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Outradii of Teichmüller spaces of finitely generated Fuchsian groups of the second kind

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By

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§ 1. Introduction.

A Fuchsian group Γ is said to be of the first kind (resp. second kind) if its region of discontinuity is not connected (resp. connected). The outradius $o(\Gamma)$ which is defined in §2 is strictly greater than 2 (Earle [7]) and not greater than 6 (Nehari [13]). This constant 6 cannot be replaced by any smaller one (Chu [6], Kalme [9]). In [14] the former author proved that $o(\Gamma)$ is strictly less than 6 for a finitely generated Fuchsian group Γ of the first kind. In this paper we prove the following.

Theorem. *If Γ is a finitely generated Fuchsian group of the second kind, then $o(\Gamma)$ is equal to* 6.

This theorem answers a question raised by Lipman Bers to the former author in U.S. -Japan Seminar on Kleinian Groups and Riemann Surfaces which was held at the East–West Center in Honolulu, Hawaii, during January 15–19, 1979. In §3 we state three lemmas without proofs. A proof of Theorem is given in §4. The rest of this note is devoted to prove lemmas stated in §3.

The authors would like to express their hearty thanks to Professor L. Bers for his kind indication of the problem to them.

§ 2. Definitions and notations.

Let $\hat{C}$ be the Riemann sphere. Let Δ be the open unit disc and Δ^* be the exterior of Δ in $\hat{C}$. Let $j(z)=1/\bar{z}$ be the reflection in $\partial\Delta$. For each μ in the open unit ball $L_\infty(\Delta)_1$ of $L_\infty(\Delta)$ we define two quasiconformal automorphisms w_μ and w^μ of $\hat{C}$. Let w_μ be the unique quasiconformal automorphism of $\hat{C}$ with fixed points $1, \sqrt{-1}$ and -1 which is μ-conformal in Δ and which satisfies $w_\mu \circ j = j \circ w_\mu$. In particular, w_μ keeps Δ invariant. Let w^μ be the unique quasiconformal automorphism of $\hat{C}$ with fixed points 1, $\sqrt{-1}$ and -1 which is μ-conformal in Δ and conformal in Δ^*.

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Let Γ be a Fuchsian group keeping Δ invariant. Denote by $L_\infty(\Delta, \Gamma)_1$ the subset of $L_\infty(\Delta)_1$ consisting of those $\mu \in L_\infty(\Delta)_1$ which satisfy

$$(\mu\circ\gamma)\bar{\gamma}'/\gamma' = \mu \qquad \text{for all} \quad \gamma \in \Gamma.$$

Then $\mu \in L_\infty(\Delta)_1$ belongs to $L_\infty(\Delta, \Gamma)_1$ if and only if $w_\mu\circ\gamma\circ(w_\mu)^{-1}$ (or $w^\mu\circ\gamma\circ(w^\mu)^{-1}$) is a Möbius transformation for every $\gamma \in \Gamma$ (Ahlfors [1, p. 121 and p. 123]).

Let $\lambda(z) = (|z|^2 - 1)^{-1}$ be a Poincaré density of Δ^*. Denote by $B_2(\Delta^*, \Gamma)$ the Banach space of holomorphic functions ϕ defined in Δ^* which satisfy

$$(\phi\circ\gamma)(\gamma')^2 = \phi \qquad \text{for all} \quad \gamma \in \Gamma$$

and

$$\|\phi\| = \sup_{z\in\Delta^*} \lambda(z)^{-2}|\phi(z)| < \infty.$$

For each $\mu \in L_\infty(\Delta, \Gamma)_1$ let $\phi^\mu = \{w^\mu|\Delta^*, z\}$, where $\{w^\mu|\Delta^*, z\}$ denotes the Schwarzian derivative of w^μ restricted to Δ^*. Then ϕ^μ belongs to $B_2(\Delta^*, \Gamma)$ (Ahlfors [1, p. 126]). The Teichmüller space $T(\Gamma)$ of Γ is the set $\{\phi^\mu; \mu \in L_\infty(\Delta, \Gamma)_1\}$. It is well known that $T(\Gamma)$ is a bounded domain of $B_2(\Delta^*, \Gamma)$ for a Fuchsian group Γ with $\dim B_2(\Delta^*, \Gamma) > 0$ (Bers [3]). For such a group Γ the outradius $o(\Gamma)$ of $T(\Gamma)$ is defined to be the radius of the smallest ball about the origin containing $T(\Gamma)$, that is, $o(\Gamma) = \sup \|\phi\|$, where the supremum is taken over all ϕ in $T(\Gamma)$.

§ 3. Three lemmas.

In this section we state three lemmas without proofs. Lemma 1 is due to Chu [6]. Lemmas 2 and 3 are proved in §§5–7. Let $k(z) = z + z^{-1}$. Then k maps Δ^* conformally onto $\hat{C}$ with the closed real segment $[-2, 2]$ removed. Let S_r be the circle of radius $r(>1)$ about the origin. The image of S_r under k is the ellipse

$$E_r\colon \xi^2/(r + r^{-1})^2 + \eta^2/(r - r^{-1})^2 = 1,$$

where $\zeta = k(z)$ and $\zeta = \xi + \eta\sqrt{-1}$.

For two Jordan loops J_1 and J_2 in the complex plane C we define the Fréchet distance $\delta(J_1, J_2)$ as $\inf \max_{0\leq t\leq 1} |z_1(t) - z_2(t)|$, where the infimum is taken over all possible parametrizations $z_i(t)$ of J_i $(i = 1, 2)$.

Lemma 1 (Chu [6]). *For each positive ε there exist constants $r_1 > 1$ and $d_1 > 0$ so that if $E_{r_1} = k(S_{r_1})$ and J is a Jordan loop in C with $\delta(J, E_{r_1}) \leq d_1$, then a conformal mapping f of Δ^* onto the exterior of J statisfies $\|\{f, z\}\| > 6 - \varepsilon$.*

A quasidisc is the image of an open disc under a quasiconformal automorphism of $\hat{C}$.

Lemma 2. *Let Γ be a finitely generated Fuchsian group of the second kind keeping Δ invariant. Then for each $r > 1$ and $d > 0$ there exist a sequence $\{\beta_n\}_{n=1}^\infty$*

of Möbius transformations and a sequence $\{\Omega_n\}_{n=1}^{\infty}$ *of quasidiscs which satisfy the following.*

(i) $\Omega_n \ni \infty$, $\gamma(\Omega_n)=\Omega_n$ *for all* $\gamma \in \Gamma$ *and* $\delta(\partial\Omega_n, \partial\Delta) \leqq 1/n$.

(ii) $\beta_n(\Omega_n) \ni \infty$ *and* $\delta(\beta_n(\partial\Omega_n), E_r) \leqq d$.

Lemma 3. *Let* $\{\Omega_n\}_{n=1}^{\infty}$ *be the sequence of quasidiscs in Lemma 2. Then there exist a sequence* $\{F_n\}_{n=1}^{\infty}$ *of quasiconformal automorphisms of* $\hat{C}$, *a sequence* $\{\mu_n\}_{n=1}^{\infty}$ *in* $L_\infty(\Delta, \Gamma)_1$ *and a sequence* $\{\sigma_n\}_{n=1}^{\infty}$ *of Möbius transformations keeping* Δ *invariant which satisfy the following.*

(i) F_n *maps* Δ^* *conformally onto* Ω_n.

(ii) $(F_n \circ \sigma_n \circ w_{\mu_n}) \circ \gamma \circ (F_n \circ \sigma_n \circ w_{\mu_n})^{-1} = \gamma$ *for all* $\gamma \in \Gamma$.

(iii) $\lim_{n\to\infty} \|\mu_n\|_\infty = 0$.

§4. Proof of Theorem.

Now we begin to make a proof of Theorem. For each $\varepsilon>0$ let r_1 and d_1 be the constants in Lemma 1. Lemma 2 shows that there exist a sequence $\{\Omega_n\}_{n=1}^{\infty}$ of quasidiscs and a sequence $\{\beta_n\}_{n=1}^{\infty}$ of Möbius transformations satisfying

$$\beta_n(\Omega_n) \ni \infty \tag{4.1}$$

and

$$\delta(\beta_n(\partial\Omega_n), E_{r_1}) \leqq d_1/2. \tag{4.2}$$

By Lemma 3 there exist a sequence $\{F_n\}_{n=1}^{\infty}$ of quasiconformal automorphisms of $\hat{C}$, a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $L_\infty(\Delta, \Gamma)_1$ and a sequence $\{\sigma_n\}_{n=1}^{\infty}$ of Möbius transformations keeping Δ invariant which satisfy

$$F_n \text{ maps } \Delta^* \text{ conformally onto } \Omega_n, \tag{4.3}$$

$$(F_n \circ \sigma_n \circ w_{\mu_n}) \circ \gamma \circ (F_n \circ \sigma_n \circ w_{\mu_n})^{-1} = \gamma \quad \text{for all } \gamma \in \Gamma \tag{4.4}$$

and

$$\lim_{n\to\infty} \|\mu_n\|_\infty = 0. \tag{4.5}$$

Denote by ν_n the Beltrami coefficient $\mu[F_n \circ \sigma_n \circ w_{\mu_n} | \Delta]$ of $F_n \circ \sigma_n \circ w_{\mu_n}$ restricted to Δ. Then (4.4) implies that ν_n belongs to $L_\infty(\Delta, \Gamma)_1$.

Let τ_n be the Möbius transformation so that

$$W_n = \tau_n \circ w^{\nu_n} \circ (\beta_n \circ F_n \circ \sigma_n \circ w_{\mu_n})^{-1} \tag{4.6}$$

keeps 0, 1 and ∞ fixed. Since $\mu[\tau_n \circ w^{\nu_n} | \Delta]$ and $\mu[\beta_n \circ F_n \circ \sigma_n \circ w_{\mu_n} | \Delta]$ are both equal to ν_n, (4.6) shows that $\mu[W_n | \beta_n \circ F_n(\Delta)]$ vanishes (Ahlfors [1, p. 9]). This together with (4.6) shows $\|\mu[W_n^{-1}]\|_\infty = \|\mu[W_n^{-1} | \tau_n \circ w^{\nu_n}(\Delta^*)]\|_\infty = \|\mu[w_{\mu_n} | \Delta^*]\|_\infty$. On the other hand $\|\mu[W_n^{-1}]\|_\infty = \|\mu[W_n]\|_\infty$ and $\|\mu[w_{\mu_n} | \Delta^*]\|_\infty = \|\mu_n\|_\infty$ (Ahlfors [1, p. 9 and p. 99]). Hence $\|\mu[W_n]\|_\infty = \|\mu_n\|_\infty$. By (4.5) we see $\lim_{n\to\infty} \|\mu[W_n]\|_\infty = 0$. Let K be the compact set $\{z \in C;\ \mathrm{dist}(z, E_{r_1}) \leqq d_1/2\}$. A result on quasiconformal mappings (Ahlfors and Bers [2, Lemma 17]) yields the existence of a positive integer n_1 so that

$$|W_n(z)-z|\leqq d_1/2 \qquad \text{for all} \quad z\in K \quad \text{and all} \quad n>n_1.$$

This together with (4.2) shows

$$\delta(W_n\circ\beta_n(\partial\Omega_n),\ \beta_n(\partial\Omega_n))\leqq d_1/2. \tag{4.7}$$

On the other hand by (4.6) $\tau_n\circ w^{\nu_n}(\partial\Delta)=W_n\circ(\beta_n\circ F_n\circ\sigma_n\circ w_{\mu_n})(\partial\Delta)=W_n\circ\beta_n(\partial\Omega_n)$. Hence it follows from (4.2) and (4.7) that

$$\begin{aligned}\delta(\tau_n\circ w^{\nu_n}(\partial\Delta),\ E_{r_1})&=\delta(W_n\circ\beta_n(\partial\Omega_n),\ E_{r_1})\\ &\leqq\delta(W_n\circ\beta_n(\partial\Omega_n),\ \beta_n(\partial\Omega_n))+\delta(\beta_n(\partial\Omega_n),\ E_{r_1})\leqq d_1/2+d_1/2=d_1.\end{aligned}$$

Since W_n keeps ∞ fixed and since both w_{μ_n} and σ_n keep Δ^* invariant, (4.6), (4.1) and (4.3) imply

$$\begin{aligned}(\tau_n\circ w^{\nu_n})^{-1}(\infty)&=(\beta_n\circ F_n\circ\sigma_n\circ w_{\mu_n})^{-1}\circ W_n^{-1}(\infty)\\ &=w_{\mu_n}^{-1}\circ\sigma_n^{-1}\circ F_n^{-1}\circ\beta_n^{-1}(\infty)\in w_{\mu_n}^{-1}\circ\sigma_n^{-1}\circ F_n^{-1}(\Omega_n)=\Delta^*.\end{aligned}$$

Hence $\tau_n\circ w^{\nu_n}(\Delta^*)$ is the exterior of $\tau_n\circ w^{\nu_n}(\partial\Delta)$. Now Lemma 1 shows $\|\{\tau\circ w^{\nu_n}, z\}\|>6-\varepsilon$. Since $\{\tau_n\circ w^{\nu_n}, z\}=\{w^{\nu_n}, z\}=\phi^{\nu_n}$ (Ahlfors [1, p. 125]), $\|\phi^{\nu_n}\|>6-\varepsilon$. Recall that ν_n is in $L_\infty(\Delta, \Gamma)_1$. Then we see that ϕ^{ν_n} is in $T(\Gamma)$. Hence $o(\Gamma)>6-\varepsilon$. Since $\varepsilon>0$ is arbitrary, $o(\Gamma)\geqq 6$. On the other hand $o(\Gamma)\leqq 6$ (Nehari [13]). Therefore $o(\Gamma)=6$. This completes the proof of Theorem.

§5. Construction of a quasiconformal mapping.

In this section we construct a quasiconformal mapping which we use to prove Lemma 2 in §6. Let r and s be real numbers with $r>1$ and $0<s<r+r^{-1}$. Let T be the vertical line in $\hat{C}$ passing through s. Let U be the upper half plane. Then E_r and T intersect at exactly two points $\zeta\in U$ and $\bar{\zeta}$. Let I be the bounded closed subarc of T joining ζ to $\bar{\zeta}$. Let P be the component of $\hat{C}-T$ containing the origin. Denote by J the Jordan loop $(E_r\cap P)\cup I$. Let C be the circle with the diameter I and let B be the exterior of C. Note that T and P depend on s and that ζ, I, J, C and B all depend on both r and s.

Let D be an open disc in $\hat{C}$. It is known that every quasiconformal automorphism w of D can be extended to a homeomorphism $\hat{w}$ of the closure of D onto itself (Ahlfors [1, p. 47]).

Lemma 4. *There exists a quasiconformal automorphism v of B satisfying the following.*

(i) $v(T-I)=J-I$.

(ii) *$\hat{v}$ keeps every point of C fixed.*

Proof. Let Y be the imaginary axis. Let α be a Möbius transformation which maps B onto U and which sends ζ and $\bar{\zeta}$ to ∞ and 0, respectively. Then we see that $\alpha(T-I)=Y\cap U$ and that $\alpha(J-I)\subset U$ is an open smooth Jordan arc joining ∞ to 0. For each $y>0$ let H_y be the horizontal line in $\hat{C}$ which passes through $y\sqrt{-1}$. Then $C_y=\alpha^{-1}(H_y)$ is a circle which passes through ζ and which crosses $T-I$ orthogonally. It is elementary that C_y interesects with $J-I$ at exactly one point ζ_y. Let $h(y)=\operatorname{Re}\alpha(\zeta_y)$. Set $u(z)=z+h(y)$ for $z=x+y\sqrt{-1}\in U$. Then clearly u is a homeomorphism of U onto itself.

We show that u is a quasiconformal automorphism of U such that $u(Y\cap U)=\alpha(J-I)$ and that $\hat{u}$ keeps every point of ∂U fixed. Since $u(y\sqrt{-1})=y\sqrt{-1}+h(y)=(\operatorname{Im}\alpha(\zeta_y))\sqrt{-1}+\operatorname{Re}\alpha(\zeta_y)=\alpha(\zeta_y)$ for $y\in(0,\infty)$ and since $\cup_{y>0}\zeta_y=(J-I)\cap(\cup_{y>0}C_y)=(J-I)\cap(B\cup\{\zeta\})=J-I$, we see $u(Y\cap U)=\alpha(J-I)$. Let θ_y and $\pi-\theta_y$ be the angles between C_y and $J-I$ at ζ_y. Since α is conformal, the angles between $H_y=\alpha(C_y)$ and $\alpha(J-I)$ at $\alpha(\zeta_y)$ are also θ_y and $\pi-\theta_y$. It is easily seen that there exists a positive constant θ so that $\theta\leqq\theta_y\leqq\pi-\theta$ for all $y\in(0,\infty)$. On the other hand $\alpha(J-I)=\{h(y)+y\sqrt{-1};\ y\in(0,\infty)\}$. Hence $h(y)$ is differentiable in $(0,\infty)$ and $|h'(y)|=|\cot\theta_y|\leqq|\cot\theta|$. Therefore u is a diffeomorphism of U and satisfies

$$\begin{aligned}|(\partial u/\partial\bar{z})/(\partial u/\partial z)|&=|h'(y)/(2-h'(y)\sqrt{-1})|\\&\leqq\cot\theta(4+\cot^2\theta)^{-1/2}<1.\end{aligned}$$

Thus u is a quasiconformal automorphism of U. Since $\lim_{y\to0}h(y)=0$, $\hat{u}$ keeps every point of ∂U fixed. Clearly $v=\alpha^{-1}\circ u\circ\alpha$ is a quasiconformal automorphism of B which satisfies (i) and (ii). q.e.d.

§6. Proof of Lemma 2.

In this section we prove Lemma 2. Let T, I, P, J, C and B be as in §5. Fix an $s\in(0, r+r^{-1})$ sufficiently near to $r+r^{-1}$ so that

$$\operatorname{diam}C\leqq d/2 \tag{6.1}$$

and

$$\delta(J, E_r)\leqq d/2, \tag{6.2}$$

where diam C denotes the Euclidean diameter of C.

First we construct $\{\beta_n\}_{n=1}^{\infty}$. Let D_0 be a Dirichlet region for Γ in Δ. Let D be the union of D_0, the reflection of D_0 in $\partial\Delta$ and the free sides of D_0. Let A be an open circular arc whose closure is contained in a free side of D_0. Let β_A be a Möbius transformation which maps P and $T-I$ onto Δ and A, respectively. Then $\beta_A(C)$ is orthogonal to $\partial\Delta$ and $\beta_A(B)\cap\partial\Delta=A$. Hence $\beta_A(B)\subset D$. This shows that the family $\{\gamma(\beta_A(B))\}_{\gamma\in\Gamma}$ of open discs are mutually disjoint. For each positive integer n at most a finite number of them, say $\gamma_1(\beta_A(B)),\dots,\gamma_l(\beta_A(B))$, have diameters greater than $1/n$. We can replace A by a sufficiently small open subarc A_n of A so that diam $\gamma_i(\beta_{A_n}(B))\leqq1/n$ for $i=1,\dots,l$. Set $\beta_n=\beta_{A_n}^{-1}$. Then

$$\beta_n(\Delta)=P,\ \beta_n(A_n)=T-I \tag{6.3}$$

and

$$\operatorname{diam} \gamma(\beta_n^{-1}(B)) \leqq 1/n \qquad \text{for all} \quad \gamma \in \Gamma. \tag{6.4}$$

Secondly we construct $\{\Omega_n\}_{n=1}^{\infty}$. Define

$$V_n = \begin{cases} \gamma \circ \beta_n^{-1} \circ v \circ \beta_n \circ \gamma^{-1} & \text{in} \quad \gamma(\beta_n^{-1}(B)) \qquad \text{for all} \quad \gamma \in \Gamma \\ \text{the identity mapping in } \hat{C} - \bigcup_{\gamma \in \Gamma} \gamma(\beta_n^{-1}(B)), & \end{cases} \tag{6.5}$$

where v is the quasiconformal automorphism of B obtained in Lemma 4. The derived set of $\bigcup_{\gamma \in \Gamma} \gamma(\beta_n^{-1}(C))$ coincides with the limit set $\Lambda(\Gamma)$ of Γ. Clearly V_n is a bijection of $\hat{C}$ onto itself and both V_n and V_n^{-1} are continuous in $\hat{C} - \Lambda(\Gamma)$. Since V_n keeps every point of $\hat{C} - \bigcup_{\gamma \in \Gamma} \gamma(\beta_n^{-1}(B))$ fixed and since V_n maps $\gamma(\beta_n^{-1}(B))$ onto itself for every $\gamma \in \Gamma$, we see that V_n and V_n^{-1} are also continuous at each point of $\Lambda(\Gamma)$. Hence V_n is a homeomorphism of $\hat{C}$ onto itself. Furthermore V_n is quasiconformal off the circle $\partial \Delta$. Therefore V_n is a quasiconformal automorphism of $\hat{C}$ (Lehto and Virtanen [12, p. 45]). Set $\Omega_n = V_n(\Delta^*)$. Then Ω_n is a quasidisc.

Thirdly we prove (i). The definition (6.5) of V_n implies that $V_n(\infty) = \infty$ and $V_n \circ \gamma = \gamma \circ V_n$ for all $\gamma \in \Gamma$. Hence Ω_n contains ∞ and Ω_n is Γ-invariant. Also (6.5) implies

$$\partial \Omega_n \subset \partial \Delta \cup \Big[\bigcup_{\gamma \in \Gamma} \gamma(\beta_n^{-1}(B)) \Big]. \tag{6.6}$$

By (6.4) and (6.6) we see $\delta(\partial \Omega_n, \partial \Delta) \leqq 1/n$.

Finally we prove (ii). Using (6.5), (6.3) and Lemma 4 (i), we obtain

$$\partial \Omega_n \cap \beta_n^{-1}(B) = \beta_n^{-1} \circ v \circ \beta_n(A_n) = \beta_n^{-1} \circ v(T - I) = \beta_n^{-1}(J - I). \tag{6.7}$$

Since $J - I \subset P \cap B$, (6.7) and (6.3) show $\partial \Omega_n \cap \beta_n^{-1}(B) = \beta_n^{-1}(J - I) \subset \Delta \cap \beta_n^{-1}(B)$. This together with (6.6) implies $A_n \subset \Omega_n$. Hence by (6.3) we see $\infty \in \beta_n(\Omega_n)$. By (6.3) and (6.7) we have

$$\beta_n(\partial \Omega_n) \subset \beta_n([\partial \Omega_n \cap \beta_n^{-1}(B)] \cup [\hat{C} - \beta_n^{-1}(B)]) = (J - I) \cup (\hat{C} - B).$$

Therefore (6.1) shows $\delta(\beta_n(\partial \Omega_n), J) \leqq d/2$. Combining this with (6.2), we get

$$\delta(\beta_n(\partial \Omega_n), E_r) \leqq \delta(\beta_n(\partial \Omega_n), J) + \delta(J, E_r) \leqq d/2 + d/2 = d.$$

Thus Lemma 2 is proved.

§7. Proof of Lemma 3.

Let M_Δ be the group of all Möbius transformations which keep Δ invariant. Let $\Gamma \subset M_\Delta$ be a finitely generated Fuchsian group generated by $\gamma_1, \dots, \gamma_m$. A homomorphism χ of Γ into M_Δ is said to be allowable if $(\operatorname{trace} \chi(\gamma))^2 = (\operatorname{trace} \gamma)^2$ for all elliptic and parabolic $\gamma \in \Gamma$. Let $\{\chi_n\}_{n=1}^{\infty}$ be a sequence of homomorphisms of Γ into M_Δ. Then $\{\chi_n\}_{n=1}^{\infty}$ is said to converge to the identity if $\{\chi_n(\gamma_i)\}_{n=1}^{\infty}$ converges to γ_i for each i. Now we state a basic result on stability of Fuchsian groups due to

Bers [4, §3, Remark] in the following form, which is convenient for our proof of Lemma 3 (see also Bers [5, p. 15]).

Lemma 5. *Let Γ be a finitely generated Fuchsian group keeping Δ invariant. Let $\{\chi_n\}_{n=1}^{\infty}$ be a sequence of allowable homomorphisms $\chi_n\colon \Gamma \to M_\Delta$ which converges to the identity. Then there exist a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $L_\infty(\Delta, \Gamma)_1$ and a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in M_Δ which satisfy*

(i) $\chi_n(\gamma) = (\sigma_n \circ w_{\mu_n}) \circ \gamma \circ (\sigma_n \circ w_{\mu_n})^{-1}$ *for all* $\gamma \in \Gamma$

and

(ii) $\lim\limits_{n\to\infty} \|\mu_n\|_\infty = 0.$

Though Bers [4] proved Lemma 5 for non-elementary groups, obviously it also holds for elementary groups.

Proof of Lemma 3. Let $\{\Omega_n\}_{n=1}^{\infty}$ be the sequence of quasidiscs in Lemma 2. Let f_n be the conformal mapping of Δ^* onto Ω_n normalized so that $f_n(\infty) = \infty$ and $f_n'(\infty) > 0$. Since Ω_n is Γ-invariant, $\gamma_n = f_n^{-1} \circ \gamma \circ f_n$ is a conformal automorphism of Δ^* for every $\gamma \in \Gamma$. Hence $\Gamma_n = f_n^{-1} \Gamma f_n$ is a finitely generated Fuchsian group keeping Δ invariant. Let χ_n be the isomorphism of Γ onto Γ_n defined by $\chi_n(\gamma) = \gamma_n$ for each γ.

First we construct $\{F_n\}_{n=1}^{\infty}$. Since $f_n \circ \gamma_n = \gamma \circ f_n$, we see $\{f_n, \gamma_n(z)\}(\gamma_n')^2 = \{f_n, z\}$ (Ahlfors [1, p. 125]). This together with a theorem of Nehari [13] yields $\{f_n, z\} \in B_2(\Delta^*, \Gamma_n)$. On the other hand since Ω_n is a quasidisc, f_n can be extended to a quasiconformal automorphism of $\hat{C}$ (Ahlfors [1, p. 75]). Hence $\{f_n, z\}$ also belongs to the universal Teichmüller space $T(1)$. Therefore $\{f_n, z\}$ is an element of $T(\Gamma_n) = T(1) \cap B_2(\Delta^*, \Gamma_n)$, where the equality is due to Kra [10]. This implies the existence of $\kappa_n \in L_\infty(\Delta, \Gamma_n)_1$ such that $\{f_n, z\} = \{w^{\kappa_n}|\Delta^*, z\}$. Hence there exists a Möbius transformation ρ_n so that $f_n = \rho_n \circ w^{\kappa_n}$ in Δ^* (Bers [4, p. 589]). Set $F_n = \rho_n \circ w^{\kappa_n}$ on $\hat{C}$. Then F_n is a quasiconformal automorphism of $\hat{C}$ and F_n maps Δ^* conformally onto Ω_n.

Secondly we show that $\{\chi_n\}_{n=1}^{\infty}$ converges to the identity. For two points z_1 and z_2 in $\hat{C}$ we denote by $[z_1, z_2]$ the spherical distance from z_1 to z_2. For each $z \in \Delta^*$ we have

$$(7.1) \qquad \begin{aligned}[\gamma_n(z), \gamma(z)] &\leq [\gamma_n(z), f_n \circ \gamma_n(z)] + [f_n \circ \gamma_n(z), \gamma(z)] \\ &= [\gamma_n(z), f_n(\gamma_n(z))] + [\gamma(f_n(z)), \gamma(z)].\end{aligned}$$

Since f_n is the conformal mapping of Δ^* onto Ω_n with $f_n(\infty) = \infty$ and $f_n'(\infty) > 0$ and since $\lim_{n\to\infty} \delta(\partial\Omega_n, \partial\Delta) = 0$ by Lemma 2 (i), a classical result on conformal mappings (Goluzin [8, p. 59]) shows that $\lim_{n\to\infty} [f_n(z), z] = 0$ uniformly on the closure of Δ^*. Hence (7.1) implies $\lim_{n\to\infty} [\gamma_n(z), \gamma(z)] = 0$ for each $z \in \Delta^*$. Now by a theorem on convergence of Möbius transformations (Lehner [11, p. 73]) we see that $\{\chi_n(\gamma)\}_{n=1}^{\infty}$ converges to γ. Therefore $\{\chi_n\}_{n=1}^{\infty}$ converges to the identity.

Finally we prove (ii) and (iii). Since $\kappa_n \in L_\infty(\Delta, \Gamma_n)_1$, we see that $\chi_n(\gamma) = F_n^{-1} \circ \gamma \circ F_n$ for all $\gamma \in \Gamma$. Hence χ_n is an allowable isomorphism. Now by Lemma 5 there exist a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $L_\infty(\Delta, \Gamma)_1$ and a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in M_Δ satisfying both $\chi_n(\gamma)=(\sigma_n\circ w_{\mu_n})\circ\gamma\circ(\sigma_n\circ w_{\mu_n})^{-1}=\gamma_n$, that is, $(F_n\circ\sigma_n\circ w_{\mu_n})\circ\gamma\circ(F_n\circ\sigma_n\circ w_{\mu_n})^{-1}=\gamma$ and $\lim_{n\to\infty}\|\mu_n\|_\infty=0$. This completes the proof of Lemma 3.

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