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On the integral cohomology of *BSpin*(*n*)

By

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§1. Introduction.

In [2] Borel and Hirzebruch proved that "*the torsion elements of* $H^*(BO(n); \boldsymbol{Z})$ *and* $H^*(BSO(n); \boldsymbol{Z})$ *are of order* 2" (cf. 30.5 of [2]). The purpose of this paper is to show the following:

Theorem. *The torsion elements of* $H^*(BSpin(n); \boldsymbol{Z})$ *are of order* 2.

To prove Theorem we need the structure of $H^*(BSpin(n); \boldsymbol{Z}/2)$ which was determined by Quillen [3]. Therefore we review it in section 2. In section 3, we compute the Sq^1-cohomology of $H^*(BSpin(n); \boldsymbol{Z}/2)$ and prove Theorem by making use of the method of 30.5 of [2].

Throughout the paper $H^*(\ )$ denotes the mod 2 cohomology $H^*(\ ; \boldsymbol{Z}/2)$.

§2. The structure of $H^*(BSpin(n))$.

In this section we review the result of Quillen [3]. As is well known, as a graded algebra

$$H^*(BSO(n)) = \boldsymbol{Z}/2[w_2, \cdots, w_n]\,,$$

where $w_j \in H^j(BSO(n))$ is the j-th universal Stiefel Whitney class. Put $r_1 = w_2$ and

$$r_{j+1} = Sq^{2^{j-1}} Sq^{2^{j-2}} \cdots Sq^1 w_2 \qquad (j \geq 1)\,.$$

Define a function $h: \boldsymbol{Z} \to \boldsymbol{Z}$ by

$$h(8k+1) = 4k\,, \quad h(8k+2) = 4k+1\,, \quad h(8k+3) = h(8k+4) = 4k+2\,,$$

$$h(8k+5) = h(8k+6) = h(8k+7) = h(8k+8) = 4k+3\,.$$

For a positive integer n, the number $2^{h(n)}$ is called the Radon-Hurewitz number and is equal to the dimension of the spin representation of $Spin(n)$.

The following is Theorem 6.3 and Theorem 6.5 of [3]:

Theorem 2.1. (1) *For a positive integer n, the sequence*

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$$r_1, r_2, \cdots, r_{h(n)}$$

is a regular sequence in $H^*(BSO(n))$.

(2) *If* π: $BSpin(n) \to BSO(n)$ *is the natural projection, then* $\operatorname{Ker} \pi^*$ *is generated by* $r_1, \cdots, r_{h(n)}$ *and as a graded algebra*

$$H^*(BSpin(n)) = H^*(BSO(n))/(r_1, r_2, \cdots, r_{h(n)}) \otimes \boldsymbol{Z}/2[e],$$

where $e \in H^{2^{h(n)}}(BSpin(n))$ *is the* $2^{h(n)}$*-th Stiefel Whitney class of the spin representation* Δ: $BSpin(n) \to BO(2^{h(n)})$.

§3. The Sq^1-cohomology of $H^*(BSpin(n))$.

The purpose of this section is to prove Theorem. Define $Q_n(t) = \sum_{k=0}^{\infty} q_{n,k} t^k \in \boldsymbol{Z}[[t]]$ by

$$Q_{2n}(t) = \left(\prod_{j=1}^{n-1} (1-t^{4j})\right)^{-1} \cdot (1-t^{2n})^{-1}$$

and

$$Q_{2n+1}(t) = \left(\prod_{j=1}^{n} (1-t^{4j})\right)^{-1}.$$

As is well known the coefficient of t^k in $Q_n(t)$ is equal to the k-th Betti number of $BSpin(n)$. For a graded vector space $V = \sum_{k=0}^{\infty} V_k$ of finite type over a field $\boldsymbol{Z}/2$, The Poincaré series $P.S.(V)$ is defined by

$$P.S.(V) = \sum_{k=0}^{\infty} (\dim V_k) t^k \in \boldsymbol{Z}[[t]].$$

Put $R_0 = H^*(BSO(n))$ and $R_k = H^*(BSO(n))/(t_1, \cdots, r_k)$ and denotes the natural projection $H^*(BSO(n)) \to R_k$ by p_k. Since the sequence $r_1, \cdots, r_{h(n)}$ is a regular sequence, there are exact sequences of graded vector spaces over $\boldsymbol{Z}/2$

$$(*)_k \qquad 0 \to \Sigma^{2^k+1} R_k \xrightarrow{\cdot r_{k+1}} R_k \to R_{k+1} \to 0$$

for $k < h(n)$, where $(\Sigma^{2^k+1} R_k)_j = (R_k)_{j-2^k-1}$.

Lemma 3.1. *If* $k \leq h(n)$, *then*

$$P.S.(R_k) = \left(\prod_{j=0}^{k-1} (1-t^{2^j+1})\right) \cdot \left(\prod_{j=2}^{n} (1-t^j)\right)^{-1}.$$

Proof. By the exact sequence $(*)_k$, we have

$$P.S.(R_{k+1}) + t^{2^k+1} P.S.(R_k) = P.S.(R_k).$$

Since $P.S.(R_0) = \left(\prod_{j=2}^{n} (1-t^j)\right)^{-1}$, we have the result. Q.E.D.

By Theorem 2.1, $\operatorname{Im} \pi^* = R_{h(n)}$ and as a graded algebra

$$H^*(BSpin(n)) = R_{h(n)} \otimes \boldsymbol{Z}/2[e],$$

where $e \in H^{2^{h(n)}}(BSpin(n))$.

Lemma 3.2. *In $H^*(BSO(n))$, $Sq^1 r_1 = r_2$, $Sq^1 r_2 = 0$ and $Sq^1 r_{j+1} = r_j^2$ for $j \geq 2$.*

Proof. $Sq^1 r_1 = r_2$ is the definition of r_2 and using the Adem relation $Sq^1 Sq^1 = 0$, we have $Sq^1 r_2 = Sq^1 Sq^1 r_1 = 0$. If $j \geq 2$, then by the Adem relation, $Sq^1 Sq^{2^{j-1}} = Sq^{2^{j-1}+1}$. Since $r_j \in H^{2^{j-1}+1}$, we have

$$Sq^1 r_{j+1} = Sq^1 Sq^{2^{j-1}} r_j = Sq^{2^{j-1}+1} r_j = r_j^2. \qquad \text{Q.E.D.}$$

In particular, the ideal $(r_1, \cdots, r_k)$ is closed under the action of Sq^1 for $k \geq 2$ and therefore we have

Lemma 3.3. *If $k \neq 1$, Sq^1 induces a derivation d_k: $R_k \to R_k$ satisfying $d_k \circ p_k = p_k \circ Sq^1$.*

Moreover we have

Lemma 3.4. *If $2 \leq k < h(n)$, then the exact sequence $(*)_k$ is an exact sequence of cochain complexes.*

The following was proved in [2] (see 30.5 of [2]):

Lemma 3.5. *$P.S.(H^*(H^*(BSO(n)), Sq^1)) = Q_n(t)$.*

Note that by Wu formula, $Sq^1 w_{2j} = w_{2j+1}$ and $Sq^1 w_{2j+1} = 0$ for $j \geq 1$ (cf. [1]). Therefore as a cochain complex,

$$(H^*(BSO(n)), Sq^2) = (\boldsymbol{Z}/2[w_2, w_3], d) \otimes (R_2, d_2)$$

where $d(w_2) = w_3$ and $d(w_3) = 0$. By an easy computation we have

$$P.S.(H^*(\boldsymbol{Z}/2[w_2, w_3], d)) = (1 - t^4)^{-1}$$

and

$$P.S.(H^*(R_2, d_2)) = (1 - t^4) \cdot Q_n(t).$$

The exact sequence $(*)_k$ induces a long exact sequence

$$\cdots \to H(\Sigma^{2^k+1} R_k) \to H^i(R_k) \to H^i(R_{k+1}) \to H^{i+1}(\Sigma^{2^k+1} R_k) \to$$

which is equivalent to

$$\cdots \to H^{i-2^k-1}(R_k) \to H^i(R_k) \to H^i(R_{k+1}) \to H^{i-2^k}(R_k) \cdots.$$

By the induction on k, we have $H^{2i+1}(R_k) = 0$ and

$$H^{2i}(R_{k+1}) = H^{2i}(R_k) \oplus H^{2i-2^k}(R_k).$$

Thus we have

Lemma 3.6. (1) *If* $2\leq k<h(n)$,

$$P.S.(H^*(R_{k+1}, d_{k+1})) = (1+t^{2^k})\cdot P.S.(H^*(R_k, d_k)) .$$

(2) $P.S.(H^*(R_{h(n)}, d_{h(n)}))=(1-t^{2^{h(n)}})\cdot Q_n(t)$.

Next we prove the following:

Lemma 3.7. $Sq^1e=0$.

Proof. Since $e\in H^{2^{h(n)}}(BSpin(n))$ is the $2^{h(n)}$-th Stiefel Whitney class of the spin representation Δ: $BSpin(n)\to BO(2^{h(n)})$, we have

$$Sq^1e = Sq^1w_{2^{h(n)}}(\Delta) = w_{2^{h(n)}}(\Delta)w_1(\Delta)$$

by the Wu formula, where $w_j(\Delta)$ is the j-th Stiefel Whitney class of Δ. But $w_1(\Delta)=0$, since $H^1(BSpin(n))=0$. Therefore the lemma is proved. Q.E.D.

Proof of Theorem. Since $R_{h(n)}$ is $\mathrm{Im}\,\pi^*$, $(R_{h(n)}, d_{h(n)})$ is a subcomplex of $(H^*(BSpin(n)), Sq^1)$. On the other hand $Sq^1e=0$ by Lemma 3.7. Therefore as a cochain complex

$$(H^*(BSpin(n), Sq^1) = (R_{h(n)}, d_{h(n)})\otimes(\boldsymbol{Z}/2[e], 0) .$$

Thus we have

$$P.S.(H^*(H^*(BSpin(n)), Sq^1)) = P.S.(H^*(R_{h(n)}, d_{h(n)})\cdot(1-t^{2^{h(n)}})^{-1} = Q_n(t) .$$

Now Theorem is proved by making use of the method of 30.4 of [2]. Q.E.D.

Put $P_n(t)=P.S.(H^*(BSpin(n))=P.S.(R_{h(n)})\cdot(1-t^{2^{h(n)}})^{-1}$. Then there exists $R_n(t)=\sum_{k=0}^{\infty} r_{n,k}t^k\in\boldsymbol{Z}[[t]]$ such that $r_{n,k}\geq 0$ and

$$P_n(t) = (1+1/t)\cdot R_n(t)+Q_n(t)$$

(cf. 30.4 and 30.5 of [2]). As a corollary of Theorem we have

Corollary 3.8. (1) *As an abelian group*

$$H^k(BSpin(n); \boldsymbol{Z})=(\boldsymbol{Z})^{q_{n,k}}\oplus(\boldsymbol{Z}/2)^{r_{n,k}} .$$

(2) *The kernel of* Sq^1 *on* $H^*(BSpin(n))$ *is the reduction* mod 2 *of* $H^*(BSpin(n); \boldsymbol{Z})$ *and its image is the reduction* mod 2 *of the torsion elements of* $H^*(BSpin(n); \boldsymbol{Z})$.
(3) *An element of* $H^*(BSpin(n); \boldsymbol{Z})$ *is completely determined by its canonical images in* $H^*(BSpin(n); \boldsymbol{R})$ *and* $H^*(BSpin(n); \boldsymbol{Z}/2)$.

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