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Local energy integrals for effectively hyperbolic operators II

By

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1. Introduction.

In the previous paper [2], we have derived the local energy estimates for effectively hyperbolic operators of second order of type $(1)_p$. In this note, we shall derive the local energy estimates for effectively hyperbolic operators of other types. That is of types $(2)_p$ and $(1)_p$ with $\phi_p = x_{p+1}$. We follow [2] quite far and obtain the estimates which indicate the loss of regularity of solutions when they transverse a certain hypersurface in $T^*(\boldsymbol{R}^{d+1})$. The only difference in the proof is the treatment of the quantitative condition in $(2)_p$. We use the same notations as in [2].

Let us consider the operators of following types.

$$(1)'_p \begin{cases} \xi_0^2 - \sum_{i=0}^{p-1} (x_i - x_{i+1})^2 q_i(X, \xi) - \sum_{i=1}^{p-1} \xi_i^2 r_i(X, \xi) - \phi_p(x^{(p)}, \xi^{(p)}) q_p(X, \xi), \\ \text{with } (\partial^2 \phi_p / \partial \xi_p^2)(0, \hat{\xi}^{(p)}) = 0, \ 1 \leqq p \leqq d, \ \hat{\xi} = (0, \dots, 0, \hat{\xi}_d), \end{cases}$$

$$(2)_p \begin{cases} \xi_0^2 - \sum_{i=0}^{p-1} (x_i - x_{i+1})^2 q_i(X, \xi) - \sum_{i=1}^{p} \xi_i^2 r_i(X, \xi) - g_p(x^{(p)}, \xi^{(p+1)}) r_{p+1}(X, \xi), \\ \text{with } \sum_{i=1}^{p} r_i(0, \hat{\xi})^{-1} > 1, \ (\partial^2 g_p / \partial x_p^2)(0, \hat{\xi}^{(p+1)}) = 0, 1 \leqq p \leqq d-1, \\ \qquad \hat{\xi} = (0, \dots, 0, \hat{\xi}_d), \end{cases}$$

where q_i, r_i is positive and homogeneous of degree 2, 0 and ϕ_p, g_p is non-negative, vanishing at $(0, \hat{\xi})$, homogeneous of degree 0, 2. Without loss of generality, we may assume that all the above functions are defined in $I \times \boldsymbol{R}^d \times U$, where U is a conic neighborhood of $\hat{\xi}$, $I = (-T, T)$, and do not depend on x if $|x| > R$.

We make the change of scale of the variables; $\eta_0 = \mu^{-1}\xi_0$, $\eta = \mu^{-1}\xi$, $y_0 = \mu x_0$, $y = \mu x$, and we extend $q_i(\mu X, \xi)$, $r_i(\mu X, \xi)$, $\phi_p(\mu x^{(p)}, \xi^{(p)})$, $g_p(\mu x^{(p)}, \xi^{(p+1)})$ to $q_i(X, \xi, \mu)$, $r_i(X, \xi, \mu)$, $\phi(x, \xi, \mu)$, $g(x, \xi, \mu)$ coinciding with the original ones in

$$U(\mu, \hat{\xi}) = \{(x, \xi) ; |x| < 1, \ |\xi|\xi|^{-1} - \hat{\xi}| < \mu, \ |\xi| > \mu^{-2}\}.$$

These extensions will be clarified in the next section. Then we are led to the following operators.

$$(1)'_p\begin{cases}P_{(\mu)}=\xi_0^2-\mu^2\sum_{i=0}^{p-1}Y_i(X)^2q_i(X,\xi,\mu)-\sum_{i=1}^{p-1}\xi_i^2r_i(X,\xi,\mu)-\psi(x,\xi,\mu)q_p(X,\xi,\mu)+\\ \qquad +\mu T_1(X,\xi,\mu)+\mu T_0(X,\xi,\mu)\xi_0,\end{cases}$$

$$(2)_p\begin{cases}P_{(\mu)}=\xi_0^2-\mu^2\sum_{i=0}^{p-1}Y_i(X)^2q_i(X,\xi,\mu)-\sum_{i=1}^{p}\xi_i^2r_i(X,\xi,\mu)-g(x,\xi,\mu)r_{p+1}(X,\xi,\mu)+\\ \qquad +\mu T_1(X,\xi,\mu)+\mu T_0(X,\xi,\mu)\xi_0,\end{cases}$$

where $T_i(X,\xi,\mu)$ denotes an extension of $T_i(\mu X,\xi)$ with $T_i(X,\xi)$ being positively homogeneous of degree i in ξ defined in $I\times\boldsymbol{R}^d\times U$, independent of x if $|x|>R$. Obviously $Y_i(X)=x_i-x_{i+1}$.

From [1], there exist real numbers $\{\varepsilon_i\}_{i=1}^p$ such that

$$(1.1)\qquad \sum_{i=1}^p\varepsilon_i^2r_i(0,\hat{\xi})<1,\ \sum_{i=1}^p\varepsilon_i=1,$$

in both cases $(2)_p$ and $(1)'_p$. Especially, in the case $(1)'_p$, we take $\varepsilon_j=0$, $1\leqq j\leqq p-1$, $\varepsilon_p=1$. Using these numbers, we put

$$(1.2)\qquad Y(X)=x_0-\sum_{i=1}^p\varepsilon_ix_i.$$

Here we note that

$$(1.3)\qquad Y(X)=\sum_{i=0}^{p-1}\alpha_iY_i(X),\ \text{with some constants }\alpha_i.$$

Using $Y(X)$ in (1.3), we define $J(X,\xi,\mu)$, $J_\pm(X,\xi,\mu)$, $\alpha_n^\pm(X,\xi,\mu)$ and $|||u|||_{n+k,r}^2$ following the same formulas in [2]. We denote by $P_{(\mu)}^s(X,\Xi)$ the subprincipal symbol of $P_{(\mu)}$. Since $\psi_p(0,\hat{\xi}^{(p)})=0$, $\operatorname{grad}\psi_p(0,\hat{\xi}^{(p)})=0$, $g_p(0,\hat{\xi}^{(p+1)})=0$, $\operatorname{grad}g_p(0,\hat{\xi}^{(p+1)})=0$, it is clear that $\mu^{-1}P_{(\mu)}^s(0,0,0,\hat{\xi})$ does not depend on μ. Hence we denote it by $P^s(0,0,0,\hat{\xi})$. Then we have

Theorem 1.1. *For any* $L\in\boldsymbol{N}$, *we have*
$C(n,\mu,L)\int e^{-2x_0\theta}\|P_{(\mu)}u\|_{-2L}^2dx_0+\int e^{-2x_0\theta}|||P_{(\mu)}u|||_{n,0}^2dx_0\geqq c_1n\int e^{-2x_0\theta}|||D_0u|||_{n,1}^2dx_0+$
$+c_2n\int e^{-2x_0\theta}|||u|||_{n+1,0}^2dx_0+c_3\theta\int e^{-2x_0\theta}|||D_0u|||_{n,1/2}^2dx_0+c_3\theta\int e^{-2x_0\theta}|||u|||_{n+1,-1/2}^2dx_0+$
$+c_4\theta^3\int e^{-2x_0\theta}\|u\|_{-L}^2dx_0+c_4\theta^{3/2}\int e^{-2x_0\theta}\|D_0u\|_{-L}^2dx_0$, *for* $n\geqq C_0C$, $0<\mu\leqq\mu_0(n)$, $\theta\geqq\theta_0(n,\mu,L)$, $u\in C_0^\infty(I\times\boldsymbol{R}^d)$, *where* $C=|P^s(0,0,0,\hat{\xi})|+1$, $C_0=C_0(q_i(0,\hat{\xi}))$ *in the case* $(1)'_p$ *and* $C_0=C_0(q_i(0,\hat{\xi}),r_i(0,\hat{\xi}))$ *in the case* $(2)_p$.

Theorem 1.2. *For any* $s\in\boldsymbol{R}$, *we have*
$C(n,s)\int e^{-2x_0\theta}\|P_{(1)}u\|_{n+s+1}^2dx_0\geqq\theta^{3/2}\int e^{-2x_0\theta}\|D_0u\|_s^2dx_0+\theta^3\int e^{-2x_0\theta}\|u\|_s^2dx_0$, *for* $n\geqq C_0C$, $\theta\geqq\theta_0(n,s)$, $u\in C_0^\infty(I\times\boldsymbol{R}^d)$.

Before to proceed to the next section, we make some general remarks on symbols which will be used in this note. From the definition, it is easily verified that $J_{\pm}(X,\xi,\mu)^n$ satisfy the following estimates

$$(1.4)\qquad |(J_{\pm}(X,\xi,\mu)^n)^{(\gamma)}_{(\alpha)}|\leqq C_{\alpha,\gamma}J(X,\xi,\mu)^{n-|\alpha|}\langle\xi\rangle^{-|\gamma|},$$

for all multi-indexes $\alpha\in\boldsymbol{N}^{d+1}, \gamma\in\boldsymbol{N}^d$ with $C_{\alpha,\gamma}$ independent of μ, $0<\mu\leqq 1$. Since $J(X,\xi,\mu)\leqq\langle\mu\xi\rangle^{-1/2}$, it is obvious that

$$(1.5)\qquad ((J_{\pm}(X,\xi,\mu)^n)^{(\gamma)}_{(\alpha)}\in J^{n-|\alpha|}S^{-|\gamma|,0},\quad n\in\boldsymbol{R}.$$

It is also clear that $(\alpha_n^{\pm}(X,\xi,\mu))^{(\gamma)}_{(\alpha)}$ belongs to $J^{-|\alpha|}S^{-|\gamma|,0}$. In [2], we have uesed $J_{\pm}(X,\xi,\mu)$, $\alpha_n^{\pm}(X,\xi,\mu)$ which do not depend on $x'=(x_1,\dots,x_p)$, whereas in the case $(1)'_p$, these symbols depend only on (x_0, x_p, ξ) and in the case $(2)_p$, $J_{\pm}(X,\xi,\mu)$, $\alpha_n^{\pm}(X,\xi,\mu)$ does not depend on $x''=(x_{p+1},\dots,x_d)$. Thus when we apply the arguments in sections 5, 6 and 7 in [2], we must exchange the role of variables suitably.

2. Preliminaries.

We define the extensions of $q_i(\mu X,\xi)$, $r_i(\mu X,\xi)$, $\psi_p(\mu x^{(p)},\xi^{(p)})$ and $g_p(\mu x^{(p)},\xi^{(p+1)})$. Take $\eta_1(s)\in C_0^\infty(\boldsymbol{R}^d)$ with $0\leqq\eta_1(s)\leqq 1$, $\eta_1(s)=1$ for $|s|\leqq 1$, $\eta_1(s)=0$ for $|s|\geqq 2$. Set

$$(2.1)\qquad \begin{aligned}&q_{i,1}(X,\xi,\mu)=\{q_i(\mu X,\xi)-q_i(0,\hat{\xi})|\xi|^2\}\eta_1(\mu^{-1}(\xi|\xi|^{-1}-\hat{\xi}))\eta_1(x),\\ &q_i(X,\xi,\mu)=q_{i,1}(X,\xi,\mu)+q_i(0,\hat{\xi})|\xi|^2.\end{aligned}$$

Obviously, it follows that

$$(2.2)\qquad \begin{aligned}&q_{i,1}(X,\xi,\mu)\in(Es)^{2,1}_{1,0},\quad |q_i(X,\xi,\mu)-q_i(0,\hat{\xi})|\xi|^2|\leqq C\mu|\xi|^2,\\ &(\mu^2Y_i(X)^2q_i(X,\xi,\mu))^{(\gamma)}_{(\alpha)}\in(Es)^{2-|\gamma|,1}_{1,0},\ \text{for}\ |\gamma|\leqq 2,\end{aligned}$$

where C does not depend on μ. In addition, taking into account that $q_i(0,\hat{\xi})|\xi|^2$ depends only on ξ, it follows easily that

$$(2.3)\qquad \mathrm{op}\{\mu^2Y_i(X)^2q_i\}-\mathrm{op}\{\mu^2Y_i(X)^2q_i\}^*\in(Es)^{1,2}_{1,0}.$$

Following the formula (2.1), $r_i(\mu X,\xi)$ will be extended to $r_i(X,\xi,\mu)=$ $=r_{i,1}(X,\xi,\mu)+r_i(0,\hat{\xi})$. Then it is clear that

$$(2.4)\qquad \begin{aligned}&\xi_i^2r_{i,1}(X,\xi,\mu)\in(Es)^{2,3}_{1,0},\quad |r_i(X,\xi,\mu)-r_i(0,\hat{\xi})|\leqq C\mu,\\ &\partial_x^\alpha\partial_{\xi_k}^j\{\xi_i^2r_i(X,\xi,\mu)\}\in(Es)^{1,3-j}_{1,0},\ \text{if}\ k\neq i.\end{aligned}$$

Furthermore, we see that

$$(2.5)\qquad \mathrm{op}\{\xi_i^2r_{i,1}\}-\mathrm{op}\{\xi_i^2r_{i,1}\}^*\in(Es)^{1,2}_{1,0}.$$

Next, we extend ψ_p, g_p. Choose $\eta_2(s)\in C_0^\infty(\boldsymbol{R}^{d-p+1})$ such that $\eta_2(s)=1$, for $|s|\leqq 1$ and $\eta_2(s)=0$ for $|s|\geqq 2$. Let $\eta_3(s)\in C_0^\infty(\boldsymbol{R})$ be equal to s for $|s|\leqq 2$ and equal to 0 for $|s|\geqq 3$ and set

$$Z(x)=(\eta_3(x_1),\dots,\eta_3(x_d)),\quad \Gamma(\xi,\mu)=\mu(\eta_3(\mu^{-1}(\xi_1|\xi|^{-1}- -\hat{\xi}_1),\dots,\eta_3(\mu^{-1}(\xi_d|\xi|^{-1}-\hat{\xi}_d))).$$

We define $\phi(x, \xi, \mu) = \phi_1(x, \xi, \mu) + \phi_2(x, \xi, \mu)$ by

$$(2.6)\quad \phi_1(x, \xi, \mu) = \sum_{|\alpha+\beta|=2} \frac{1}{\alpha!\beta!} \mu^{|\beta|} Z(x)^\beta \Gamma(\xi, \mu)^\alpha \partial_x^\beta \partial_\xi^\alpha \phi_p(0, \hat{\xi}^{(p)}),$$

$$\phi_2(x, \xi, \mu) = \{\phi_p(\mu x^{(p)}, \xi^{(p)}) - \sum_{|\alpha+\beta|=2} \frac{1}{\alpha!\beta!} \mu^{|\beta|} x^\beta (\xi|\xi|^{-1} - \hat{\xi})^\alpha \partial_x^\beta \partial_\xi^\alpha \phi_p(0, \hat{\xi}^{(p)})\} \eta_2(\mu^{-1}(\xi^{(p)}|\xi|^{-1} - \hat{\xi}^{(p)})) \eta_2(x^{(p)}).$$

Since $\phi_p(0, \hat{\xi}^{(p)}) = 0$, $\phi_p(x^{(p)}, \xi^{(p)}) \geqq 0$ and $(\partial^2 \phi_p / \partial \xi_p^2)(0, \hat{\xi}^{(p)}) = 0$, $(\partial^2 \phi_p / \partial x_i \partial \xi_d)(0, \hat{\xi}^{(p)}) = 0$, it follows that $\phi_1(x, \xi, \mu) \geqq 0$, $\partial_{\xi'}^\gamma \phi_1 \in (Es)_{1,0}^{-|\gamma|,2}$, where $x' = (x_1, \dots, x_p)$, $\xi' = (\xi_1, \dots, \xi_p)$. If $\phi_2 = 0$, $\phi(x, \xi, \mu)$ coincides with $\phi_p(x^{(p)}, \xi^{(p)})$ and hence $\phi(x, \xi, \mu) \geqq 0$. As for $\phi_2(x, \xi, \mu)$, it is clear that it belongs to $(Es)_{1,0}^{0,3}$ and then we see that $\partial_x^\alpha \partial_{\xi'}^\gamma \phi$ belongs to $(Es)_{1,0}^{2-|\gamma|,1}$ for $|\gamma| \leqq 2$. From this, it follows that

$$(2.7)\quad \partial_x^\alpha \partial_{\xi'}^\gamma (\phi q_p) \in (Es)_{1,0}^{2-|\gamma|,1}, \text{ for } |\gamma| \leqq 2.$$

Following (2.6), we extend $g_p(\mu x^{(p)}, \xi^{(p+1)})$ to $g(x, \xi, \mu) = g_1(x, \xi, \mu) + g_2(x, \xi, \mu)$ which is non-negative. Since $g_p(\mu x^{(p)}, \xi^{(p+1)})$ depends only on $(x^{(p)}, \xi^{(p+1)})$, it is clear that $\partial_x^\alpha \partial_{\xi'}^\gamma g \in (Es)_{1,0}^{2-|\gamma|,1}$ for $|\gamma| \leqq 2$, and hence

$$(2.8)\quad \partial_x^\alpha \partial_{\xi'}^\gamma (g r_{p+1}) \in (Es)_{1,0}^{2-|\gamma|,1} \text{ for } |\gamma| \leqq 2.$$

Proposition 2.1.
$\partial_{x_i} \partial_{\xi_i} \phi(x, \xi, \mu) = \mu C_i(x, \xi, \mu) \partial_{x_i} \partial_{\xi_i} \phi_p(0, \hat{\xi}^{(p)}) |\xi|^{-1} + \tilde{C}_i(x, \xi, \mu) |\xi|^{-1}$,
$\partial_{x_i} \partial_{\xi_i} g(x, \xi, \mu) = \mu E_i(x, \xi, \mu) \partial_{x_i} \partial_{\xi_i} g_p(0, \hat{\xi}^{(p+1)}) |\xi| + \tilde{E}_i(x, \xi, \mu) |\xi|$,
where $C_i(x, \xi, \mu)$, $E_i(x, \xi, \mu) \in (Es)_{1,0}^{0,0}$ *are equal to* 1 *on* $U(2\mu, \hat{\xi})$ and $\tilde{C}_i(x, \xi, \mu)$, $\tilde{E}_i(x, \xi, \mu) \in (Es)_{1,0}^{0,2}$.

In view of this proposition and (2.2), (2.4), it follows that

$$(2.9)\quad \mathrm{op}\{\phi q_p\} - \mathrm{op}\{\phi q_p\}^* - i \sum_{j=1}^d \mathrm{op}(\mu C_j \partial_{x_j} \partial_{\xi_j} \phi_p(0, \hat{\xi}^{(p)}) |\xi|^{-1} q_p) \in (Es)_{1,0}^{1,2},$$

$$\mathrm{op}\{g r_{p+1}\} - \mathrm{op}\{g r_{p+1}\}^* - \sum_{j=1}^d \mathrm{op}(\mu E_j \partial_{x_j} \partial_{\xi_j} g_p(0, \hat{\xi}^{(p+1)}) |\xi| r_{p+1}) \in (Es)_{1,0}^{1,2}.$$

Now we define $T_i(X, \xi, \mu)$. Let $\eta_4(s) \in C_0^\infty(\boldsymbol{R}^d)$ be equal to 1 for $|s| \leqq 1$ and equal to 0 for $|s| \geqq 2$. Set

$T_{i,1}(X, \xi, \mu) = \{T_i(\mu X, \xi) - T_i(0, \hat{\xi}) |\xi|^i\} \eta_4(\mu^{-1}(\xi|\xi|^{-1} - \hat{\xi})) \eta_4(x)$,
$T_0(X, \xi, \mu) = T_{0,1}(X, \xi, \mu) + T_0(0, \hat{\xi})$
$T_1(X, \xi, \mu) = T_{1,1}(X, \xi, \mu) + T_1(0, \hat{\xi}) |\xi| - \frac{i}{2} \sum_{j=1}^d (1 - C_j) \partial_{x_j} \partial_{\xi_j} \phi_p(0, \hat{\xi}^{(p)}) q_p |\xi|^{-1}$,
(in the case $(1)_p'$),
$T_1(X, \xi, \mu) = T_{1,1}(X, \xi, \mu) + T_1(0, \hat{\xi}) |\xi| - \frac{i}{2} \sum_{j=1}^d (1 - E_j) \partial_{x_j} \partial_{\xi_j} g_p(0, \hat{\xi}^{(p+1)}) |\xi| r_{p+1}$
(in the case $(2)_p$),

where C_j, E_j are the same ones in proposition 2.1. It is clear that

$$(2.10)\quad T_i(X, \xi, \mu) \in (Es)_{1,0}^{i,0}, \quad |T_{i,1}(X, \xi, \mu)| \leqq C\mu |\xi|^i,$$
$$T_1(X, \xi, \mu) = T_i(\mu X, \xi) \text{ in } U(\mu, \hat{\xi}).$$

In the case $(1)'_p$, $Y(X)$ depends only on x_0, x_p and if we note the fact which follows from (2.4) that

$$\partial_x^\alpha \partial_{\xi_p}^j \{\xi_i^2 r_i(X, \xi, \mu)\} \in (Es)_{1,0}^{2-j,1},\ 0 \leqq j \leqq 2,\ 1 \leqq i \leqq p-1,$$

one can proceed exactly in the same way as in sections 5, 6 and 7 in [2].

Consider the case $(2)_p$. Since $Y(X)$ does not depend on x'', one can apply the same roasoning to following operator,

$$\tilde{P}_{(\mu)} = \xi_0^2 - \mu^2 \sum_{i=0}^{p-1} Y_i(X)^2 q_i(X, \xi, \mu) - \sum_{i=1}^{p} \xi_i^2 r_{i,1}(X, \xi, \mu) -$$
$$-g(x, \xi, \mu) r_{p+1}(X, \xi, \mu) + \mu T_1(X, \xi, \mu) + \mu T_0(X, \xi, \mu) \xi_0.$$

Thus, the only term which must be considered is $\sum_{i=1}^{p} \xi_i^2 r_i(0, \hat{\xi})$. In the next section, we shall treat this term.

3. Observations on $r_i(0, \hat{\xi}) D_i^2$.

We start with the following egality.

$$\begin{aligned} (3.1)\qquad & 2\mathrm{Im}(I_n(n-1/2) D_i^2 w, I_n(n-1/2)(D_0 - i\theta) w) = \\ & = -\Phi_n^-(D_i^2 w, (D_0 - i\theta) w) = 2\theta \| I_n(n-1/2) D_i w \|^2 - \\ & -2\mathrm{Re}(\partial_0 I_n(n-1/2) D_i w, I_n(n-1/2) D_i w) + \\ & +2\mathrm{Im}([I_n(n-1/2),\ D_i] D_i w, I_n(n-1/2)(D_0 - i\theta) w) + \\ & +2\mathrm{Im}([I_n(n-1/2), D_i](D_0 - i\theta) w, I_n(n-1/2) D_i w). \end{aligned}$$

Consider the second term of the right hand side of (3.1). If we note that

$$\partial_0 I_n(n-1/2) D_i \alpha_n^- + (n-1/2) I_n(n-3/2) D_i \alpha_n^- \equiv 0, \text{ for } n \geqq 16,$$

it follows that

$$\begin{aligned} & -2\mathrm{Re}(\partial_0 I_n(n-1/2) D_i \alpha_n^- u, I_n(n-1/2) D_i \alpha_n^- u) \equiv \\ & \equiv (2n-1)\mathrm{Re}(I_n(n-3/2) D_i \alpha_n^- u, I_n(n-1/2) D_i \alpha_n^- u). \end{aligned}$$

On the other hand, from the expression

$$I_n(n-1/2)^* I_n(n-3/2) \equiv I_n(n-1)^*(1+b) I_n(n-1), \text{ with } b \in \mu S_{1/2}^{0,0},$$

this is estimated from below by

$$(3.2)\qquad (2n-1-C(n)\mu) \| I_n(n-1) D_i \alpha_n^- u \|^2, \text{ modulo } C(\mathrm{n}, \mu, L) \| u \|_{-L}^2.$$

Now consider the last two terms of the right hand of (3.1). Since

$$\sigma([I_n(n-1/2), D_j]) = i(n-1/2)\varepsilon_j L(X, \xi, \mu) J_-(X, \xi, \mu)^{n-3/2},$$

where $L(X, \xi, \mu) = \{2\chi_0(-Y(X)\langle \mu\xi \rangle^{1/2}) - 1\} - 2\chi_0^{(1)}(-Y(X)\langle \mu\xi \rangle^{1/2}) Y(X) \langle \mu\xi \rangle^{1/2}$, it follows that

$$I_n(n-1/2)^*[I_n(n-1/2), D_j] \equiv i\varepsilon_j(n-1/2)I_n(n-1)^*(L+b)I_n(n-1),$$
$$b\in \mu S^{0,0}_{1/2}.$$

On the other hand, taking into account that $L(X, \xi, \mu)=1$ on the support of α_n^-, for $n\geqq 16$, we get

$$\begin{aligned}&I_n(n-1/2)^*[I_n(n-1/2), D_j]D_j\alpha_n^- \equiv\\&\equiv i\varepsilon_j(n-1/2)I_n(n-1)^*(1+b)I_n(n-1)D_j\alpha_n^-,\ \ b\in\mu S^{0,0}_{1/2},\end{aligned}$$

and this implies that

$$\begin{aligned}(3.3)\qquad &|r_j([I_n(n-1/2), D_j]D_j\alpha_n^-u, I_n(n-1/2)(D_0-i\theta)\alpha_n^-u)|\leqq\\&\leqq r_j\varepsilon_j(n-1/2)\|I_n(n-1)D_j\alpha_n^-u\|\|I_n(n-1)(D_0-i\theta)\alpha_n^-u\|+\\&+C(n)\mu\|I_n(n-1)D_j\alpha_n^-u\|^2+C(n)\mu\|I_n(n-1)(D_0-i\theta)\alpha_n^-u\|^2,\end{aligned}$$

where $r_j=r_j(0, \hat{\xi})$. Thus, the last two terms are estimated by

$$\begin{aligned}(3.4)\qquad &2r_j\varepsilon_j(2n-1)\|I_n(n-1)D_j\alpha_n^-u\|\|I_n(n-1)(D_0-i\theta)\alpha_n^-u\|+\\&+C(n)\mu\|I_n(n-1)D_j\alpha_n^-u\|^2+C(n)\mu\|I_n(n-1)(D_0-i\theta)\alpha_n^-u\|^2.\end{aligned}$$

From the assumption (1.1), there exists $0<\bar{\delta}<1$ such that

$$\bar{\delta}^{-1}\Big(\sum_{j=1}^{p} r_j\varepsilon_j^2\Big)=\hat{\delta}<1,$$

then we see from (3.2) and (3.4) that

$$\begin{aligned}&-\sum_{j=1}^{p}\Phi_n^-(D_j^2\alpha_n^-u, (D_0-i\theta)\alpha_n^-u)\geqq 2\theta\sum_{j=1}^{p}r_j\|I_n(n-1/2)D_j\alpha_n^-u\|^2+\\&+(2n-1)(1-\bar{\delta}-C(n)\mu)\sum_{j=1}^{p}r_j\|I_n(n-1)D_j\alpha_n^-u\|^2-\\&-(2n-1)(\hat{\delta}+C(n)\mu)\|I_n(n-1)(D_0-i\theta)\alpha_n^-u\|^2.\end{aligned}$$

After having done the same arguments for $\Phi_n^+(D_j^2\alpha_n^+u, (D_0-i\theta)\alpha_n^+u)$, it follows that

Proposition 3.1.

$$\begin{aligned}&-\sum\sum_{j=1}^{p}\Phi_n^{\pm}(r_jD_j^2\alpha_n^{\pm}u, (D_0-i\theta)\alpha_n^{\pm}u)\geqq 2\theta\sum\sum_{j=1}^{p}r_j\ \{\|I_n(n-1/2)D_j\alpha_n^{\pm}u\|^2+\\&+\|J_+(-n-1/2)D_j\alpha_n^{\pm}u\|^2\}+(2n-1)(1-\bar{\delta}-C(n)\mu)\sum\sum_{j=1}^{p}r_j\{\|I_n(n-1)D_j\alpha_n^{\pm}u\|^2+\\&+\|J_+(-n-1)D_j\alpha_n^{\pm}u\|^2\}-(2n-1)(\hat{\delta}+C(n)\mu)\|u\|^2_{D,n,1},\ \textit{modulo}\ C(n,\mu,L)\,|u|^2_{-L}.\end{aligned}$$

We proceed to the next step. Consider $[\alpha_n^-, D_j]$. Here, obviously

$$\sigma([\alpha_n^-, D_j])=-i\varepsilon_jn^{1/2}\chi^{(1)}(-n^{1/2}Y(X)\langle\mu\xi\rangle^{1/2})\langle\mu\xi\rangle^{1/2}=-i\varepsilon_jl(X, \xi, \mu),$$

and we remark that $l(X, \xi, \mu)\in J^{-1-2N}S^{0,-N}$, for any $N\in \boldsymbol{R}$, Then it is clear that one can express

$$\begin{aligned}(3.5)\qquad &I_n(n-1/2)^*I_n(n-1/2)[\alpha_n^-, D_j]\equiv\varepsilon_jn^{1/2}I_n(n-1)^*(l_1+b_1)I_n\ (n-1)\equiv\\&\equiv\varepsilon_jn^{1/2}I_n(n-1)^*(l_2+b_2)J_+(-n-1),\ \ b_i\in\mu S^{0,0}_{1/2},\end{aligned}$$

where

$$l_1(X,\xi,\mu)=\chi^{(1)}(-n^{1/2}Y(X)\langle\mu\xi\rangle^{1/2})\langle\mu\xi\rangle^{1/2}J_-(X,\xi,\mu),$$
$$l_2(X,\xi,\mu)=\chi^{(1)}(-n^{1/2}Y(X)\langle\mu\xi\rangle^{1/2})(\langle\mu\xi\rangle J_-(X,\xi,\mu)J_+(X,\xi,\mu))^n\times$$
$$\times\langle\mu\xi\rangle^{1/2}J_+(X,\xi,\mu).$$

From proposition 6.1 in [2], we know that

(3.6) $\quad |l_i(X,\xi,\mu)|\leqq C$, with C independent of n,

then this fact and (3.5) show that (since $\alpha_n^+ + \alpha_n^- =1$),

(3.7) $$|\Phi_n^-([\alpha_n^-,D_j]D_ju,(D_0-i\theta)\alpha_n^-u)|\leqq n^{1/2}(C+C(n)\mu)\{||I_n(n-1)D_j\alpha_n^-u||^2+$$
$$+||J_+(-n-1)D_j\alpha_n^+u||^2\}+n^{1/2}(C+C(n)\mu)||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2.$$

On the other hand, one can write

$$[\alpha_n^-,D_j^2]=2[\alpha_n^-,D_j]D_j+\varepsilon_j^2r,$$

where $r(X,\xi,\mu)=n\chi^{(2)}(-n^{1/2}Y(X)\langle\mu\xi\rangle^{1/2})\langle\mu\xi\rangle\in J^{-2-2N}S^{0,-N}$, for any $N\in\boldsymbol{R}$. Therefore we get

$$I_n(n-1/2)^*I_n(n-1/2)r\equiv nI_n(n-1)^*(\tilde{l}_1+b_1)I_n(n-2)\equiv$$
$$\equiv nI_n(n-1)^*(\tilde{l}_2+b_2)J_+(-n-2),$$

where $l_i(X,\xi,\mu)$ belongs to $J^0S^{0,0}$ and satisfies the estimate (3.6). Thus the same procedure gives that

(3.8) $$|\Phi_n^-(ru,(D_0-i\theta)\alpha_n^-u)|\leqq n^{1/2}(C+C(n)\mu)||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2+$$
$$+n^{3/2}(C+C(n)\mu)|||u|||_{n,2}^2.$$

Summing up, from (3.7) and (3.8), it follows that

$$|\Phi_n^-([\alpha_n^-,D_j^2]u,(D_0-i\theta)\alpha_n^-u)|\leqq n^{1/2}(c_1+C(n)\mu)||I_n(n-1)(D_0-i\theta)\alpha_n^-u||^2+$$
$$+n^{1/2}(c_1+C(n)\mu)\{||I_n(n-1)D_j\alpha_n^-u||^2+||J_+(-n-1)D_j\alpha_n^-u||^2\}+$$
$$+n^{3/2}(c_1+C(n)\mu)|||u|||_{n,2}^2,$$
modulo $C(n,\mu,L)\,|u|_{-L}^2$.

Since the estimate for $\Phi_n^+(\cdots)$ is obtained similarly, we have

Proposition 3.2.
$\sum|\Phi_n^\pm([\alpha_n^\pm,D_j^2]u,(D_0-i\theta)\alpha_n^\pm u)|\leqq n^{1/2}(c_1+C(n)\mu)||u||_{D,n,1}^2+n^{1/2}(c_1+C(n)\mu)\times$ $\times\{||I_n(n-1)D_j\alpha_n^-u||^2+||J_+(-n-1)D_j\alpha_n^+u||^2\}+n^{3/2}(c_1+C(n)\mu)|||u|||_{n,2}^2,$ *modulo* $C(n,\mu,L)\,|u|_{-L}^2$.

Combining propositions 3.1 and 3.2, we obtain finally

Proposition 3.3.

$$-\sum\sum_{j=1}^{p}\Phi_n^\pm(\alpha_n^\pm r_jD_j^2u,(D_0-i\theta)\alpha_n^\pm u)\geqq-(2n-1)(\hat{\delta}+c_2n^{1/2}+C(n)\mu)||u||_{D,n,1}^2-$$
$$-n^{3/2}(c_1+C(n)\mu)|||u|||_{n,2}^2,$$

for $n \geqq n_0(c_1)$, $0<\mu\leqq\mu_0(n)$, *modulo* $C(n, \mu, L)\,|u|^2_{-L}$.

Now we complete the proof of theorem 1.1 in case $(2)_p$. In the inequality of proposition 7.5 in [2], we take δ, $\delta_1>0$ sufficiently small so that $\delta+\delta_1+\hat{\delta}<1$. Then one can absorb the right hand side of the inequailty in proposition 3. 3, and this fact proves theorem 1. 1. Here we recall that $\hat{\delta}$ depends on $\{r_i(0, \hat{\xi})\}$, and accordingly, C_0 (in theorem 1. 1) may depend also on $\{r_i(0, \hat{\xi})\}$.

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