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A remark on automorphisms of Kummer surfaces in characteristic p

By

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Introduction. Let k be an algebraically closed field and C a non-singular complete curve of genus two defined over k. The Jacobian variety $J(C)$ of the curve C is defined over k and has a natural involution ι. If char. $k \neq 2$, the quotient variety $J(C)/\langle\iota\rangle$ has sixteen rational double points of type A_1. Resolving these singularities minimally, we obtain the Kummer surface $\mathrm{Kum}(C)$ associated with the curve C. The Kummer surface $S=\mathrm{Kum}(C)$ is a $K3$ surface, that is, the canonical bundle K_S of S is trivial and $H^1(S, \mathcal{O}_S)=0$. In §1 of the present paper we shall prove the following theorem.

Theorem. *If char. $k \neq 2$, the Kummer surface $\mathrm{Kum}(C)$ carries a structure of an elliptic surface π: $\mathrm{Kum}(C) \to \boldsymbol{P}_k^1$ which has the following property: the elliptic fibration π has two sections $\mathfrak{o}$ and $\mathfrak{s}$ such that if $\mathrm{Kum}(C)$ is considered as a one-dimensional abelian variety $\mathcal{E}$ over $K=k(\boldsymbol{P}^1)$ through π and $\mathfrak{o}$, then the K-rational point $\tilde{\mathfrak{s}}$ of $\mathcal{E}$ corresponding to $\mathfrak{s}$ is of infinite order in the group $\mathcal{E}(K)$ of K-rational points of $\mathcal{E}$.*

Moreover, it will be shown that there are many elliptic fibrations of $\mathrm{Kum}(C)$ which have the property in the above theorem. (See Remark in §1 below.)

The translation of $\mathcal{E}$ by $\tilde{\mathfrak{s}}$ induces an automorphism σ of $\mathrm{Kum}(C)$. Hence we have the following corollary.

Corollary. *If char. $k \neq 2$, $\mathrm{Aut}(\mathrm{Kum}(C))$ contains an element σ of infinite order which preserves a structure of an elliptic surface and σ acts trivially on $H^0(S, \Omega_S^2)$. Moreover, $\mathrm{Kum}(C)$ contains infinitely many non-singular rational curves.*

Note that $\mathrm{Aut}(\mathrm{Kum}(C))$ is a discrete group. In case $k=\boldsymbol{C}$, it is easy to show that $\mathrm{Aut}(\mathrm{Kum}(C))$ contains an element of infinite order by using the theory of the $K3$ lattice. Our approach is more primitive and it works for all characteristics except 2. We use the fact that $\mathrm{Kum}(C)$ contains thirty two distinguished non-singular ratinal curves which form the Kummer configuration. (For details, see §1 below.)

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In §2, as an application of the above mentioned results, we shall prove a lemma which was used in Ueno [9] to construct certain non-Kähler threefolds. We also construct new non-Kähler threefolds related to these threefolds.

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§1. Let C be a non-singular complete curve of genus 2 defined over an algebraically closed field k with char.$k\neq 2$. There is a two sheeted ramified covering $\pi: C\to \boldsymbol{P}^1$ with ramification points $\mathfrak{p}_1, \cdots, \mathfrak{p}_6$. Then we can define an embedding of C into its Jacobian variety $J(C)$ by

$$\begin{array}{rccc} \varphi: & C & \to & J(C) \\ & \Psi & & \Psi \\ & \mathfrak{p} & \mapsto & [\mathfrak{p}-\mathfrak{p}_1] \end{array}$$

where $[\delta]$ means the linear equivalence class containing a divisor δ. Since $2(\mathfrak{p}_i-\mathfrak{p}_1)$ is linearly equivalent to zero, the point $[\mathfrak{p}_i-\mathfrak{p}_1]=e_i$ in $J(C)$ is a 2-torsion point. It is easy to show that $\varphi(C)\cap J(C)_2$ consists of these six 2-torsion points, where $J(C)_2$ is the set of all 2-torsion points of $J(C)$. The covering π gives an involution ι of C and $J(C)$ has a natural involution ι: $z\mapsto -z$. The embedding φ is compatible with these involutions, that is $\varphi(\iota(\mathfrak{p}))=-\varphi(\mathfrak{p})$. For each 2-torsion point e of $J(C)$, the morphism

$$\begin{array}{rccc} \varphi_e: & C & \to & J(C) \\ & \Psi & & \Psi \\ & \mathfrak{p} & \mapsto & \varphi(\mathfrak{p})+e \end{array}$$

is also an embedding and $\varphi_e(\iota(\mathfrak{p}))=-\varphi_e(\mathfrak{p})$. Note that $\varphi_e(C)$ contains precisely six 2-torsion points e_i+e, $i=1, 2, \cdots, 6$. Conversely for each 2-torsion point e there are six 2-torsion points x_i, $i=1, \cdots, 6$ with $\varphi_{x_i}(C)\ni e$, where $x_i=e-e_i$.

Let Kum(C) be the minimal non-singular model of the quotient variety $J(C)/\langle\iota\rangle$. The surface Kum(C) contains sixteen non-singular rational curves (E-curves) appearing by the resolution of singularities corresponding to 2-torsion points of $J(C)$. On the orther hand, the strict transform of the image of $\varphi_e(C)$ in $J(C)/\langle\iota\rangle$ to Kum(S) is also a non-singular rational curve and there are sixteen such curves (Q-curves). From the above consideration it follows that each Q-curve intersects six E-curves. Moreover, it is easy to show that they intersect transversally. Similarly, each E-curve intersect six Q-curves transversally. Moreover, we number E-curves and Q-curves in such a way that

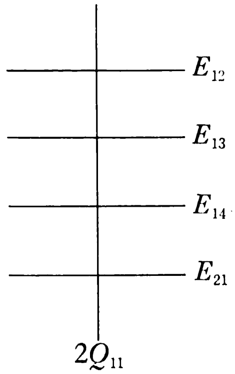
$$Q_{ij}\cdot E_{kl}=\begin{cases} 1, & \text{if } k\neq i,\ l=j \text{ or } k=i,\ l\neq j, \\ 0, & \text{otherwise,} \end{cases}$$

$$1\leq i, j, k, l\leq 4,$$

where $Q\cdot E$ means the intersection number of divisors Q and E[1]. Note that we have $Q_{i,j}^2=E_{i,j}^2=-2$, where D^2 is the self-intersection number of a divisor D. For further interesting properties of these rational curves, we refer the reader to the books [1], [2].

Using these thirty two curves we shall construct an elliptic fibration. The following lemma is an easy consequence of the Riemann-Roch theorem and well-known.

Lemma. *On a K3 surface S if there exists an effective divisor D consisting of non-singular rational curves which has the same type as a singular fibre of an elliptic surface, then there exists a unique elliptic fibration* ψ: $S\to \boldsymbol{P}^1$ *such that D is a singular fibre of* ψ.



Let us consider divisors D, D' on Kum(C) defined by

$$D = 2Q_{11}+E_{12}+E_{13}+E_{14}+E_{21}$$

$$D' = 2Q_{21}+E_{22}+E_{23}+E_{24}+E_{11}\,.$$

Then we have $D\cdot D'=0$ and by the above lmema there exists an elliptic fibration π: $S=\mathrm{Kum}(C)\to\boldsymbol{P}^1$ such that D and D' are singular fibres of π. As we have

$$D\cdot E_{ij} = 0\,, \qquad i=3,\,4, \quad j=2,\,3,\,4\,,$$

these six curves E_{ij} are contained in singular fibres. On the other hand, as we have

$$D\cdot Q_{ij} = 1\,, \qquad i = 3,\,4, \quad j = 2,\,3,\,4\,,$$

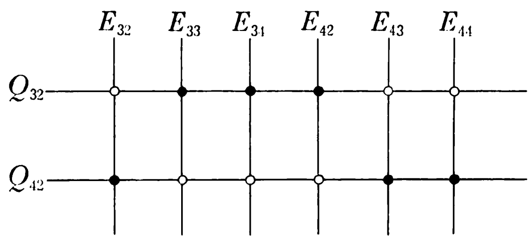
these six curves Q_{ij} are sections of π. We take Q_{32} as the zero section $\mathfrak{o}$ of π. Hence our ellitpic fibration π: $S\to\boldsymbol{P}^1$ with the zero section $\mathfrak{o}$ can be considered as an elliptic curve $\mathcal{E}$ over the function field $K=k(\boldsymbol{P}^1)$. Then other five sections corresponding to the curves Q_{ij} define K-rational points $s_{ij}\in\mathcal{E}(K)$. Put $\mathfrak{z}=s_{42}$ and let us consider the order of $\mathfrak{z}$ in $\mathcal{E}(K)$. For that purpose we need to study singular fibres of π.

From above, the elliptic fibration π contains two singular fibres of type I_0^* and other singular fibres which contain six curves E_{ij}, $i=3$, 4, $j=2$, 3, 4. The

1) For example, we can use the argument in "Two Configurations Revisited" in [1], p784~787 and Figure 21 there. Note that the argument there is purely algebraic and works for all characteristics except 2. Using the notation there, we have, for example, the following numbering.

$$\begin{array}{llll}
Q_{11}=\Theta, & Q_{12}=\Theta_1, & Q_{13}=\Theta_2, & Q_{14}=\Theta_{12},\\
Q_{21}=\Theta_{45}, & Q_{22}=\Theta_{23}, & Q_{23}=\Theta_{13}, & Q_{24}=\Theta_3,\\
Q_{31}=\Theta_{35}, & Q_{32}=\Theta_{24}, & Q_{33}=\Theta_{14}, & Q_{34}=\Theta_4,\\
Q_{41}=\Theta_{34}, & Q_{42}=\Theta_{25}, & Q_{43}=\Theta_{15}, & Q_{44}=\Theta_5,\\
E_{11}=\mu_{12}, & E_{12}=\mu_2, & E_{13}=\mu_1, & E_{14}=\mu,\\
E_{21}=\mu_3, & E_{22}=\mu_{13}, & E_{23}=\mu_{23}, & E_{24}=\mu_{45},\\
E_{31}=\mu_4, & E_{32}=\mu_{14}, & E_{33}=\mu_{24}, & E_{34}=\mu_{35},\\
E_{41}=\mu_5, & E_{42}=\mu_{15}, & E_{43}=\mu_{25}, & E_{44}=\mu_{34}.
\end{array}$$

intersections of these six curves and two curves Q_{32}, Q_{42} are as follows.



Here the symbol ● means that two curves intersect transversally and the symbol ○ means that two curves do not intersect. Since curves Q_{32}, Q_{42} are sections of π, from this configuration, we conlcude the following facts.

1.1) All six curves E_{ij}, $i=3, 4, j=2, 3, 4$, are components of singular fibres with multiplicity one.

1.2) The curves E_{33}, E_{34}, E_{42} (resp. E_{32}, E_{43}, E_{44}) are contained in different singular fibres. Hence there are at least three singular fibres other than D and D'.

1.3) Three of these six curves are not contained in a singular fibre.

From these informations we can detemrine all possible types of singular fibres. Let us recall the list of singular fibres appearing in an elliptic fibration with a section. We use the nation of Kodaira [3].

Type	I_b	I_b^*	II	II*	III	III*	IV	IV*
Euler number	b	$b+6$	2	10	3	9	4	8

One of the important fact of an elliptic surface $\pi: S \to \boldsymbol{P}^1$ with a section is that the Euler number of S is equal to the sum of the Euler numbers of all singular fibres, if char. $k \neq 2, 3$. (By the Euler number $\varepsilon(F_\lambda)$ of a singular fibre F_λ, we mean the number $n+\varepsilon-1$ where n is the number of irreducible component of F_λ and ε is 1 or 2 according as F_λ is of type I_b or other type, respectively. Note that this number is equal to the usual Euler number when $k=\boldsymbol{C}$.) In our case, since the Euler number of a $K3$ surface is 24, if char. $k \neq 3$, then we have

$$24 = \sum_{\lambda \in \Lambda} \varepsilon(F_\lambda)\,,$$

where $\{F_\lambda\}_{\lambda \in \Lambda}$ is the set of all singular fibres of $\pi: S \to \boldsymbol{P}^1$. If char. $k=3$, then we have

$$24 = \sum_{\lambda} (\varepsilon(F_\lambda)+\delta(F_\lambda))\,,$$

where $\delta(F_\lambda)$ is a non-negative integer which measures the degree of wild ramification. (See Ogg [5]. The number $\delta(F_\lambda)$ may be positive, if F is a singular fibre of one of the types II, IV, II*, IV*.)

Now assume that two of the six curves E_{ij} are contained in the same fibre.

Then this singular fibre F is of type I_b, $b\geq 4$ or type I_b^*, $b\geqq 0$ or type III* or type IV*. Hence we hvae $\varepsilon(F)\geq 4$. Moreover the equality holds if and only if F is of type I_4. From the above 1.1), 1.2), 1.3), we have the following possibilities of singular fibres of π containing our six curves E_{ij}.

①

S	F_1	F_2	F_3
N	2	2	2
ε	≥4	≥4	≥4

②

S	F_1	F_2	F_3	F_4
N	2	2	1	1
ε	≥4	≥4	≥2	≥2

③

S	F_1	F_2	F_3	F_4	F_5
N	2	1	1	1	1
ε	≥4	≥2	≥2	≥2	≥2

④

S	F_1	F_2	F_3	F_4	F_5	F_6
N	1	1	1	1	1	1
ε	≥2	≥2	≥2	≥2	≥2	≥2

S: singular fibre, N: the number of curves E_{ij} contained in the singular fibre.
ε : Euler number of the singular fibre.

Since we have

$$\sum \varepsilon(F_\lambda)\geq 12\,,$$

in the above table for the Euler numbers, only equalities hold and there are no singular fibres other than two singular fibres of type I_0^*. Thus we obtain the following Table.

	types of singular fibres	ρ'
case 1)	I_0^*, I_0^*, I_4, I_4, I_4	17
2)	I_0^*, I_0^*, I_4, I_4, I_2, I_2	16
3)	I_0^*, I_0^*, I_4, I_2, I_2, I_2, I_2	15
4)	I_0^*, I_0^*, I_2, I_2, I_2, I_2, I_2, I_2	14

Here $\rho'=\sum_\lambda (n_\lambda-1)$ and n_λ is the number of irreducible components of a singular fibre F_λ.

From this table, we conclude that the moduli of elliptic curves appearing in smooth fibres of our elliptic fibration π are not constant. We let ν: $N\to\boldsymbol{P}^1$ be the Néron minimal model of the elliptic fibration π. This is a group scheme over $\boldsymbol{P}^1$. Put $x=\pi(D)$. The fibre $N_x=\nu^{-1}(x)$ of ν over x is a group isomorphic to $\boldsymbol{G}_a\times\boldsymbol{Z}/(4)$. For a point $y\in\boldsymbol{P}^1$ where $\pi^{-1}(y)$ is a singular fibre of type I_b, the fibre $N_y=\nu^{-1}(y)$ is a group isomorphic to $\boldsymbol{G}_m\times\boldsymbol{Z}/(b)$. By the above table, we can always find such a point y with $b=2$ or 4. There is natural group homomorphisms ψ_x: $\mathcal{E}(K)\to N_x$ and ψ_y: $\mathcal{E}(K)\to N_y$. Now let us consider the orders of $\psi_x(\tilde{z})$ and $\psi_y(\tilde{z})$ where $\tilde{z}$ is a K-rational point of $\mathcal{E}$ corresponding to the curve Q_{42}. Note that Q_{32} and Q_{42} intersect E_{12} in the singular fibre D. Hence $\psi_x(\tilde{z})$ belongs to the identity component N_x^0 of N_x. As N_x^0 is isomorphic to $\boldsymbol{G}_a$ this implies that if char. $k=0$, then $\psi_x(\tilde{z})$ is of infinite order. On the other hand, if char. $k=p\geq 3$ assume that $\tilde{z}$ is of finite order, say $n=p^a m$, $(p, m)=1$. As $\psi_x(\tilde{z})$ is contained in the identity component N_x^0, $\psi_x(p^a\tilde{z})$ is also contained in N_x^0. By a theorem of Tate [7], ψ_x is injective on the torsion elements $\mathcal{E}(K)_{\text{tor}}$ whose orders are prime to p. Hence $\psi_x(p^a\tilde{z})$ is of order m. This implies that $m=1$. Since the intersection points of Q_{43}, Q_{22} and the singular fibre $\pi^{-1}(y)$ are different and N_y is isomorphic to $\boldsymbol{G}_m\times \boldsymbol{Z}/(b)$ with $b=2$ or 4 the order of $\psi_y(\tilde{z})$ is prime to p. This is a contradiction. Therefore, $\tilde{z}$ must be of infinite order. Thus we have proved the theorem in the introduction.

Let σ be a birational mapping of $S=\text{Kum}(C)$ induced from a translation of $\mathcal{E}$ by $\tilde{z}$ over $K=k(\boldsymbol{P}^1)$. Since S contains no exceptional curves of the first kind, σ is an automorphism of S. As $\tilde{z}$ is of infinite order in $\mathcal{E}(K)$, σ is also of infinite order in $\text{Aut}(S)$. Hence $\sigma^k(Q_{32})$, $k=1, 2, 3, \cdots$ are disinct non-singular rational curves. Moreover, since σ comes from a translation of E, it operates trivially on $H^0(S, \Omega_S^2)$. This proves the corollary of the introduction.

From the above proof we obtain the following

Corollary. *Let $\pi\colon S\rightarrow \boldsymbol{P}^1$ be the same as above. Assume that the Picard number $\rho(J(C))=1$, then we have the case* 4) *of the above table.*

Note that, if the curve C is generic, then $\rho(J(C))=1$.

Proof. By a theorem of Shioda [6], we have

$$\rho(S) = \text{rank}\,\mathcal{E}(K)+2+\rho',$$

where $\rho'=\sum_{\lambda\in\Lambda}(n_\lambda-1)$ and n_λ is the number of the irreducible components of a singular fibre F_λ. On the other hand, we have

$$\rho(S) = 16+\rho(J(C)) = 17\,,$$

and as we proved above,

$$\text{rank}\,\mathcal{E}(K)\geq 1\,.$$

Hence, only the caes 4) is possible. Q.E.D.

Remark 1. There are many possibilities to find the above divisor of type I_0^*. If we start from the curve Q_{11} as a component of multiplicity 2, four other components of multiplicity one are taken from four of six curves E_{1i}, E_{i1}, $i=2, 3, 4$. Hence there are 15 different D's. Hence there are 15×16 different choice of D, but we saw above that D and D' give the same elliptic fibration. Thus we have 120 elliptic fibrations. On the other hand, since the translation T_a of $J(C)$ by a two-torsion point of $J(C)$ is commutative with the canonical involution ι of $J(C)$, T_a induces an automorphism t_a of $\text{Kum}(C)$. Hence $\text{Aut}(\text{Kum}(C))$ contains a subgaroup G of order 16 isomorphic to the group of two-torsion points of $J(C)$. For any element t_a of G, $t_a(D)$ is again a divisor of the same type. Hence t_a induces an isomorphism between elliptic fibrations obtained from the divisors D

and $t_a(D)$. Moreover we can always find an element t_b of G with $t_b(D)=D'$. Hence we have 15 different elliptic fibrations which are not isomorphic under the action of G. The author does not know whether there exist isomorphic elliptic fibrations among these 15 elliptic fibrations.

2. Interchanging the role of Q-curves and E-curves in the above construction, we obtain an elliptic fibration assoicated with a divisor $\tilde{D}=2E_{11}+Q_{12}+Q_{13}+Q_{14}+Q_{21}$. This fibration is isomorphic to the elliptic fibration associated with D. This can be seen as follows. The singular surface $J(C)/\langle\iota\rangle$ is realized as a quartic surface S' with 16 ordinary double points in $\boldsymbol{P}^3$. The dual mapping ψ

$$S'\ni x\mapsto \text{the tangent plane of } S' \text{ at } x\in\check{\boldsymbol{P}}^3$$

induces an automorphism $\tilde{\psi}$ of $\mathrm{Kum}(C)$ which maps E-curves (resp. Q-curves) to Q-curves (resp. E-curves).

3. The Kummer surface $S=\mathrm{Kum}(C)$ has many other elliptic fibrations. For example we have the following divisors on S.

(1) $2Q_{11}+2E_{41}+E_{21}+E_{31}+Q_{42}+Q_{43}$, type I_1^*.

(2) $Q_{11}+E_{12}+Q_{22}+E_{21}$, type I_4.

(3) $Q_{11}+E_{13}+Q_{12}+E_{22}+Q_{32}+E_{34}+Q_{44}+E_{41}$, type I_8.

(4) $6Q_{11}+5E_{12}+4Q_{22}+3E_{23}+2Q_{24}+E_{34}+3E_{13}+4E_{31}+2Q_{41}$, type II^*.

If $\mathrm{char}.\,k=0$, from the divisor (1) we can construct an elliptic fibration $\pi\colon S\to\boldsymbol{P}^1$ which satisfies the above theorem.

§2. For the notation in this section we refer the reader to Ueno [9]. Let C be a non-singular curve of genus 2 defined over $\boldsymbol{C}$. Put $S=\mathrm{Kum}(C)$. Then by the above Remark 1, there exist two different elliptic fibrations $\pi_i\colon S\to\boldsymbol{P}^1$, $i=1, 2$ which satisfy the above Theorem. Let σ_i be the analytic automorphism of S of infinite order constructed by the elliptic fibration π_i as above.

Lemma 2.1. *Let Δ be a subgroup of* $\mathrm{Aut}(S)$ *generated by σ_1 and σ_2. If the group Δ fixes the first Chern class $c_1(L)$ of a line bundle L on S, then we have $\kappa(S, L)\leq 0$.*

Proof. Assume $\kappa(S, L)>0$. If $\kappa(S, L)=2$, then it is well known that any subgroup G of $\mathrm{Aut}(S)$ which preserves $c_1(L)$ is finite. Therefore, by our assumption we have $\kappa(S, L)=1$. We let m be a sufficiently large positive integer such that the image of the rational mapping $\Phi_{|mL|}$ associated with the complete linear system $|mL|$ is one-dimensional. Let $|M|$ (resp. Z) be the varialbe (resp. fixed) part of $|mL|$. If $M^2>0$, then by the Riemann-Roch theorem we have $\kappa(S, M)=2$. On the other hand the group Δ fixes $c_1([M])$ and $c_1([Z])$, respectively, where $[M]$ and $[Z]$ are the corresponding line bundles to M and Z, respectively. Therefore we have $M^2=0$. Hence we may assume that $L^2=0$, $|L|$ is free from base points and fixed componets, $\Phi_{|L|}(S)=C$ is a non-singular curve and $\Phi=\Phi_{|L|}\colon S\to C$ has connected fibres. Then $\Phi\colon S\to C$ is an elliptic surface. Moreover, we may assume $L=[F]$ where F is a general fibre of Φ and the elliptic fibration Φ is different from the elliptic fibration π_1. Hence we have $F\cdot F_1>0$ where F_1 is a general fibre of π_1. Since $c_1([F])$ is fixed by the group Δ and $c_1([F_1])$ is fixed by the group $\langle\sigma_1\rangle$, $c_1([F+F_1])$ is fixed by the group $\langle\sigma_1\rangle$. On the other hand, as we have $(F+F_1)^2=2F\cdot F_1>0$, we have $\kappa(S, F+F_1)=2$. This contradicts the fact that σ_1 is of infinite order. q.e.d.

Let G be a finitely generated subgroup of $\mathrm{Aut}(S)$ containing the group Δ such that G operates trivially on $H^0(S, \Omega_S^2)$. Let $\{g_1, g_2, \cdots, g_n\}$ be a system of generators of G. Take a non-singular curve A of genus n with a system of generators $\{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\}$ of the fundamental group of A with the relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}=e$. The group homomorphism ρ: $\pi_1(A)\to G$ given by

$$\alpha_i \mapsto g_i, \ \beta_i \mapsto g_i^{-1}, \qquad i = 1, 2, \cdots, n,$$

defines an analytic fibre bundle f: $X\to A$ whose fibre is S and whose structure group is G. In Ueno [9], by using the above Lemma, it was shown that the algebraic dimension $a(X)$ of X is one and X is not a holomorphic image of a compact Kähler manifold. From our construction we have $K_X=f^*K_A$, since G operates trivially on $H^0(S, \Omega_S^2)$. The threefold X has the following numerical invariants.

$$p_g(X) = n\,, \quad \kappa(X) = 1\,, \quad h^{0,1}(X) = h^{1,0}(X) = n\,.$$

Moreover, the Albanese variety of X is isomorphic to the Jacobian variety $J(A)$ of A.

Since f: $X\to A$ is locally trivial and S has a structure of an elliptic surface, using the technique given in Ueno [10], we can construct an analytic threefold Y with a surjective holomorphic mapping h: $Y\to A$ which has the following properties.

1) There exists a Zariski open subset A' of A such that $h'=h|_{Y'}$: $Y'=h^{-1}(A')\to A'$ is isomorphic to $f'=f|_{X'}$: $X'=f^{-1}(A')\to A'$.

2) h: $Y\to A$ is *not* bimeromorphic to an analytic fibre bundle over A whose fibre is S.

3) Each singular fibre of h is reduced and a normal crossing divisor $\sum_{i=0}^{a_\lambda} F_{\lambda,i}$ where $F_{\lambda,0}$ is a non-singular $K3$ surface and other $F_{\lambda,i}$ are elliptic Hopf surfaces.

4) $K_Y=h^*K_A+\sum_\lambda \sum_{i=1}^{a_\lambda} F_{\lambda,i}$.

The analytic threefold Y is not a holomorphic image of a compact Kähler manifold and has the following numerical invariants.

$$a(Y) = 1, \quad p_g(Y) = n, \quad \kappa(Y) = 1, \quad h^{0,1}(Y) = h^{1,0}(Y) = n\,.$$

Also $\mathrm{Alb}(Y)$ is isomorphic to $J(A)$.

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