in W_r^{\nu+1}=W_{r-1}^{\nu}\ominus(-W_1)$, then $u-W_1\subseteqq W_{r-1}^{\nu}$ and $u\in W_{r-1}^{\nu}+W_1$. This proves $l(D)=\nu$.

$$\dim(W_r^\nu+W_1)=\dim W_r^\nu+1\,.$$

Proof. Since $W_r^\nu+W_1$ is the image of $W_r^\nu\times W_1$ is the image of $W_r^\nu\times W_1$ under the addition mapping $J(M)\times J(M)\to J(M)$, we have $\dim(W_r^\nu+W_1)\leqq\dim W_r^\nu+1$. If $\dim(W_r^\nu+W_1)=\dim W_r^\nu$, then select an irreducible component V of W_r^ν with $\dim V=\dim W_r^\nu$. Since $V\subseteqq V+W_1$ and $V+W_1$ is an irreducible component of $W_r^\nu+W_1$ it follows that $\dim(V+W_1)\leqq\dim(W_r^\nu+W_1)=\dim V$ so that $V=V+W_1$. But then $V=V+W_1=\cdots=V+W_g=J(M)$, which is impossible.

We prove at last the useful formula which was proved by Martens [11].

Proposition 7. *For* $1\leqq\nu\leqq r$ *and* $w_1,\ w_2\in W_1$ *with* $w_1\neq w_2$,

$$(W_r^\nu+w_1)\cap(W_r^\nu+w_2)=(W_{r+1}^{\nu+1})\cup(W_{r-1}^\nu+w_1+w_2)\,.$$

Proof. We first prove the formula for $\nu=1$. Since $W_{r-1}+w_1+w_2\subseteqq W_r+w_i$ and $W_{r+1}^2=W_r\ominus(-W_1)=\bigcap_{x\in W_1}(W_r+x)\subseteqq(W_r+w_1)$ $(i=1,\ 2)$, we only have to show that

$$(W_r+w_1)\cup(W_r+w_2)\subseteqq W_{r+1}^2\cup(W_{r-1}+w_1+w_2)\,.$$

Let $w_1=\varphi(p)$, $w_2=\varphi(q)$ with $p,\ q\in M$ and $p\neq q$. Then any point $x\in(W_r+w_2)$ can be written

$$x=\varphi(P_1+\cdots+p_r+p)=\varphi(q_1+\cdots+q_r+q)$$

for some points $p_i,\ q_i\in M$ so that

$$p_1+\cdots+p_r+p\sim q_1+\cdots+q_r+q$$

by Abel's theorem. If these two divisors are identical, we may assume that $p_r=q$ and $q_r=p$. Then we have

$$x=\varphi(p_1+\cdots+p_{r-1}+q+p)\in W_{r-1}+w_1+w_2\,.$$

If these two divisors are distinct, then $l(p_1+\cdots+p_r+p)\geqq2$ so that

$$x=\varphi(p_1+\cdots+p_r+p)\in W_{r+1}^2\,.$$

To get the formula for $\nu\geqq2$, the next lemma is necessary.

Lemma 3. *Let A be an irreducible subvariety and B, C be two subvarieties of $J(M)$. Then*

$$(B\cap C)\ominus A=(B\ominus A)\cap(C\ominus A)\,,$$

$$(B\cup C)\ominus A=(B\ominus A)\cup(C\ominus A)\,.$$

Proof. These are immediate consequences of the definition of $A\ominus B$ and the fact that

$$u\in(B\cup C)\ominus A \text{ if and only if } A+u\subseteqq B\cup C$$

$$\text{if any only if } A+u\subseteqq B \text{ or } A+u\subseteqq C\,.$$

We have just proved that

$$(W_{r-\nu+1}+w_1)\cap(W_{r-\nu+1}+w_2)=W^2_{r-\nu+2}\cup(W_{r-\nu}+w_1+w_2)\,.$$

If we operate on both sides of the above equation with $\ominus(-W_1)$ and use Lemma 3, we have

$$\begin{aligned}(W^2_{r-\nu+2}+w_1)\cap(W^2_{r-\nu+2})&=((W_{r-\nu+1}\ominus(-W_1))+w_1)\cup((W_{r-\nu+1}\ominus(-W_1))+w_2)\\&=(W^2_{r-\nu+2}\ominus(-W_1))\cup((W_{r-\nu}\ominus(-W_1))+w_1+w_2)\\&=W^3_{r-\nu+3}\cup(W^2_{r-\nu+1}+w_1+w_2)\,.\end{aligned}$$

Repeating this process we have the desired result.

§ 3. The hyperelliptic Riemann surfaces.

A hyperelliptic Riemann surface M is defined to be one that can be represented as a two-sheeted branched covering of the Riemann sphere, and have the hyperelliptic involution θ corresponding to the interchage of sheets in the representation. Since $l(p+\theta p)=2$ and $p+\theta p\sim q+\theta q$ for any points $p, q\in M$, the common image $e=\varphi(p+\theta p)$ is contained in W^2_2 and called the hyperelliptic point of $J(M)$. It is evident that if $W^2_2\neq 0$ for a Riemann surface M, then M is hyperelliptic, and thus hyperelliptic Riemann surfaces can be characterized as those for which $W^2_2\neq 0$.

If $u\in W^2_2$, then $u-W_1\subseteqq W_1$ since $W^2_2=W_1\ominus(-W_1)$. But since both are irreducible and of same dimension, $u-W_1=W_1$. There is only one point having this property so that for hyperelliptic Riemann surfaces $W^2_2=e$ and $-W_1=W_1-e$. Iterating the last relation, we have $-W_{\nu-1}=W_{\nu-1}-(\nu-1)\cdot e$ so that

$$W^{\nu}_{2\nu-2}=\{(\nu-1)\cdot e\}\qquad(2\leqq\nu\leqq g)\,.$$

We also have that

$$\begin{aligned}W^{\nu}_{r}&=W_{r-\nu+1}\ominus(-W_{\nu-1})=W_{r-\nu+1}\ominus(W_{\nu-1}-(\nu-1)\cdot e)\\&=(W_{r-\nu+1}\ominus W_{\nu-1})+(\nu-1)\cdot e\\&=W_{r-2\nu+2}+(\nu-1)\cdot e\qquad(1\leqq\nu\leqq r\leqq g)\,.\end{aligned}$$

If $2\leqq\nu\leqq r\leqq g$ and $r>2(\nu-1)$, it follows that

$$W^{\nu}_{r}=W_1+W_{r-2\nu+1}+(\nu-1)\cdot e=W_1+W^{\nu}_{r-1}$$

so that $\mathring{W}^{\nu}_{r}=0$. In the special case that $r=2(\nu-1)$, we have $W^{\nu}_{r}=(\nu-1)\cdot e$, and $W^{\nu}_{r-1}=0$ by Proposition 2 so that $\mathring{W}^{\nu}_{2\nu-2}\neq 0$. From Corollary 2 together with these facts, it follows that on a hyperelliptic Riemann surface of genus g there exist no meromorphic functions of odd order $n\leqq g$ and the meromorphic functions of even order $n\leqq g$ are the lifts of the rational functions on the Riəmann of genus 0.

We can see from the formula $W^{\nu}_{r}=W_{r-2\nu+2}+(\nu-1)\cdot e$ that W^{ν}_{r} is an irreducible subvariety and of maximal dimension $r-2\nu+2$ in Proposition 4. We also can characterize the hyperelliptic Riemann surfaces by attainment of those maximal values. To do this we first prove the next Cliflord's Theorem.

Theorem 1 (Clifford's Theorem). *If* $W^{\nu}_{2\nu-2}\neq 0$ *for some index* ν $(2\leqq\nu\leqq g-1)$ *for a Riemann surface* M *of genus* g, *then* M *is hyperelliptic.*

Proof. If we deduce $W^{\lambda}_{2\lambda-2}\neq 0$ for some index λ $(2\leqq\lambda<\nu)$ from our hypothesis, we shall finally reach $W^{2}_{2}\neq 0$, which means that M is hyperelliptic. From Lemma 1, it follows that $k-W^{\nu}_{2\nu-2}=W^{\mu}_{2\mu-2}$ $(\mu+\nu=g+1)$ and we can assume that $\nu\leqq\mu$. For a point $x\in W^{\nu}_{2\nu-2}$, we set $y=k-x\in W^{\mu}_{2\mu-2}$. Then we can choose a divisor D_x of degree $2\nu-2$ and a divisor D_y of degree $2\mu-2$ such that $\varphi(D_x)=x$, $\varphi(D_y)=y$, and that at least one point of D_x also appears in D_y and at least one point of D_x does not appear in D_y. Let $D_z=D_x\cap D_y$ be a divisor of degree r $(1\leqq r<2\nu-2)$ and set $z=\varphi(D_z)$. We will denote $L(D)$ the complex vector space of meromorphic functions on M whose divisors are multiples of $-D$. Since $L(D_x)\cap L(D_y)=L(D_z)$, we have

$$\dim[L(D_x)+L(D_y)]=l(D_x)+l(D_y)-l(D_z)\,,$$

and from the evident inclusion relation that

$$L(D_x)+L(D_y)\subseteqq L(D_x+D_y-D_z)$$

it follows that

$$\dim[L(D_x)+K(D_y)]\leqq\dim L(D_x+D_y--D_z)\,.$$

Then we have

$$\begin{aligned} l(D_x)+l(D_y)-l(D_z)&\leqq l(D_x+D_y-D_z)\\ &=l(K-D_z)=i(D_z)\,, \end{aligned}$$

and substituting the values $l(D_y)=g+1-\nu$ and $i(D_z)=l(D_z)+g-1-r$, and applying Proposition 2, we get the next inequalities.

$$2\leqq 2l(D)-r\leqq 2\left(\left[\frac{r}{2}\right]+1\right)-r\,.$$

Thus r must be an even number, say $r=2\lambda-2$, and then $l(D_z)=\lambda$. Hence $z\in W^{\lambda}_{2\lambda-2}$ and this completes the proof.

Remark 1. In general either $W^{\nu}_{2\nu-2}=0$ or $W^{\nu}_{2\nu-2}$ consists of the unique point $u\in J(M)$ such that $-W_{\nu-1}=W_{\nu-1}-u$ $(2\leqq\nu\leqq g)$, see Gunning [5, p. 50], and for $\nu=g$ it follows from the Riemann-Roch theorem that $W^{g}_{2g-2}=\{k\}$. Clifford's Theorem asserts that $W^{\nu}_{2\nu-2}=0$ $(2\leqq\nu\leqq g-1)$ for non-hyperelliptic Riemann surfaces.

To attain our purpose we here have to introduce some notions.

Let $M^{(r)}$ denote the r-fold symmetric product of a compact Riemann surface M. Let ψ: $M^{(r)}\rightarrow J(M)$ be a complex analytic mapping such that $\varphi=\psi\circ\tau$, where τ : $M^{r}\rightarrow M^{(r)}$ is the natural quotient mapping. At each point $D\in M^{(r)}$ the image of $T_D(M^{(r)})$ (tangent space at D in $M^{(r)}$ under the differential of the mapping ϕ is the linear subspace $d\phi_D(T_D(M^{(r)}))$ of $T_{\phi(D)}(J(M))$ dual to the subspace L_D^* of the complex cotangent space $T^*_{\phi(D)}(J(M))$ defined by

$$L_D^*=\{w \mid w\in\Omega(M) \text{ and } D(w)\geqq D\},$$

where we identify $T^*_{\phi(D)}(J(M))$ with the space $\Omega(M)$ of holomorphic differentials on M and $D(\omega)\geqq D$ means that the divisor of ω is a multiple of D. (Gunning [5], p. 87)

Let V be an analytic subvariety of $J(M)$. To any point $x\in V$ associate the linear subspace $T_x^*(V)\subseteqq T_x^*(J(M)$ spanned by all covectors of the form d_xf, where f is any analytic function in an open neighborhood of x in $J(M)$ which vanishes identically on V. The natural dual to the subspace $T_x^*(V)\subseteqq T_x^*(J(M))$ is a linear subspace $T_x(V)\subseteqq T_x(J(M))$ called the tangent space to the subvariety V at x. Its dimension is called the imbedding dimension of V at x. The points of V at which V is a regular analytic submanifold of $J(M)$ are called the regular points of V and the set of such points is denoted by $\mathcal{R}(V)$, and the remaining points of V are called the singular points of V and the set of such points is denoted by $\mathcal{S}(V)$. The points of $\mathcal{R}(V)$ are precisely those points at which the imbedding dimension is equal to the dimension of V in a small neighborhood of the point.

It is proved by Weil that $\mathcal{S}(W_r)=W_r^2$, and more generally by Mayer that for a proper subvariety $W_r^\nu\subseteqq J(M)$, $W_r^{\nu+1}\subseteqq\mathcal{S}(W_r^\nu)$.

Proposition 8. *For any point $x\in W_r^2\backslash W_r^3$ $(2\leqq r\leqq g)$*) such that $x=x'+x''$ where $x'\in\mathring{W}_s^2$ and $x'\in W_{r-s}$, let $D\in M^{(r)}$, $D'\in M^{(s)}$ be any positive divisors such that $\phi(D)=x$ and $\phi(D')=x'$. Then for any two points p_1, $p_2\in M$ such that $x-\varphi(p_1)$ and $x-\varphi(p_2)$ are regular points of W_{r-1}, the tangent spaces $T_{x-\varphi(p_1)}(W_{r-1})$ and $T_{x-\varphi(p_2)}(W_{r-1})$ either coincide or intersect in a linear subspace of dimension $l(D+D')+r-s-3$, and the imbedding dimension of W_r^2 at x is not greater than $l(D+D')+r-s-3$.*

Proof. Since $x-\varphi(p_1)$ is a regular point of W_{r-1} so that $x\in\varphi(p_i)\notin W_{r-1}^2$ there exists a unique positive divisor $C_i\in M^{(r-1)}$ such that $x=\varphi(p_i+D_i)$, and $T_{x-\varphi(p_i)}(W_{r-1})=T_{\varphi(D_i)}(W_{r-1})$ can be identified with $d\phi_{D_i}(T_{D_i}(M^{(r-1)}))$ $(i=1, 2)$.**) As is previously stated, the dual spaces $L_{D_i}^*\subseteqq T^*(J(M))$ to these tangent spaces are defined by

$$L_{D_i}^*=\{\omega \mid \omega\in\Omega(M) \text{ and } D(\omega)\geqq D_i\}.$$

From above $\dim L_{D_i}^*=g-r+1$, and $\dim L_{D_i+p_i}^*=g-r+1$ from the Riemann-Roch theorem so that $L_{D_i}^*=L_{D_i+p_i}^*$. The spaces $T_{x-\varphi(p_1)}(W_{r-1})$ and $T_{x-\varphi(p_2)}(W_{r-1})$ can be seen as subspaces of the tangent space to $J(M)$ at a point, and the intersection of these tangent spaces is just the dual space to $L_{D_1}^*\cup L_{D_2}^*=L_{D_1+p_1}^*\cup L_{D_2+p_2}^*$.

* Note that $x-\varphi(P)\in W_{r-1}^2$ for all $P\in M$ if and only if $x\in W_{r-1}^2\ominus(-W_1=W_r^3$.

**At any point $D\in M^{(r)}$ such that $l(D)=\nu$, the differential of the analytic mapping ϕ: $M^{(r)}\to J(M)$ has rank given by rank $d\phi_D=r+1-\nu$ (Cunning [5], Th. 10(b)).

Thus its dimension is $g-\dim(L^*_{D_1}+L^*_{D_2})=g-\dim L^*_{D_1}-\dim L^*_{D_2}+\dim(L^*_{D_1}\cap L^*_{D_2})=2(r-1)-g+\dim(L^*_{D_1}\cap L^*_{D_2})$. If $P_1+D_1=P_2+D_2$, then $L^*_{D_1}=L^*_{D_2}$. If not, let $p_1+D_1=D'_1+D''$ and $p_2+D_2=D'_2+D''$ where the divisors D'_1 and D'_2 have no common points. Then $\phi(D'_i)=x'$ $(i=1, 2)$, $\phi(D'')=x''$, and

$$\begin{aligned}L^*_{D_1}\cap L^*_{D_2}&=L^*_{D_1+p_1}\cap L^*_{D_2+p_2}\\&=\{\omega\,|\,\omega\in\Omega(M),\ D(\omega)\geqq D_1+p_1 \text{ and } D(\omega)\geqq D_2+p_2\}\\&=\{\omega\,|\,\omega\in\Omega(M) \text{ and } D(\omega)\geqq D'_1+D'_2+D''\}.\end{aligned}$$

By the Riemann-Roch theorem we have

$$\begin{aligned}\dim(L^*_{D_1}\cap L^*_{D_2})&=i(D'_1+D'_2+D'')\\&=l(D'_1+D'_2+D'')+g-1-s-r.\end{aligned}$$

Each calculations show that the dimension of the intersection is as desired.

To prove the last assertion, recall from Lemma 2. 3) that $W^2_r=W_{r-1}\ominus(-W_1)=\bigcap_{p\in M}[W_{r-1}+\varphi(p)]$. From this it follows that

$$T_x(W^2_r)\subseteqq T_{x-\varphi(p_1)}(W_{r-1})\cap T_{x-\varphi(p_2)}(W_{r-1}),$$

and the first assertion just proved gives the proof of the last.

We now prove the generalized Clifford's theorem which characterize the hyperelliptic Riemann surfaces.

Theorem 2. *If* $\dim W^\nu_r=r-2\nu+2$ *for some pair of indices* ν, r $\left(2\leqq\nu\leqq\frac{1}{2}r+1,\ r\leqq g-1\right)$ *for a Riemann surface* M *of genus* g, *then* M *is hyperelliptic.*

Proof. By Proposition 3, $r-2\nu+2=\dim W^\nu_r<\dim W^{\nu-1}_{r-1}$, and $\dim W^{\nu-1}_{r-1}<r-2\nu+3$ by Proposition 4 so that $\dim W^{\nu-1}_{r-1}=(r-1)-2(\nu-1)+2$. Repeating the argument we have $\dim W^2_{r-\nu+2}=r-\nu$. Thus we have only to prove the Theorem for the case that $\dim W^2_r=r-2$ for some index r $(2\leqq r\leqq g-1)$, and we suppose that r is the smallest among such indices, and M is non-hyperelliptic.

Since M is non-hyperelliptic, $r>2$. If $W^2_r=W^2_{r-1}+W_1$, then $\dim W^2_{r-1}=\dim W^2_r-1=(r-1)-2$ by Proposition 6, and this is a contradiction. Thus we have $\mathring{W}^2_r\neq\emptyset$ for some index $r>2$. Choosing a regular point $x\in\mathring{W}^2_r$ at which W^2_r has dimension $r-2$ and a divisor $D\in M^{(r)}$ such that $\phi(D)=x$, it follows from Proposition 8 that $r-2=\dim W^2_r=\dim T_x(W^2_r)\leqq l(2D)-3$ so that $l(2D)\geqq r+1$. Then $\phi(2D)\in W^{r+1}_{2r}$, and $W^{r+1}_{2r}\neq\emptyset$. If $r\leqq g-2$, then M would be hyperelliptic by Theorem 1. If $r=g-1$, $\phi(2D)\in W^g_{2g-2}=\{k\}$ so that $2x=\phi(2D)=k$. But the last equation can only be satisfied by a finite number of points of $J(M)$, and $\dim W^2_r=r-2\geqq 1$. We can thus choose the point $x\in\mathring{W}^2_r$ in the above argument which do not satisfy the equation, and this gives a contradiction.

Remark 2. For a hyperelliptic Riemann surface,

$$\dim W_r^\nu = r-2\nu+2\,,$$

since $W_r^\nu = W_{r-2\nu+2}+(\nu-1)\cdot e \qquad (1\leqq \nu \leqq r \leqq g)$.

At last note that for hyperelliptic Riemann surfaces

$$\begin{aligned}\mathcal{S}(W_r^\nu) &= \mathcal{S}(W_{r-2\nu+2})+(\nu-1)\cdot e = W_{r-2\nu+2}^2+(\nu-1)\cdot e \\ &= W_{r-2\nu}+e+(\nu-1)\cdot e = W_{r-2\nu}+\nu\cdot e \\ &= W_r^{\nu+1}\,.\end{aligned}$$

§ 4. The trigonal Riemann surfaces.

A Riemann surface of genus $g\geqq 3$ which admits meromorphic functions of order 3 will be called a trigonal Riemann surface. It is easy to check that a Riemann surface is trigonal if and only if $W_2^2=0$ and $W_3^2\neq 0$ so that $\mathring{W}_3^2\neq 0$. It follows from Meis's results [16] that the non-hyperelliptic Riemann surfaces of genus $g=3$, 4 are trigonal. The Riemann surfaces of genus $g\geqq 5$ are generically not trigonal, and for any $g\geqq 3$ there are trigonal Riemann surfaces of genus g.

We shall give some interesting examples of non-trigonal Riemann surfaces. It was shown by Accola [1] and Farkas [3] that the order $n<g+1-2\tilde{g}$ of a meromorphic function on a $\tilde{g}$-hyperelliptic Riemann surface (two-sheeted covering of a compact Riemann surface of genus $\tilde{g}$) of genus $g\geqq 5$ is even.

We have invoke Martens's remark [14] to show that the above $\tilde{g}$-hyperelliptic Riemann surface is not hyperelliptic if the base surface is non-hyperelliptic. Using Abel's theorem he proved that if there exists a meromorphic function of order $n\geqq 2$ on a covering of a compact Riemann surface of positive genus, then there also exists a meromorphic function of order n on the base surface. Thus a $\tilde{g}$-hyperelliptic Riemann surface of genus g $(g+1-2\tilde{g}>3)$ whose base surface is non-hyperelliptic is non-hyperelliptic and admits no meromorphic functions of order 3.

Another such example is a normal covering of a hyperelliptic Riemann surface of genus $g\geqq 3$ which is not hyperelliptic. We know that such coverings exist [8]. Since there exist no meromorphic functions of order 3 on the base surface, we see by Martens's remark that there exist no meromorphic functions of order 3 on these coverings.

The third example is a Riemann surface of genus $g>6$ admitting a meromorphic function of order 4. Its non-trigonality follows from the classical result that if two meromorphic functions f, h of order m, n on a Riemann surface M of genus g generate the full field of functions on M, then $g\leqq (m-1)(n-1)$ (Accola [1], Prop. I).

It is classical that for a trigonal Riemann surface of genus $g\geqq 5$, W_3^2 contains only one point [7, 9, 12]. In general for a non-hyperelliptic Riemann surface of genus $g\geqq 4$,

$$W^{\nu}_{2\nu-1}=0 \quad \left(3\leqq\nu\leqq\frac{1}{2}g\right)$$

except possible the case $g=6$ and $\nu=3$ where if $u\in W^2_3$, then $2u=k$ [12].

We shall investigate some properties of the subvarieties W^2_r of $J(M)$ for trigonal Riemann surfaces M.

Lemma 4. *If* $\mathring{W}^2_3\neq 0$, *then* $\dim W^2_r=r-3 \quad (2\leqq r\leqq g-1)$.

Proof. From Theorem 2 and Proposition 3 it follows that

$$0=\dim W^2_3<\dim W^2_4<\cdots<\dim W^2_{g-1}=g-4 \qquad (g\geqq 5).$$

Thus we hace $\dim W^2_r=r-3$, and this holds also for $g=3, 4$.

Lemma 5. *For a trigonal Riemann surface of genus g it holds that*

$$W^2_r\subseteqq(W^2_3+W_{r-3})\cup(k-W^2_3-W^{g-r}_{2g-r-5}) \qquad (3\leqq r\leqq g-1).$$

Proof. Let u_0 be any point of W^2_3 and u be any point of W^2_r such that $\varphi(D_0)=u_0$ and $\varphi(D)=u$, where D_0 and D are positive divisors of degree 3 and r respectively and $l(D_0)=2$ and $l(D)\geqq 2$. Then we have two meromorphic functions f and h such that

$$(f)_\infty=D_0 \quad\text{and}\quad -(h)_\infty\geqq -D.$$

If 1, f, h and fh are linearly independent, then it follows that $l(D_0+D)\geqq 4$ and $u_0+u\in W^4_{r+3}=k-W^{g-r}_{2g-r-5}$ by Lemma 1 so that $u\in k-u_0-W^{g-r}_{2g-r-5}$. If 1, f, h and fh are linearly dependent, then there exist constants a, b, c such that $(f+a)(h+b)=c$, and $D\sim D_0+D'$ for some positive divisor D' of degree $r-3$ so that $u\in W^2_3+W_{r-3}$.

Theorem 3. *For a trigonal Riemann surface of genus* $g\geqq 4$ *it holds that*

$$W^2_r=(W^2_3+W_{r-3})\cup(k-W^2_3-W^{g-r}_{2g-r-5}) \qquad (3\leqq r\leqq g-1).$$

Proof. It follows that $W^2_3=W^2_r\ominus W_{r-3}$ from Lemma 2, 4) and hence $W^2_r\supseteqq W^2_3+W_{r-3}$. Since $W^2_r=k-W^{g-r+1}_{2g-r-2}$ by Lemma 1 and W^{g-r+1}_{2g-r-5}, we have $W^2_r\supseteqq k-W^2_3-W^{g-r}_{2g-r-5}$ and thus we have the opposite inclusion relation to that in Lemma 5.

Corollary 4. *For a trigonal Riemann surface of genus* $g\geqq 5$, W^2_3 *consists of only one point* [7, 9, 12].

Proof. If we let $r=3$ in the proof of Lemma 5, then either $u_0+u\in W^4_6$ or $u_0=u$. But for $g\geqq 5$ the first case is excluded by Clifford's Theorem.

Corollary 5 (Andreotti-Mayer [2]**).** *For a trigonal Riemann surface of genus* $g\geqq 4$ *it holds that*

$$W^2_{g-1}=(W^2_3+W_{g-4})\cup(k-W^2_3-W_{g-4})$$

so that W_{g-1}^{2} *consists of two irreducible components components if* $g\geqq 5$*).

Corollary 6. *For a trigonal Riemann surface of genus* 4, W_{3}^{2} *consists of two points which may coincide, when* W_{3}^{2} *is the image of a half canonical divisor under* φ.

Corollary 7. *For a trigonal Riemann surface of genus g it holds that*

$$W_{g-2}^{2}=(W_{3}^{2}+W_{g-5})\cup(k-W_{3}^{2}-W_{g-3}^{2}) \qquad (g\geqq 5),$$

$$W_{g-3}^{2}=(W_{3}^{2}+W_{g-6})\cup(k-W_{3}^{2}-W_{g-2}^{3}) \qquad (g\geqq 6) \text{ and}$$

$$W_{g-4}^{2}=(W_{3}^{2}+W_{g-7})\cup(k-W_{3}^{2}-W_{g-1}^{4}) \qquad (g\geqq 7).$$

Corollary of Corollary 7. *For a trigonal Riemann surface of genus* $g\geqq 6$ *it holds that*

$$W_{g-2}^{2}=(W_{3}^{2}+W_{g-5})\cup(k-2W_{3}^{2}-W_{g-6})\cup W_{g-2}^{3}.$$

Proposition 9. *For a trigonal Riemann surface of genus* $g\geqq 8$, W_{g-2}^{2} *consists of at least two irreducible components.* W_{g-2}^{2} *is irreducible only if* $k=4W_{3}^{2}$ *for* $g=7$, *and* $k\in 3W_{3}^{2}+W_{1}$ *for* $g=6$.

Proof. If W_{g-2}^{2} is irreducible, then it necessarily holds that $W_{3}^{2}+W_{g-5}\supseteqq k-2W_{3}^{2}\in W_{g-5}\ominus(-W_{g-6})=W_{2g-11}^{g-5}=\emptyset$ for $g\geqq 8$ by the result of Martens stated before Lemma 4, and this prove the assertion for $g\geqq 8$. The rest follows from the above.

Corollary 8. *A trigonal Riemann surface of genus* 6 *admits no meromorphic function of order* 4 *if and only if* $k\in 3W_{3}^{2}+W_{1}$. *A trigonal surface of genus* 7 *admits no meromorphic function of order* 5 *if and only if* $k=4W_{3}^{2}$**).

Proof. The proofs of "only if" parts for $g=6, 7$ are done in Proposition 9, and the "if" parts follow from the facts than $W_{3}^{4}=\emptyset$, and $W_{5}^{2}=\emptyset$ for $g=7$.

Lemma 6. *If* $W_{r}^{2}=W_{3}^{2}+W_{r-3}$ *for a trigonal Riemann surface of genus* $g\geqq 5$, *then* $W_{s}^{2}=W_{3}^{2}+W_{s-3}$ $(3\leqq s\leqq r)$.

Proof. $W_{r-1}^{2}=W_{r}^{2}\ominus(W_{1})=\bigcap_{u\in W_{1}}((W_{3}^{2}+W_{r-3})-u)$

$$=W_{3}^{2}+(W_{r-3}\ominus W_{1})=W_{3}^{2}+W_{r-4}.$$

This process can be repeated till $s=3$.

Lemma 7. *If* $W_{r}^{2}=W_{3}^{2}+W_{r-3}$ $(r\geqq 4)$ *for a trigonal Riemann surface of genus* $g\geqq 5$, *then* $W_{s}^{3}=W_{3}^{2}+W_{s-3}^{2}$ *so that* $\dim W_{s}^{3}=\dim W_{s-3}^{2}=s-6$ $(5\leqq s\leqq r+1)$.

---

* If $W_{3}^{2}+W_{g-4}=k-W_{3}^{2}-W_{g-4}$, then $k-2W_{3}^{2}\in W_{g-4}\ominus(W_{g-4})=W_{2g-8}^{g-3}=\emptyset$ by Clifford's Theorem.

**cf. Kato [10].

Proof. $W^3_{r+1}=W^2_r\ominus(-W_1)=\bigcap_{u\in W_1}(W^2_r+u)=\bigcap_{u\in W_1}(W^2_3+W_{r-3}+u)$

$$=W^2_3+(W_{r-3}\ominus(-W_1))=W^2_3+W^2_{r-2}.$$

From this and Lemma 4 and 6 we have desired result.

Corollary 9. *If $W^2_4=W^2_3+W_1$ for a trigonal Riemann surface of genus 6, then $W^3_5=\emptyset$.*

Corollary 10. *If $W^2_r=W^2_3+W_{r-3}$ $(r\geqq 4)$ for a trigonal Riemann surface of genus $g\geqq 5$, then $W^3_s=2W^2_3+W_{s-6}$ $(6\leqq s\leqq r+1)$, and more generally $W^\nu_s=(\nu-1)\cdot W^2_3+W_{s-3\nu+3}$ $(3\nu-3\leqq s\leqq r+\nu-2,\ \nu\geqq 2)$.*

Proof. By applying the proofs of Lemma 6 and 7 repeatedly we can get the desired formulae.

Remark 3. The formulae of the subvarieties W^ν_r of $J(M)$ for a trigonal Riemann surface M in Corollary 10 are analogous to those for a hyperelliptic Riemann surface which was stated at the beginning of §3. It should also be noted that $W^2_r=W^2_3+W_{r-3}$ $\left(3\leqq r\leqq\frac{1}{2}(g-1)\right)$, which we shall prove by using the next Lemma.

Lemma 8. *If $\mathring{W}^2_r\neq\emptyset$ for a trigonal Riemann surface, then the surface admits meromorphic functions of order $r+1$ and $r+2$.*

Proof. Let f be a meromorphic function of order 3 and let h be a meromorphic function of order r which corresponds to a point $u\in\mathring{W}^2_r$. If

(1) $$(f)=D-D' \text{ and } (h)=(D+E)-(D'+E')$$

where D and D' are positive divisors of degree 3, E and E' are positive divisors of degree $r-3$, then $u\in W^2_3$, which is a contradiction. If

(2) $$(f)=(D+P)-(D'+P') \text{ and}$$
$$(h)=(D+E)-(D'+E')$$

where D and D' are positive divisors of degree 2, P and P' are points, E and E' are positive divisors of degree $r-2$ such that D, E and D', E' are relatively prime respectively, then after a suitable linear transformation we can get a function h' such that

(3) $$(f)=(D+P)-(D'+P') \quad \text{and}$$
$$(h')=(F+F)-(F'+P')$$

where F and F' are positive divisors of degree $r-1$ and D, F and D', F' are relatively prime respectively. Then $(f/h')=(D+F)-(D'+F)$ so that f/h' is a meromorphic function of order $r+1$.

Similarly we can get a function h'' such that

$$(4)\qquad (f)=(D+P)-(D'+P') \text{ and}$$

$$(h'')=(F+P)-D''$$

where D'' is a positive divisor of degree r such that $D'+P'$ and D'' are relatively prime. Then $(f/h')=(D+D'')-(D'+P'+F)$ so that f/h' is a meromorphic function of order $r+2$.

By applying linear transformations to functions f and h, other situations can be reduced to the above cases (2), (3) and (4).

Since there exists no meromorphic function of order n $\left(4\leqq n\leqq\frac{1}{2}(g+1)\right.$ and $\left.(n, 3)=1\right)$ by the classical result stated in the third example in §4, it follows from Lemma 8 that $W_r^2=W_3^2+W_{r-3}$ $\left(3\leqq r\leqq\frac{1}{2}(g+1)\right)$ for a trigonal Riemann surface.

Proposition 10. *For a trigonal Riemann surface of genus $g\geqq 7$ it holds that*

$$\dim W_r^\nu=r-3\nu+3\ \left(3\nu-3\leqq r\leqq\frac{1}{2}(g-5)+\nu,\ \nu\geqq 2\right)\quad \textit{and}$$

$$\mathcal{S}(W_r^\nu)=W_r^{\nu+1}\ \left(3\nu\leqq r\leqq\frac{1}{2}(g-5)+\nu,\ \nu\geqq 2\right).$$

Proof. The first equalities follow from Corollary 10 and the above considerations. The second assertions also follow from Corollary 10 and the observation that if $r-3\nu+3\geqq 3$, then

$$\begin{aligned}\mathcal{S}(W_r^\nu)&=(\nu-1)\cdot W_3^2+\mathcal{S}(W_{r-3\nu+3})\\&=(\nu-1)\cdot W_3^2+W_{r-3\nu+3}^2\\&=(\nu-1)\cdot W_3^2+W_3^2+W_{r-3\nu}\\&=\nu\cdot W_3^2+W_{r-3\nu}\\&=W_r^{\nu+1}.\end{aligned}$$

Compare this Proposition with Remark 2 and the following of it in §3.

Finally we shall prove the existence of meromorphic functions of order $g-1$ on a trigonal Riemann surface of genus g.

Theorem 4. *A trigonal Riemann surface of genus $g\geqq 4$ admits meromorphic functions of order $g-1$.*

Proof. For $g=4$, the trigonality is assumed. For $g\geqq 5$, $W_{g-2}^2=(W_3^2+W_{g-5})\cup(k-W_3^2-W_{g-3}^2)$ by Corollary 7, and $W_{g-1}^2=(W_3^2+W_{g-4})\cup(k-W_3^2-W_{g-4})$ by Corollary 5. Since $W_3^2+W_{g-4}$ and $k=W_3^2-W_{g-4}$ are distinct irreducible components and $\dim(k-W_3^{2}-W_{g-4})=g-4>g-5=\dim((k-W_3^2-W_{g-3}^2)+W_1)$ by Proposition 6 and Lemma 4, we have $\mathring{W}_{g-1}^2\neq\emptyset$, and this completes the proof.

Corollary 11. *A non-hyperelliptic Riemann surface of genus* 5 *admits meromorphic functions of order* 4 [10].

Proof. If $W_3^2=\emptyset$, then $W_4^2\neq\emptyset$ by the known result that $W_r^2\neq\emptyset$ if $r\geqq\frac{g+1}{3}$ (Meis [16]). If $W_3^2\neq\emptyset$, then we can apply Theorem 4, and the proof is completed.

§5. The elliptic-hyperelliptic Riemann surfaces.

A two-sheets covering of a compact Riemann surface of genus 1 will be called an elliptic-hyperelliptic Riemann surface. On such a surface of genus g every meromorphic function of order $n\leqq g-2$ is a lift of a meromorphic function on the elliptic base surface if n is even, and no meromorphic function of order $n\leqq g-2$ exists if n is odd, as is stated in §4.

From these facts we can deduce the following lemmas.

Lemma 9. *For an elliptic-hyperelliptic Riemann surface of genus* $g\geqq 6$, $W_2^2=W_3^2=\emptyset$ *and* $\dim W_{2r}^r=1$ $\left(2\leqq r\leqq\frac{1}{2}(g-2)\right)$.

Proof. Since the elliptic-hyperelliptic Riemann surfaces of genus $g>3$ are not hyperelliptic (Farkas [3]), we have $W_2^2=\emptyset$. In addition $W_3^2=\emptyset$, for otherwise $\mathring{W}_3^2\neq\emptyset$ and the surface would admit a meromorphic function of order 3, which contradicts to the above facts. At last we have $\dim W_{2r}^r=1$, since any meromorphic function of order $2r$ on the surface is a lift of a meromorphic function of order r on the elliptic base surface.

Lemma 10. *For an elliptic-hyperelliptic Riemann surface of genus* $g\geqq 6$, $W_r^2=W_4^2+W_{r-4}$, *so that* $\mathring{W}_r^2=\emptyset$ $(5\leqq r\leqq g-2)$.

Proof. As in the proof of Lemma 4 in §4, it follows that

$$1=\dim_4^2<\cdots<\dim W_{g-1}^2=g-4$$

so that we have $\dim W_r^2=r-3$. We also have $\dim(W_4^2+W_{r-4})=r-3$ by Proposition 6 so that $W_4^2+W_{r-4}$ is an irreducible subvariety of W_r^2 of maximal dimension. We shall prove by induction. We suppose that $W_t^2=W_4^2+W_{t-4}$ $(4\leqq t\leqq s-1)$. If W_s^2 has another irreducible component, then $\mathring{W}_s^2\neq\emptyset$ $(5\leqq s\leqq g-2)$. But if s is odd, the surface would admit meromorphic functions of odd order s. If s is even, then $\mathring{W}_s^2\subseteqq W_{s/2}^s$ since a meromerphic function of even order $s\leqq g-2$ is a lift of a meromorphic function of order $\frac{s}{2}$ on elliptic base surface. As $W_{s/2}^s=W_{s-1}^{(s/2)-1}\ominus(-W_1)$ so that $\mathring{W}_s^2-W_1\subseteqq W_{s-2}^{(s/2)-1}\subseteqq W_{s-1}^2$, we have $\mathring{W}_s^2\subseteqq\mathring{W}_s^2+W_1-W_1\subseteqq W_{s-1}^2+W_1$, which contradicts the fact that $\mathring{W}_s^2\neq\emptyset$.

Lemma 11. *For an elliptic-hyperelliptic Riemann surface of genus* $g\geqq 6$, $\mathring{W}^{r}_{g-1}=\emptyset$ $(r\geqq 3)$.

Proof. If $\mathring{W}^{r}_{g-1}\neq\emptyset$ for some $r\geqq 3$, there would exist a meromorphic function f of order s $(4\leqq s\leqq g-2)$ such that $-(f)_{\infty}$ is a multiple of $-D$ where D is a positive divisor of degree $g-1$ chosen so as not to contain any branch point of the elliptic-hyperelliptic covering and $\varphi((f)_{\infty})\in\mathring{W}^{2}_{s}$ and $\varphi(D)\in\mathring{W}^{r}_{g-1}$. But by Lemma 10, s must be 4, and D has two pairs of symmetric points P, $T(P)$; Q, $T(Q)$, where T is the elliptic-hyperelliptic involution. If h is a meromorphic function of order $g-1$ whose polar divisor is D, then the function $H=h-h\circ T$ would have greater number of zeros than the poles, which is absurd.

Corollary 12. *On an elliptic-hyperelliptic Riemann surface of genus* $g\geqq 6$ *there exists at least one meromorphic function of order* $g-1$ *if any only if* $\mathring{W}^{2}_{g-1}\neq 0$.

Now we shall prove the existence of meromorphic functions of order $g-1$ on an elliptic-hyperelliptic Riemann surface of genus g.

Theorem 5. *An elliptic-hyperelliptic Riemann surface of genus* $g\geqq 6$ *admits meromorphic functions of order* $g-1$.

*Proof**. Since $W^{3}_{g+1}=k-W_{g-3}$, $W^{4}_{g+1}=k-W^{2}_{g-3}$ by Lemma 1, and the dimension of the subvariety $W^{3}_{g}+W_{1}=k-W^{2}_{g-2}+W_{1}$ is not greater than $g-4$, there is a positive divisor D of degree $g+1$ consisting of points in general position and $l(D)=3$ such that $\varphi(D)\in\mathring{W}^{3}_{g+1}$. Let $1, f, h$ be linearly independent meromorphic functions on the surface such that $-(f)_{\infty}\geqq -D$ and $(h)_{\infty}=D$, and the degree of f be r $(4\leqq r\leqq g)$.

Since the divisor D consists of points in general position, we can have a biholomorphic mapping of the Riemann surface to a plane curve C of degree $g+1$. By the formula for the genus of a plane curve we have

$$g=\frac{g(g-1)}{2}-\sum\frac{r_i(r_i-1)}{2}$$

where r_i are the multiplicities of the singularities of the curve C. Thus the curve C must have at least one singularity. Let S be an s-fold singularity among its singularities; then $\varphi(D)=\varphi(S+S')$ where S' is a divisor of degree $g+1-s$ and $\varphi(S')\in\mathring{W}^{2}_{g+1-s}$. Therefore we have a function of degree $g+1-s\leqq g-1$.

If $r=g-1$, then we have done.

If $4\leqq r\leqq g-2$, then r must be 4 by Lemma 10 and D has two pairs of points symmetric with respect to T. Since $(h-h(P))_0$ has at least one pairs of points

* The author is grateful to Professor Accola who kindly informed him a proof of this theorem due to J. Harris. This proof is another version of it.

symmetric with respect to T for a zero P of the function $h-h\circ T$, the meromorphic function $\dfrac{h-h(P)}{f-f(P)}$ is of order at most $g-1$. But the polar divisor of it is the multiple of $-D$ and does not consist of pairs of points symmetric with respect to T. Thus its degree must be $g-1$. This completes the proof.

Corollary 13. *For an elliptic-hyperelliptic Riemann surface of genus* $g\geqq 6$, W^2_{g-1} *has at least two irreducible components.*

Proof. This immediately follows from Lemma 10, Corollary 12 and Theorem 5.

§6. The existence of meromorphic functions of order g.

It was stated in Hensel-Landsberg [7, p. 508] that on a non-hyperelliptic Riemann surface of genus g there exist infinitely many meromorphic functions of order g. We give here another proof from our point of view.

Theorem 6. *A non-hyperelliptic Riemann surface of genus g admits infinitely many meromorphic functions of order g.*

Proof. For a non-hyperelliptic Riemann surface of genus g, $\dim W^2_{g-1}=g-4$ by Theorem 2, and $W^2_g=k-W_{g-2}$ by Lemma 1 so that $\dim W^2_g=g-2$. Since $\dim(W^2_{g-1}+W_1)=g-3$ by Proposition 6, it follows that $\overset{\circ}{W}{}^2_g\neq\emptyset$, and $\dim \overset{\circ}{W}{}^2_g=g-2$ $\geqq 1$. Thus there exist infinitely many moromorphic functions on the surface.

Remark 4. We can prove the next weaker statement for $g=4$ by making use of Weierstraß points: *A Riemann surface of genus* 4 *admits at least one meromorphic function of order* 4.

Proof. The statement holds trivially for hyperelliptic surfaces and the surfaces with at least one Weierstraß point whose first nongap is 4. Thus we have only to consider surfaces all whose Weierstraß points have 3 as their first nongaps.

Let P be a Weierstraß point on the surface and let f be a meromorphic function such that divisor of f is

$$(f)=(Q+D)-3P,$$

where Q is another Weierstraß point, D is a positive divisor of degree 2, and Q and D are relatively prime. Such a function exist. For there are at least 15 Weierstraß points since gap sequence of the Weierstraß points are $\{1, 2, 4, 5\}$ or $\{1, 2, 4, 7\}$, and the total degree of ramification of the covering represented by f is 12. There also exists a function h such that

$$(h)=(P+D')-3Q \text{ or } (h)=(2P+R)-3Q,$$

where D' has the same property as D and $P\neq R$. We then have

$$(fh)=(D+D')-(2P+2Q)$$

in the first case, and

$$((f+c)h)=(R+D'')-(P+3Q)$$

in the second case, where c is a constant and D'' is a positive divisor of degree 3. Hence there exists a meromorphic function of order 4 in each case. Note that the case $(h)=3P-3Q$ can be excluded.

§ 7. The existence of meromorphic functions of order $g-1$.

We have seen in § 4 and § 5 that a Riemann surface of genus $g\geqq 6$ which is either trigonal or elliptic-hyperelliptic admits meromorphic functions of order $g-1$, and here we shall prove that a non-hyperelliptic Riemann surface of genus $g\geqq 4$ admits meromorphic functions of order $g-1$. To do this, a result of Mumford [17, Appendix] is very useful.

Theorem 7 (Mumford). *Let M be a Riemann surface of genus $g\geqq 5$. If $\dim W_r^2=r-3$ for some integer r $(3\leqq r\leqq g-2)$, then M is trigonal, or elliptic-hyperelliptic, or a surface represented by a nonsingular plane quintic.*

Proof. Since $\dim W_r^2=r-2$ for hyperelliptic Riemann surface, M is non-hyperelliptic. Let us denote r the smallest r for which $\dim W_r^2=r-3$. If $r=3$, then $W_3^2\neq 0$ and M is trigonal. In the following we assume that $r\geqq 4$ so that $g\geqq 6$.

If $W_r^2=W_{r-1}^2+W_1$, then $\dim W_{r-1}^2=r-4=(r-1)-3$ by Proposition 6, which is contrary to our assumption on r. Thus $\mathring{W}_r^2\neq 0$, and choosing a regular point $x\in\mathring{W}_r^2$ at which $\mathring{W}_r^2$ has dimension $r-3$ and a positive divisor D of degree r such that $\varphi(D)=x$, it follows from Proposition 8 that

$$r-3=\dim W_r^2=\dim T_x(W_r^2)=\dim T_x(W_r^2)\leqq l(2D)-3\,.$$

Hence $l(2D)\geqq r$ and $\varphi(2D)\in W_{2r}^r$ so that $2\mathfrak{R}(\mathring{W}_r^2)\subseteqq W_{2r}^r$ and $\dim W_{2r}^r\geqq r-3$. From Theorem 2 and Lemma 1 it follows that $\dim W_{2r}^r<2r-2\cdot r+2=2$ and hence $r<5$ so that $r=4$ and $\dim W_4^2=1$.

We have assume that M is non-trigonal. Choose two distinct points u_1, u_2 $\in W_4^2$ and positive divisors D_1, D_2 of degree 4 such that $\varphi(D_1)=u_1$ and $\varphi(D_2)=u_2$. Then by the same reasoning as in the proof of Lemma 5 in § 4, we have u_1+u_2 $\in W_8^4$ so that $u_1+u_2+W_{g-6}\subseteqq W_{g+2}^4=k-W_{g-4}$. Noting that $\dim W_{g-4}^2\leqq g-7$, we can generically choose a positive divisor D_0 of degree $g-6$ such that $i(D_0+D_1+D_2)$ $=1$, and there is an abelian differential ω of the first kind divisor of which is the multiple of $D_0+D_1+D_2$. Let f be a meromorphic function of order 4 such that $(f)_\infty=D_2$, then the divisors of ω and $f\omega$ are the multiples of D_0+D_1. Since $i(D_1)=g-3$ and hence $i(D_0+D_1)=(g-3)-(g-6)=3$, we may choose three linearly independent abelian differentials ω_1, ω_2, ω_3 divisors of which are the multiples of D_0+D_1. Then we can define an analytic map $\pi: M \to \boldsymbol{P}^2$ such that the map $f: M \to \boldsymbol{P}^1$ is the composition of π and a projection π_1 of $\pi(M)$ to $\boldsymbol{P}^1$ defined by $\pi_1(\omega_1, \omega_2, \omega_3)=(\omega, f\omega)$. If d is the degree of the plane curve $\pi(M)$, then the degree of the map π_1 is either d or $d-1$, since there are infinitely many D_2 (hence f) and $\pi(M)$ has only finitely many multiple points so that we may select the projection π_1 from a point of $\boldsymbol{P}^2-\pi(M)$ or from a simple point of $\pi(M)$. Thus if we denote δ the degree of π, then we have $\delta(d-1) \leqq 4$. If $\delta=1$, then $d \leqq 5$, and since $g \geqq 6$, M must be represented by a non-singular plane quintic [4]. If $\delta=2$, then $d \leqq 3$ and M is elliptic-hyperelliptic. If $\delta=3$, then $d \leqq 2$ and M would be trigonal. If $\delta \geqq 4$, then $d=1$ and $\omega_1, \omega_2, \omega_3$ are linearly dependent, which is a contradiction.

We have seen in Lemma 8 that for an elliptic-hyperelliptic Riemann surface, $\dim W_4^2=1$, and it is known that for a non-singular plane quintic, $W_5^2 \neq 0$ [17, p. 347], and since $W_5^3=W_4^2 \ominus (W_1)$ and hence $W_5^3-W_1 \subseteqq W_4^2$ so that $\dim W_4^2=1$.

Theorem 8. *Every non-hyperelliptic Riemann surface of genus $g \geqq 4$ admits meromorphic functions of order $g-1$.*

Proof. For $g=4$, this is well-known [16]. We have proved this for $g=5$ in Corollary 9. Thus we shall prove the Theorem for $g \geqq 6$.

We have $\dim W_{g-1}^2=g-4$ by Theorem 2, and $\dim W_{g-2}^2=g-6$, or $g-5$ by Proposition 4. If $\dim W_{g-2}^2=g-6$, then $\dim(W_{g-2}^2+W_1)=g-5$ so that $\mathring{W}_{g-1}^2 \neq \emptyset$. If $\dim W_{g-2}^2=g-5=(g-2)-3$, then we can apply Mumford's Theorem 7 and the three kinds of Riemann surfaces in the Theorem remain to be considered. However a nonsingular plane quintic is of genus 6 and admits meromorphic functions of order 5 [4, 17], and also a trigonal Riemann surface and an elliptic-hyperelliptic Riemann surface admit meromorphic functions of order $g-1$ by theorem 4 and 5. This completes the proof.

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Added in proof. Theorem 8 was also proved by G. Martens in his paper: Funktionen von vorgegebener Ordnung auf komplexen Kurven, Jour. reine Angew. Math. 320 (1980).