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Scrambled sets on compact metric spaces

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1. Introduction.

In this paper, we investigate the dynamical properties of continuous maps of a compact metric space into itself. There has been much recent interest in irregular trajectories, for example, so-called strange attractors [1]. In Auslander and Yorke [2] some topological concepts of chaos are investigated. The purpose of this paper is to examine some of the consequences of the following definition due to Li and Yorke [3].

Let X be a compact metric space with metric d and $C(X)$ be the space of all continuous functions from X into itself.

Definition 1. 1. For $f \in C(X)$, we say that S is a scrambled set of (X, d, f) if S satisfies the following two conditions.

(i) S is an uncountable subset of X.

(ii) For any $x, y \in S$, $x \neq y$, and for some $\delta > 0$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta.$$

Without loss of generality, we can assume that S contains no asymptotically periodic points. Note that if S is a scrambled set of (X, d, f^m) for some positive integer m, then S is also a scrambled set of (X, d, f).

Definition 1. 2. We say that $f \in C(X)$ is λ-expanding on $X_0 \subset X$ if $d(f(x), f(y)) \geq \lambda d(x, y)$ for any $x, y \in X_0$ and for some $\lambda > 0$.

In one-dimensional case, there are various results concerning scrambled sets. Let I be a compact interval and $d(x, y) = |x - y|$. Combining the theorems of Li and Yorke [3] and Sharkovskii [4], we have the following:

Theorem 1. 3. *If $f \in C(I)$ has a point of period $k \cdot 2^m$ for some odd integer $k \geq 3$ and for some positive integer m, then there exists a scrambled set of (I, d, f).*

Concerning the stability property of functions which possess a scrambled set, Butler and Pianigiani [5] obtained the following result.

Theorem 1. 4. *The set of functions in* $C(I)$ *which possess a scrambled set contains an open dense subset of* $C(I)$.

Recently, concerning the measurements of scrambled sets, J. Smital [6] showed that

$$f(x)=\begin{cases} 2x\ , & x\in\left[0,\ \frac{1}{2}\right] \\ 2-2x, & x\in\left[\frac{1}{2},\ 1\right] \end{cases}$$

has no scrambled sets with a positive Lebesgue measure. Moreover he constructed a scrambled set with full outer Lebesgue measure under the continuum hypothesis.

In n-dimensional case, Marotto [7] obtained the following theorem. Let $\|\ \|$ be the usual Euclidean norm and $B_r(x)$ be an open ball centered at x with radius r.

Theorem 1. 5. *Assume that* $F\in C(Q)$ *satisfies the following two conditions, where* Q *is a compact set in* $\boldsymbol{R}^n$.

(i) $F(z)=z$ *and* F *is* λ*-expanding on* $B_r(z)$ *for some* $\lambda>1$.

(ii) $F^m(w)=z$ *and* F^m *is* μ*-expanding on* $B_s(w)\subset B_r(z)$ *for some positive integer* $m\geq 2$ *and for some* $\mu>0$.

Then there exists a scrambled set of $(Q,\ \|\ \|,\ F^m)$.

It is clear that the set of functions Ψ in $C(Q)$ which possess a scrambled set is dense in $C(Q)$. However, it will be an open problem whether Ψ contains an open set or not.

2. Preliminaries.

In this section, we give some notations and definitions. Let M be a metric space with metric d. Throughout the section 2-4, we suppose that a system (M, d, f) satisfies the following:

Hypothesis 2. 1. f: $M\to M$ is a continuous mapping and there exist disjoint compact subsets A_0, $A_1\subset M$ such that

$$f(A_0)\cap f(A_1)\supset A_0\cup A_1.$$

Let $\Sigma=\{0,\ 1\}^N$ be the collection of infinite one sided sequences of $0's$ and $1's$ with a metric

$$d_\Sigma(\omega,\ \omega')=\sum_{n=0}^{\infty}\frac{1}{2^n}|\omega_n-\omega'_n|$$

where $\omega=(\omega_0\omega_1\omega_2\ldots)$ and $\omega'=(\omega'_0\omega'_1\omega'_2\ldots)$. Then Σ is a compact metric space and the shift transformation $\sigma: \Sigma \to \Sigma$, where $\sigma(\omega_0\omega_1\omega_2\ldots) = (\omega_1\omega_2\omega_3\ldots)$, is a continuous onto two to one mapping.

Let $\mathscr{C}_i$ be the set of all non-empty closed subsets of A_i for $i=0, 1$ and put $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1$. If we introduce Hausdorff metric d_H on $\mathscr{C}$:

$$d_H(c, c') = \max\{\inf_{\varepsilon>0}\{N_\varepsilon(c) \supset c'\}, \inf_{\varepsilon>0}\{N_\varepsilon(c') \supset c\}\}$$

for $c, c' \in \mathscr{C}$, where $N_\varepsilon(c)$ is an ε-neighbourhood of c, then $\mathscr{C}$ is a compact metric space by virtue of the theorem of Michael [8].

For any sequence of sets $\{D_n\}$, we define the limit superior set of $\{D_n\}$ as follows:

$$\limsup_{n\to\infty} D_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} D_i}.$$

Note that if $\{D_n\}$ is a sequence in a compact metric space, then the limit superior set of $\{D_n\}$ is a non-empty compact subset.

For any $\omega \in \Sigma$, we define

$$K(\omega) = \bigcap_{n=0}^{\infty} f^{-n}(A_{\omega_n})$$

where $\omega = (\omega_0\omega_1\omega_2\ldots)$. Since it is clear that $K(\omega)$ is a non-empty closed subset of $A_0 \cup A_1$ by Hypothesis 2. 1, K is a mapping from Σ to $\mathscr{C}$.

For convenience, we define the address of $x \in A_0 \cup A_1$ such that

$$Add(x) = \begin{cases} A_0 & \text{if } x \in A_0 \\ A_1 & \text{if } x \in A_1. \end{cases}$$

Therefore we have $Add(f^n(x)) = A_{\omega_n}$ for any $n \geq 0$ and $x \in K(\omega)$.

Finally we define a mapping F: $\mathscr{C} \to 2^M$ as follows:

$$F(c) = \{f(x)\ ;\ x \in c\} \subset M$$

for $c \in \mathscr{C}$. Since f is continuous, $F(c)$ is always a compact subset.

3. Some Properties of *K*.

In this section, we discuss some fundamental praperties of K: $\Sigma \to \mathscr{C}$. If we define $M^* = \bigcup_{\omega \in \Sigma} K(\omega) \subset M$ and $\mathscr{C}^* = K(\Sigma)$, then we have the following.

Lemma 3. 1. (i) *$\omega \neq \omega'$ implies $K(\omega) \cap K(\omega') = \phi$ and therefore K: $\Sigma \to \mathscr{C}^*$ is an onto one to one mapping.*

(ii) *M^* is a compact subset of M.*

(iii) *$K(\sigma(\omega)) = F(K(\omega))$ for any $\omega \in \Sigma$.*

(iv) *F: $\mathscr{C}^* \to \mathscr{C}^*$ is continuous.*

Proof. Plainly (i) follows from the definition of K. For any $x \in M - M^*$, we have $f^n(x) \notin A_0 \cup A_1$ for some $n \geq 0$. Since f^n is continuous at x and $A_0 \cup A_1$ is closed, there exists a neighbourhood U of x such that $f^n(U) \cap$

$(A_0 \cup A_1) = \phi$. Then we have $U \cap M^* = \phi$ and therefore M^* is a closed subset of M. Thus M^* is a compact set since $M^* \subset A_0 \cup A_1$. For any $x \in K(\sigma(\omega))$, we have $Add(f^n(x)) = A_{\omega_{n+1}}$ for any $n \geq 0$. Since $x \in A_0 \cup A_1 \subset f(A_{\omega_0})$, there exists $y \in A_{\omega_0}$ such that $x = f(y)$. Then $Add(f^n(y)) = Add(f^{n-1}(x)) = A_{\omega_n}$ for any $n \geq 1$ and we have $y \in K(\omega)$. Hence $K(\sigma(\omega)) \subset F(K(\omega))$. Conversely, for any $x \in F(K(\omega))$, there exists $y \in K(\omega)$ such that $x = f(y)$. Then $Add(f^n(x)) = Add(f^{n+1}(y)) = A_{\omega_{n+1}}$ and we have $x \in K(\sigma(\omega))$. Thus we have $K(\sigma(\omega)) = F(K(\omega))$. Also this shows that $F(\mathscr{C}^*) \subset \mathscr{C}^*$. Finally, (iv) follows since f is uniformly continuous on a compact set M^*. □

The following plays an important and fundamental role in our discussions.

Lemma 3. 2. *For any sequence* $\{\omega^{(n)}\} \subset \Sigma$, *we have*

$$\limsup_{n \to \infty} K(\omega^{(n)}) \subset \bigcup_{\omega \in \Sigma_0} K(\omega)$$

where $\Sigma_0 = \limsup_{n \to \infty} \{\omega^{(n)}\}$.

Proof. Let $\limsup_{n \to \infty} K(\omega^{(n)}) = K^*$. Since $K(\omega^{(n)}) \subset M^*$ and M^* is a closed set, we have $K^* \subset M^*$. Therefore, for any $x \in K^*$, there exists some $\omega \in \Sigma$ such that $x \in K(\omega)$. By the definition of the limit superior set, there exists a subsequence $\{n_j\}$ such that $d(y_j, x) \to 0$ as $j \to \infty$ where $y_j \in K(\omega^{(n_j)})$. Since $d(A_0, A_1) > 0$, for fixed k, we have $Add(f^k(x)) = Add(f^k(y_j)) = A_{\omega_k}$ for sufficiently large j. This shows that $d_\Sigma(\omega^{(n_j)}, \omega) \to 0$ as $j \to \infty$. Hence $\omega \in \limsup_{n \to \infty} \{\omega^{(n)}\}$ and this completes the proof. □

Corollary 3. 3. *If* $d_\Sigma(\omega^{(n)}, \omega) \to 0$ as $n \to \infty$, *then we have*

$$\limsup_{n \to \infty} K(\omega^{(n)}) \subset K(\omega).$$

We say that f_1 is topologically conjugate to f_2 if $f_1 = h \circ f_2 \circ h^{-1}$ for some homeomorphism h.

Theorem 3. 4. *The following three statements are equivalent to one another.*

(i) $K: \Sigma \to \mathscr{C}^*$ *is a continuous mapping.*

(ii) $\mathscr{C}^*$ *is a closed subset of* $\mathscr{C}$.

(iii) $F: \mathscr{C}^* \to \mathscr{C}^*$ *is topologically conjugate to the shift* $\sigma: \Sigma \to \Sigma$.

Proof. First we will prove (i) is equivalent to (ii). Clearly (ii) follows if (i) holds. Assume that $\mathscr{C}^*$ is closed. For any sequence $\{\omega^{(n)}\} \subset \Sigma$ such that $d_\Sigma(\omega^{(n)}, \omega) \to 0$ as $n \to \infty$, a sequence $\{K(\omega^{(n)})\} \subset \mathscr{C}^*$ contains a subsequence which converges to some $K(\omega^*)$ since $\mathscr{C}^*$ is compact. Then it is clear that $K(\omega^*) \subset \limsup_{n \to \infty} K(\omega^{(n)})$ and using Corollary 3. 3 we have $K(\omega^*) \subset K(\omega)$. Hence $\omega^* = \omega$ and this implies that $d_H(K(\omega^{(n)}),$

$K(\omega)) \to 0$ as $n \to \infty$, that is, K is continuous at ω. Thus (i) follows. Since K is a continuous one to one mapping onto a compact space $\mathscr{C}^*$, K must be a homeomorphism. This implies (iii). Finally, it is clear that (ii) follows if (iii) holds. □

A mapping K may have a discontinuity point. Actually we have the following. The proof is straightforward.

Lemma 3. 5. *If $K(\omega)$ has an interior point, then ω is a discontinuity point of K.*

4. Existence of Scrambled Sets.

In this section, we give some theorems concerning the existence of a scrambled set.

Theorem 4. 1. *Assume that K is continuous at some point $\omega \in \Sigma$. Then there exists a scrambled set of $(\mathscr{C}^*, d_H, F)$.*

Proof. Let $\omega = (\omega_0\omega_1\omega_2\dots)$. We define for $\gamma \in [0, 1]$ and $n \geq 0$, $\omega_n^\gamma = [\gamma m] - [\gamma(m-1)]$ if $n = m^2$ and $\omega_n^\gamma = \omega_k$ if $n > [\sqrt{n}]^2$ where $k = n - [\sqrt{n}]^2 - 1$ and $[x]$ denotes the greatest integer which does not exceed x. Then it easily follows that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \omega_{j^2}^\gamma = \gamma$. Let $\omega^\gamma = (\omega_0^\gamma\omega_1^\gamma\omega_2^\gamma\dots)$. Then we have

$$\sigma^{n^2+1}(\omega^\gamma) = (\omega_0\omega_1\omega_2\dots\omega_{2n-1}\dots),$$

therefore, for any γ, $\{\sigma^{n^2+1}(\omega^\gamma)\}_{n\geq 1}$ is a sequence in Σ which converges to ω. Thus, by the assumption of K, we have $d_H(K(\sigma^{n^2+1}(\omega^\gamma)), K(\omega)) \to 0$ as $n \to \infty$.

Now we define $S = \{K(\omega^\gamma)\,;\, \gamma \in [0, 1]\} \subset \mathscr{C}^*$. Plainly S is an uncountable subset of $\mathscr{C}^*$ and for any α, $\beta \in [0, 1]$, $\alpha \neq \beta$,

$$\begin{aligned} &d_H(F^{n^2+1}(K(\omega^\alpha)), F^{n^2+1}(K(\omega^\beta))) \\ &\quad \leq d_H(K(\sigma^{n^2+1}(\omega^\alpha)), K(\omega)) + d_H(K(\sigma^{n^2+1}(\omega^\beta)), K(\omega)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence we have

$$\liminf_{n\to\infty} d_H(F^n(K(\omega^\alpha)), F^n(K(\omega^\beta))) = 0.$$

On the other hand, there exists a subsequence $\{n_j\}$ such that $\omega_{n_j^2}^\alpha \neq \omega_{n_j^2}^\beta$, and therefore

$$K(\sigma^{n_j^2}(\omega^\alpha)) \subset A_s \text{ and } K(\sigma^{n_j^2}(\omega^\beta)) \subset A_r \quad \text{for } s \neq r.$$

Hence we have

$$\limsup_{n\to\infty} d_H(F^n(K(\omega^\alpha)), F^n(K(\omega^\beta))) \geq d(A_0, A_1).$$

This completes the proof. □

Theorem 4. 2. *Suppose that $K(\omega)$ consists of only one point. Then K is continuous at ω and there exists a scrambled set of (M^*, d, f).*

Proof. Let $\{\omega^{(n)}\}$ be any sequence in Σ such that $d_\Sigma(\omega^{(n)}, \omega)\to 0$ as $n\to\infty$. By Corollary 3. 3, we have $\limsup_{n\to\infty} K(\omega^{(n)})=K(\omega)$ since $K(\omega)$ consists of only one point. Therefore $d_H(K(\omega^{(n)}), K(\omega))\to 0$ as $n\to\infty$ and this shows the continuity of K at ω. By the above theorem, there exists a scrambled set $S=\{K(\omega^\gamma); \gamma\in[0, 1]\}$ of $(\mathscr{C}^*, d_H, F)$. Choosing suitably $x^\gamma\in K(\omega^\gamma)$, we define $\mathscr{S}=\{x^\gamma; \gamma\in[0, 1]\}\subset M^*$. Then cleary $\mathscr{S}$ is a scrambled set of (M^*, d, f). □

Corollary 4. 3. *Assume that $K(\omega)$ consists of only one point for any $\omega\in\Sigma$. Then $f: M^*\to M^*$ is topologically conjugate to the shift $\sigma: \Sigma\to\Sigma$.*

Proof. By the above theorem, $K: \Sigma\to\mathscr{C}^*$ is continuous. Thus $F: \mathscr{C}^*\to\mathscr{C}^*$ is topologically conjugate to $\sigma: \Sigma\to\Sigma$ by Theorem 3. 4. Since $K(\omega)$ consists of only one point, we can identify M^* with $\mathscr{C}^*$. This completes the proof. □

Theorem 4. 4. *Assume that there exist $\nu>1$ and $s\geq 0$ such that f is ν-expanding on $A_{i_0}\cap f^{-1}(A_{i_1})\cap\dots\cap f^{-s}(A_{i_s})$ for any $\{i_0, i_1, \dots, i_s\}\in\{0, 1\}^{s+1}$. Then $f: M^*\to M^*$ is topologically conjugate to the shift $\sigma: \Sigma\to\Sigma$.*

Proof. For any $x, y\in K(\omega)$ and for any $n\geq 0$, we have $f^n(x), f^n(y)\in A_{\omega_n}\cap f^{-1}(A_{\omega_{n+1}})\cap\dots\cap f^{-s}(A_{\omega_{n+s}})$, and therefore

$$\begin{aligned} d(x, y) &\leq \frac{1}{\nu^n} d(f^n(x), f^n(y)) \\ &\leq \frac{1}{\nu^n} \operatorname{diam}(A_0\cup A_1)\to 0 \text{ as } n\to\infty. \end{aligned}$$

Hence $K(\omega)$ must consist of only one point. □

5. Marotto's Conditions.

In this section, we will prove the following theorem as an application of our results to finite dimensional case. Let Q be a compact set in $\boldsymbol{R}^n$ and $d(x, y)=\|x-y\|$.

Theorem 5. 1. *Suppose that $f\in C(Q)$ satisfies the conditions* (i) *and* (ii) *in Theorem* 1. 5. *Then there exists a compact set $Q^*\subset Q$ and a positive integer p such that $f^p: Q^*\to Q^*$ is topologically conjugate to the shift $\sigma: \Sigma\to\Sigma$.*

We remark that Shiraiwa and Kurata [9] obtained the same conclusion assuming that the differentiability of f and some transversality condition. Before proving the theorem, we need the following lemma.

Lemma 5. 2. *Suppose that f is a continuous ν-expanding mapping on $B_\alpha(x_0)$ for some $\nu>0$. Then there exists $\beta>0$ such that $B_\beta(f(x_0))\subset f(B_\alpha(x_0))$.*

Proof. For some $\delta \in (0, \alpha)$ and for any $x \in \partial B_\delta(x_0)$, we have $\|f(x) - f(x_0)\| \geq \nu\delta$ and therefore $B_\beta(f(x_0)) \cap f(\partial B_\delta(x_0)) = \phi$ for any $\beta \in (0, \nu\delta)$. Hence for any $x \in B_\beta(f(x_0))$, $\deg(f, B_\delta(x_0), x) = \deg(f, B_\delta(x_0), f(x_0))$.

Now consider the homotopy $h(x, t): \overline{B_\delta(0)} \times [0, 1] \to \boldsymbol{R}^n$ such that

$$h(x, t) = f\left(x_0 + \frac{x}{1+t}\right) - f\left(x_0 - \frac{t}{1+t}x\right).$$

Then we have $\|h(x, t)\| \geq \nu\delta$ for any $(x, t) \in \partial B_\delta(0) \times [0, 1]$ and therefore

$$\deg(h(x, 0), B_\delta(0), 0) = \deg(h(x, 1), B_\delta(0), 0).$$

Since $h(x, 1) = f\left(x_0 + \frac{x}{2}\right) - f\left(x_0 - \frac{x}{2}\right)$ is an odd mapping, $\deg(h(x, 1), B_\delta(0), 0) \neq 0$ by Borsuk's theorem ([10], p. 99). Thus we have $\deg(f, B_\delta(x_0), f(x_0)) = \deg(h(x, 0), B_\delta(0), 0) \neq 0$. This implies that $B_\beta(f(x)) \subset f(B_\delta(x_0)) \subset f(B_\alpha(x_0))$. □

Proof of Theorem 5. 1. Without loss of generality, we can assume that $s < \frac{\lambda^m - 1}{\lambda^m + 1}\|z - w\|$ and $\nu < \lambda$. By the above lemma, there exists an open ball $B_a(z)$ such that $f^m: B_s(w) \cap f^{-m}(B_a(z)) \to B_a(z)$ is an onto mapping. If we put $f_0 = f|_{B_r(z)}$, then f_0^{-1} is a contraction mapping with the Lipschitz constant $\frac{1}{\lambda}$. We define a sequence of open sets $\{B_{-n}\}_{n\geq 1}$ as follows:

$$B_{-1} = B_s(w) \cap f^{-m}(B_a(z)), \quad B_{-n} = f_0^{-m}(B_{-n+1}) \quad \text{for } n \geq 2.$$

Then note that $B_{-1} \cap B_{-n} = \phi$ if $n \geq 2$. Thus there exists a positive integer p such that $B_{-n} \subset B_a(z)$ for any $n \geq p$ and $\lambda^{p(m-1)}\mu > 1$.

Now if we put

$$A_0 = \bigcup_{n=p+1}^{\infty} B_{-n} \cup \{z\} \quad \text{and} \quad A_1 = B_{-1} \cap f_0^{-m}\left(\bigcup_{n=p}^{\infty} B_{-n} \cup \{z\}\right),$$

then it is clear that $\overline{A_0} \cap \overline{A_1} = \phi$. Moreover, we have $f^{pm}(A_0) \supset A_0 \cup A_1$ and $f^{pm}(A_1) \supset A_0 \cup A_1$. Hence

$$f^{pm}(\overline{A_i}) = \overline{f^{pm}(\overline{A_i})} \supset \overline{f^{pm}(A_i)} \supset \overline{A_0} \cup \overline{A_1}$$

for $i = 0, 1$ and f^{pm} satisfies Hypothesis 2. 1. Also one can easily verify that f^{pm} is $\lambda^{p(m-1)}\mu$-expanding on each compact set $\overline{A_i} \cap f^{-pm}(\overline{A_j})$ for $\{i, j\} \in \{0, 1\}^2$. By Theorem 4. 4, this completes the proof. □

6. De Rham Equation.

In this section, we will give another result concerning a scrambled set using a contraction principle in complete metric space instead of Cantor's intersection theorem in compact space. With this situation we will discuss the connection with De Rham's functional equations [11].

Let E be a complete metric space with metric d and T_0, T_1: $E \to E$ be two continuous mappings. Then consider the following *De Rham equation*:

$$(*)\qquad L(\omega)=T_{\omega_0}L(\sigma(\omega)),$$

where L is a mapping from Σ into E. First we have the following.

Theorem 6. 1. *Suppose that L is a continuous solution of (*) and that T_0 and T_1 are one to one mappings such that $T_0(E)\cap T_1(E)=\phi$. Then we have the followings:*

(i) *$E^*=L(\Sigma)$ is a compact subset and $E^*=T_0(E^*)\cup T_1(E^*)$.*

(ii) *If we put $f(x)=T_i^{-1}(x)$ for $x\in T_i(E^*)$, $i=0, 1$, then $f: E^*\to E^*$ is topologically conjugate to the shift $\sigma: \Sigma\to\Sigma$.*

Proof. (i) is obvious. For any $\omega\neq\omega'\in\Sigma$, there exists $n\geq 0$ such that $\omega_n\neq\omega'_n$ and $\omega_j=\omega'_j$ for $0\leq j<n$. Then we have $L(\omega)\neq L(\omega')$ since $T_{\omega_n}L(\sigma^{n+1}(\omega))\neq T_{\omega_n}L(\sigma^{n+1}(\omega'))$. Therefore $L: \Sigma\to E^*$ is a homeomorphism. Since $f(L(\omega))=T_{\omega_0}^{-1}L(\omega)=L(\sigma(\omega))$, this completes the proof. □

In the above theorem, we remark that $f: E^*\to E^*$ is continuous and a system (E^*, d, f) satisfies Hypothesis 2. 1.

Theorem 6. 2. *Suppose that there exists $s\geq 1$ such that $T_{i_1}T_{i_2}\dots T_{i_s}$ is a contraction mapping on E for any $\{i_1, i_2, \dots, i_s\}\in\{0, 1\}^s$. Then there exists a continuous solution L of (*).*

Proof. For some $x_0\in E$, we define

$$a_n=T_{\omega_0}T_{\omega_1}\dots T_{\omega_{n-1}}(x_0)\quad\text{for any }\omega\in\Sigma.$$

Then we have, for some $\lambda<1$,

$$d(a_{n+1}, a_n)=d(T_{\omega_0}\dots T_{\omega_n}(x_0),\ T_{\omega_0}\dots T_{\omega_{n-1}}(x_0))\leq\lambda^{[\frac{n}{s}]}d^*,$$

where $d^*=\underset{\substack{i_1,\dots,i_j\\ 1\leq j\leq s}}{\mathrm{Max}}\ d(T_{i_1}\dots T_{i_j}(x_0),\ T_{i_1}\dots T_{i_{j-1}}(x_0))$.

Therefore, for any $n>m$,

$$\begin{aligned} d(a_n, a_m)&\leq d(a_n, a_{n-1})+\dots+d(a_{m+1}, a_m)\\ &\leq d^*\sum_{k=m}^{n-1}\lambda^{[\frac{k}{s}]}\\ &<\frac{d^*}{1-\mu}\mu^{m-s}, \end{aligned}$$

where $\mu=\lambda^{\frac{1}{s}}<1$. Hence $\{a_n\}$ is a Cauchy sequence in E. Now we define $L(\omega)=\lim_{n\to\infty}a_n$. Note that this limit is independent of the choice of $x_0\in E$, since

$$d(T_{\omega_0}\dots T_{\omega_n}(x_0),\ T_{\omega_0}\dots T_{\omega_n}(x_1))\to 0\ \text{ as } n\to\infty.$$

Clearly L satisfies the equation (*). Let $E^*=\bigcup_{\omega\in\Sigma}L(\omega)$. Then we have $\mathrm{diam}(E^*)<\infty$ since $d(L(\omega), x_0)\leq\frac{d^*}{\lambda(1-\mu)}$ for any $\omega\in\Sigma$. If $\omega_j=\omega_j'$ for $0\leq j<sn$, then

$$d(L(\omega), L(\omega'))\leq\lambda^n d(L(\sigma^{sn}(\omega)), L(\sigma^{sn}(\omega')))$$
$$\leq\lambda^n \mathrm{diam}(E^*)$$

This shows the continuity of L. □

It should be noted that f may possess a scrambled set even if f is a linear operator. For example, if we put $E=C[0, 1]$ and

$$T_0(g)=\int_0^x p(t)g(t)dt \quad\text{and}\quad T_1(g)=\int_0^x p(t)g(t)dt+1$$

for some positive function $p\in E$, then it is easily verified that T_i: $E\rightarrow E$ is a one to one mapping for $i=0$, 1 and $\mathrm{dist}(T_0(E), T_1(E))\geq 1$. Moreover, we have

$$||T_{i_1}\dots T_{i_n}(g)-T_{i_1}\dots T_{i_n}(h)||=||T_0^n(g-h)||$$
$$\leq\frac{||p||}{n!}||g-h||$$

for any g, $h\in E$. Then, by Theorem 6. 1. and Theorem 6. 2, there exists a scrambled set of $(E^*, ||\ ||, f)$ where

$$f=\frac{1}{p(x)}\frac{d}{dx}.$$

In this case, using $T_{\omega_n}(g)=T_0(g)+\omega_n$, we have

$$L(\omega)=\sum_{n=0}^{\infty}\omega_n T_0^n(1)=\sum_{n=0}^{\infty}\frac{\omega_n}{n!}\left(\int_0^x p(t)\,dt\right)^n.$$

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