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On a problem of hypoellipticity

Dedicated to Professor SIGERU MIZOHATA on his sixtieth birthday

By

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§ 1. Introduction.

In the work [11], L. Schwartz has introduced the notion of *hypoellipticity*, and proposed the following question (see p. 146, *Remarques* 2°).

Let $a(x, D)$ *be a differential operator, and suppose that it has the following property: there is a positive integer* l *such that every* C^l *solution* $u(x)$ *of*

$$a(x, D)u(x)=0$$

belongs to C^∞. *Then, can we claim that* $a(x, D)$ *is hypoelliptic?*

We reformulate his question in the following manner.

Let $P(x, D)$ be a differential operator of order $m \geqq 1$ in an open set Ω in $\boldsymbol{R}^n$, with infinitely differentiable coefficients.

Problem I. *Assume that* $P(x, D)$ *has the following property: given any open subset* ω *of* Ω, *there is an integer* $l \geqq m$ *such that every* C^l *solution* $u(x)$ *of*

$$P(x, D)u(x)=0 \tag{1}$$

in ω *belongs to* $C^\infty(\omega)$. *Then, is* $P(x, D)$ *hypoelliptic in* Ω?

Problem II. *Let* P_0 *be a point of* Ω. *Assume that* $P(x, D)$ *has the following property: there exist an integer* $l \geqq m$ *and a neighborhood* $\mathcal{U}$ *of* P_0 *such that every* C^l *solution* $u(x)$ *of* (1) *in* $\mathcal{U}$ *belongs to* $C^\infty(\mathcal{U})$. *Then, is* $P(x, D)$ *hypoelliptic at* P_0?

Here $P(x, D)$ is said to be *hypoelliptic at* P_0 if there is a neighborhood $\mathcal{V}$ of P_0 such that, given any distribution u in $\mathcal{V}$, u is a C^∞ function in every neighborhood of P_0 where this is true of $P(x, D)u$ (we recall that $P(x, D)$ is said to be hypoelliptic in Ω if, given any distribution u in Ω, u is a C^∞ function in every open set where this is true of $P(x, D)u$).

We know that both of Problems I and II are positive when the coefficients of $P(x, D)$ are constants. But we observe that these are negative in general in case of variable ones. In fact, let $P(x, D)=x^\alpha P_0(x, D)$, where α is an arbitrary complex constant such that $|\alpha| \geqq m+1$ and $P_0(x, D)$ is an arbitrary elliptic

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operator. Then, $u(x)=\delta$ is a solution of (1). On the other hand, we see that $u(x)\in C^{\infty}$ if $u(x)$ of $(1)\in C^{m}$.

The main result of this article is that Problem I is positive for the operator $P(x, D)$ of principal type with analytic coefficients. That is to say, in § 2, we prove it in the following form.

Theorem A. *Every differential operator $P(x, D)$ of principal type with analytic coefficients in Ω, of order m, which is not hypoellptic in Ω, has the following property:*

There is an open subset ω_0 of Ω such that, given any integer $l\geqq m$, equation (1) has a solution $u(x)$ in ω_0 such that $u(x)\in C^{l}(\omega_0)\backslash C^{l+1}(\omega_0)$.

L. Nirenberg [7] showed that *every C^1 solution u of*

$$\mathcal{L}u\equiv\frac{\partial u}{\partial x}+ix(1+x\phi(x, y))\frac{\partial u}{\partial y}=0$$

in a neighborhood of the origin is constant, where $\phi(x,. y)$ is a suitably chosen infinitely differentiable real-valued function which vanishes of infinite order on the y axis. We know $\mathcal{L}$ is not hypoelliptic at the origin. Thus we see that Problem II is negative generally in the class of operators of principal type with infinitely differentiable coefficients. Nevertheless, in § 4, we present an another counter example to Problem II, because it seems to be simpler than his one. In our example the coefficients of the principal part are analytic. When the coefficients of $P(x, D)$ are analytic, a partial positive result to Problem II is obtained by Baouendi-Treves-Zachmanoglou [1].

Finally I would like to express my sincere gratitude to Professor S. Mizohata for many valuable suggestions; in particular, the formulation of Theorem A was born through the discussions of our problem with him. And my thanks go to Doctor T. Ôkaji for many advices.

§ 2. Proof of Theorem A.

The proof can be done along the lines of L. Nirenberg-F. Treves ([8], [9]) and F. Treves [13], and ultimately relies on S. Mizohata [4]. We give a little detailed process of the proof.

Let us denote by $p_m(x, \xi)$ the principal symbol of $P(x, \xi)$. Set $T=\{(x, \xi)\in \Omega\times R^n\diagdown\{0\}\,;\, p_m(x, \xi)=0\}$. Then, by virtue of F. Treves [13], we may suppose that there exists $(x_0, \xi^0)\in T$ such that $\mathrm{grad}_{\xi}\mathrm{Re}\, p_m(x_0, \xi^0)\neq 0$, and denoting by Γ_0 the null bicharacteristic strip of $\mathrm{Re}\, p_m(x, \xi)$ passing through (x_0, ξ^0), either of the following conditions (i) and (ii) holds:

(i) $\mathrm{Im}\, p_m(x, \xi)$ *has a zero of finite odd order at* (x_0, ξ^0) *along* Γ_0.

(ii) $\mathrm{Im}\, p_m(x, \xi)$ *vanishes identically on* Γ_0.

Set $A=A(x, \xi)=\mathrm{Re}\, p_m(x, \xi)$ and $B=B(x, \xi)=\mathrm{Im}\, p_m(x, \xi)$. First, assume that (i) holds. We divide our argument in the following two cases.

Case 1-1. *B has a zero of order 1 at (x_0, ξ^0) along Γ_0.*

Case 1-2. *B has a zero of odd order $k>1$ at (x_0, ξ^0) along Γ_0.*

Case 1-1. Setting $C_{2m-1}(x, \xi)=\mathrm{Im}\sum_{j=1}^{n}\frac{\partial p_m}{\partial \xi_j}\frac{\partial \bar{p}_m}{\partial x_j}$, we see that $C_{2m-1}(x_0, \xi^0)=-2H_A B(x_0, \xi^0)\neq 0$, where H_A stands for the differentiation along Γ_0. Then, by virtue of L. Hörmander [2] (Lemma 6.1.3), there is an analytic *phase function* $\varphi(x)$ of $p_m(x, \varphi_x)=0$ in a neighborhood of x_0 such that $\varphi(x_0)=0$ $\varphi_x\neq 0$, and $\mathrm{Im}\,\varphi(x)\geqq 0$.

Now we can formulate *Théorème* 4.1 of S. Mizohata [4] in the form of *Lemma A* below, and we apply it to the actual case. Thus, *there is a neighborhood $\mathcal{U}$ of x_0 such that, given any integer $l\geqq m$, equation* (1) *has a solution $u(x)$ in $\mathcal{U}$ such that $u(x)\in C^l(\mathcal{U})\setminus C^{l+1}(\mathcal{U})$.*

Lemma A. *Assume that $P(x, D)$ is of principal type with analytic coefficients. Assume that there is an analytic phase function $\varphi(x)$ of $p_m(x, \varphi_x)=0$ in a neighborhood of a such that $\varphi(a)=0$, $\varphi_x(x)\neq 0$, and* $\mathrm{Im}\,\varphi(x)\geqq 0$. *Then, there is a neighborhood $\mathcal{U}$ of a such that, given any integer $l\geqq m$, equation* (1) *has a solution $u(x)\in C^l(\mathcal{U})\setminus C^{l+1}(\mathcal{U})$.*

From now on we call this conclusion that *Mizohata phenomenon arises at a.*

Case 1-2. If there is a point $(x', \xi')\in T$ in the vicinity of (x_0, ξ^0) such that B has the non-vanishing first derivative at (x', ξ') along the null bicharacteristic strip of A passing through (x', ξ'), then, from the result of Case 1-1, *Mizohata phenomenon arises at x'.* Therefore, we can assume that:

There exists a neighborhood $\mathcal{U}$ of (x_0, ξ^0) such that, for every $(x', \xi')\in\mathcal{U}$ where $A=B=0$, B has the vanishing first derivative at (x', ξ') along the null bicharacteristic strip of A passing through (x', ξ').

Then, we can employ the argument of L. Nirenberg-F. Treves [8] (pp. 8-25), and we see that there is an analytic *phase function* $\varphi(x)$ of $p_m(x, \varphi_x)=0$ in a neighborhood of x_0 such that $\varphi(x_0)=0$, $\varphi_x\neq 0$, and $\mathrm{Im}\,\varphi(x)\geqq 0$ (cf. [8] Lemma 4.2). Therefore, by virtue of Lemma A, *Mizohata phenomenon arises at x_0.*

Next, assume that (ii) holds. If there is a point (x', ξ') in the vicinity of (x_0, ξ^0) such that B changes sign at (x', ξ') along the null bicharacteristic strip of A passing through (x', ξ'), then, it is clear that *Mizohata phenomenon arises at x'.* Hence, we can assume that:

There exists a neighborhood $\mathcal{U}_0$ of (x_0, ξ^0) such that B never change sign along any null bicharacteristic strip of A contained in $\mathcal{U}_0$.

We can assume that, in a conic neighborhood ω of (x_0, ξ^0), $p_m(x, \xi)=(\xi_n-a(x, \xi')-(\sqrt{-1}\,b(x, \xi'))g(x, \xi)$, $\xi'=(\xi_1, \cdots, \xi_{n-1})$, where $a(x, \xi')$ and $b(x, \xi')$ are real-valued, positive homogeneous in ξ' of degree 1, and analytic in a neighborhood of (x_0, ξ_0') and $g(x, \xi)$ is positive homogeneous in ξ of degree $m-1$, an analytic function nowhere vanishing in ω. ξ_0' denotes $(\xi_1^0, \cdots, \xi_{n-1}^0)$. Notice that $\xi_0'\neq 0$. Set $t=x_n$, $\tau=\xi_n$, $\tau_0=\xi_n^0$, $x'=(x_1, \cdots, x_{n-1})$, $a(x, \xi')=a(t, x', \xi')$, and $b(x, \xi')=b(t, x', \xi')$. We may suppose $x_0=0$. Let $\mathcal{O}$ be a neighborhood $\subset\mathcal{U}_0$ of (x_0, ξ^0) such that the above expression holds. Let $\mathcal{O}_0$ be the projection of $\mathcal{O}$ in the t, x', ξ'-space. Consider the integral curves defined by

$$(2.1)\qquad \begin{cases} \dfrac{\partial x'}{\partial t}=-\mathrm{grad}_{\xi'}a(t, x', \xi') \\ \dfrac{\partial \xi'}{\partial t}=\mathrm{grad}_{x'}a(t, x', \xi') \end{cases}$$

Denoting by Γ_0 the integral curve of (2.1) passing through $(0, 0, \xi_0')$, we can assume that:

(*a*) *$b(t, x', \xi')$ vanishes identically on Γ_0.*

(*b*) *$b(t, x', \xi')$ never change sign along any integral curve of* (2.1) *contained in $\mathcal{O}_0$.*

We devide our argument in the following two cases:

Case 2-1. *In the vicinity of $(0, 0, \xi_0')$ there is a point (t_1, x_1', ξ_1') such that $b(t_1, x_1', \xi_1')=0$, but $\mathrm{grad}_{\xi'}b(t_1, x_1', \xi_1')\neq 0$.*

Case 2-2. *If $b(t, x', \xi')=0$, then, $\mathrm{grag}_{\xi'}b(t, x', \xi')=0$.*

Case 2-1. From the argument of F. Treves [13] (p. 643), we see that there is a real-valued analytic solution $w(t, x')$ of

$$\frac{\partial w}{\partial t}-a\Big(t, x', \frac{\partial w}{\partial x'}\Big)-\sqrt{-1}\,b\Big(t, x', \frac{\partial w}{\partial x'}\Big)=0$$

in a neighborhood of (t_1', x_1') such that $w(t_1', x_1')=0$ and $\dfrac{\partial}{\partial x'}w(t_1', x_1')=\xi_1'$ $(\neq 0)$. This directly implies that there exists a real-valued analytic *phase function* $\varphi(x)$ of $p_m(x, \varphi_x)=0$ in a neighborhood of some point $\tilde{x}_0$ such that $\varphi(\tilde{x}_0)=0$ and $\varphi_x(x)\neq 0$. Hence, Lemma A is applicable, and *Mizohata phenomenon arises at* $\tilde{x}_0$.

Case 2-2. Following L. Nirenberg-F. Treves [8] (pp. 21-22), we straighten out the bicharacteristic strips of $\tau-a(t, x', \xi')$ passing through (t, x', ξ_0', τ_0). Namely, let $y=(y_1(t, x'), \cdots, y_{n-1}(t, x'))$ be an analytic solution of the equation

$$\frac{\partial y}{\partial t}=J(y/x')\,\mathrm{grad}_{\xi'}a(t, x', {}^tJ(y/x')\xi_0')$$

such that $y|_{t=0}=x'$, where $J(y/x')=\Big(\dfrac{\partial y_j}{\partial x_k}\Big)_{\substack{j=1,\cdots,n-1\\ k=1,\cdots,n-1}}$ and ${}^tJ(y/x')$ is the transposed matrix of $J(y/x')$. Consider the change of variables $(t, x')\to(s, y)$ such that $s=t$ and $y_j=y_j(t, x')$ $(j=1, \cdots, n-1)$. Let us denote by (σ, η), $\eta=(\eta, \cdots, \eta_{n-1})$, the associated new coordinates in the cotangent space. By this transformation we denote by $\tilde{b}(s, y, \eta)$ the function $b(t, x', \xi')$, and set

$$\tilde{a}(s, y, \eta)=a(s, x', {}^tJ(y/x')\eta)-\langle \mathrm{grad}_{\xi'}a(s, x', {}^tJ(y/x')\xi_0'), {}^tJ(y/x')\eta\rangle.$$

Then, $\tau-a(t, x', \xi')$ becomes $\sigma-\tilde{a}(s, y, \eta)$. We see that, in a neighborhood of $(s, y)=(0, 0)$,

$$(2.2)\qquad \mathrm{grad}_\eta\tilde{a}(s, y, \xi_0')=0$$

$$(2.3)\qquad \tilde{a}(s, y, \xi_0')=0.$$

Then the bicharacteristic strip of $\sigma-\tilde{a}(s, y, \eta)$ passing through $(s_0, y_0, \xi_0', \sigma_0)$

is the straight line segment parallel to the s axis such that $y=y_0$, $\eta=\xi_0'$, and $\sigma=\sigma_0$. Therefore we see that there is some positive constant δ such that

(c) $\tilde{b}(s, 0, \xi_0')=0$ in $\mathcal{J}_\delta=\{s\,;\,|s|<\delta\}$.

(d) $\mathrm{grad}_\eta \tilde{b}(s, 0, \xi_0')=0$ in $\mathcal{J}_\delta$.

Then, the following two cases are considered:

Case 2-2-1. $^\exists(s_0, 0, \xi_0')(s_0\in\mathcal{J}_\delta)$ *such that* $\mathrm{grad}_y \tilde{b}(s_0, 0, \xi_0')\neq 0$.

Case 2-2-2. $\mathrm{grad}_y \tilde{b}(s, 0, \xi_0')=0$ *in* $\mathcal{J}_\delta$.

Case 2-2-1. Treves ([13]; pp. 645-646) showed that there is an analytic *characteristic real surface* of $P(x, D)$ in a neighborhood of some point x_0, namely, he proved that there is a real-valued analytic function $\varphi(x)$ such that $p_m(x, \varphi_x)=0$ on $\varphi(x)=0$, where $\varphi(x_0)=0$ and $\varphi_x\neq 0$. Therefore, by virtue of S. Mizohata [4] (*Théorème* 3.1), *Mizohata phenomenon arises at* x_0.

Case 2-2-2. It finally holds that

$$d\tilde{b}(s, 0, \xi_0')=0 \quad \text{in} \quad \mathcal{J}_\delta. \tag{2.4}$$

From (2.2), (2.3), and (2.4), Treves ([13]; pp. 647-648) proved that there is an analytic *phase function* $\varphi(x)$ of $p_m(x, \varphi_x)=0$ in a neighborhood of x_0 such that $\varphi(x_0)=0$, $\varphi_x\neq 0$, and $\mathrm{Im}\,\varphi(x)\geqq 0$. Thus, from Lemma A, *Mizohata phenomenon arises at* x_0.

§ 3. Fundamental lemma.

Let $f(y)$ be a real-valued continuous function in an open set containing the origin in $\boldsymbol{R}^1$. For the next section, we prepare the following fundamental lemma.

Lemma 3.1. *Suppose that there exists a solution* $u=u(x, y)$ *continuous in a neighborhood* ω *of the origin and continuously differentiable in* $\omega_\pm=\omega\setminus\{x=0\}$, *satisfying the Mizohata equation*

$$\frac{\partial u}{\partial x}+ix\frac{\partial u}{\partial y}=f(y) \tag{3.1}$$

in $\omega_\pm$. *Then*, $f(y)$ *is analytic in* $\omega\cap\{x=0\}$.

This was previously proved in H. Ninomiya [5] (in Japanese). Here we shall reproduce it. It was inspired by H. Lewy [3].

Proof. Setting $u_1=u_1(x, y)=u(x, y)-u(-x, y)$, we see that u_1 is continuous in ω and continuously differentiable in $\omega_\pm$, and moreover, u_1 satisfies (3.1) in $\omega_\pm$ with the initial condition $u_1(0, y)\equiv 0$. Let P be an arbitrary point $(0, y_0)$ of $\omega\cap\{x=0\}$. We shall prove that $f(y)$ is analytic at P. We may suppose $y_0=0$. Let x' be an another independent real variable. We can consider the function $u_1(x, y)$ that of three variables x, x', and y. We may suppose $|y|<r_0$ for some positive constant r_0. Let us introduce new real variables r and θ by $x=r\cos\theta$ and $x'=r\sin\theta$, where $0<r<r_0$ and $|\theta|<\pi/2$. Setting $u_1^*=$

$u_1^*(r, \theta, y)=u_1(r\cos\theta, y)$, we see that (3.1) can be transformed as follows:

$$\cos\theta\frac{\partial u_1^*}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u_1^*}{\partial\theta}+ir\cos\theta\frac{\partial u_1^*}{\partial y}=f(y).$$

Hence, for an arbitrary sufficiently small positive constant ε, we see that

$$\int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}\Big(\cos\theta\frac{\partial u_1^*}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u_1^*}{\partial\theta}\Big)d\theta+i\int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}r\cos\theta\frac{\partial u_1^*}{\partial y}d\theta=(\pi-2\varepsilon)f(y).$$

Therefore, the integration by parts yields:

$$\int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}\Big(\cos\theta\frac{\partial u_1^*}{\partial r}+\frac{\cos\theta}{r}u_1^*\Big)d\theta-\frac{1}{r}[\sin\theta u_1^*]_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}$$
$$+i\frac{\partial}{\partial y}\int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}r\cos\theta u_1^*d\theta=(\pi-2\varepsilon)f(y).$$

Namely, it holds that in $\{(r, y); 0<r<r_0, |y|<r_0\}$

$$\frac{1}{r}\frac{\partial}{\partial r}\int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}r\cos\theta u_1^*d\theta+i\frac{\partial}{\partial y}\int_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon}r\cos\theta u_1^*d\theta$$
$$-2\frac{\cos\varepsilon}{r}u_1(r\sin\varepsilon, y)=(\pi-2\varepsilon)f(y).$$

Hence, in view of the fact that $u_1(0, y)=0$ for any $y\in\{|y|<r_0\}$, letting $\varepsilon\to 0$, we obtain the following:

$$\frac{1}{r}\frac{\partial}{\partial r}\int_{\pi/2}^{\pi/2}r\cos\theta u_1^*d\theta+i\frac{\partial}{\partial y}\int_{\pi/2}^{\pi/2}r\cos\theta u_1^*d\theta=\pi f(y). \tag{3.2}$$

Set $R=r^2/2$ and define the function $u_1^{**}=u_1^{**}(R, y)$ by:

$$u_1^{**}=\begin{cases}\displaystyle\int_{\pi/2}^{\pi/2}r\cos\theta u_1^*d\theta+i\pi\int_0^y f(\xi)d\xi & (R>0)\\ \displaystyle i\pi\int_0^y f(\xi)d\xi & (R=0)\end{cases}$$

Since u_1^{**} is purely imaginary-valued on the y axis, u_1^{**} can be holomorphically extended into $\{(R, y); |R|<r_0^2/2, |y|<r_0\}$ by *reflection principle*. Therefore, it is concluded that $f(y)$ is analytic at $y=0$.

§4. A counter example to Problem II.

In this section we present an operator $\mathcal{L}_1$ which *is not hypoelliptic at the origin such that* $\mathcal{L}_1 u=0$ *admits only zero* C^1 *solution in a neighborhood of the origin.*

Let $f(y)$ be a real-valued $C^\infty(\boldsymbol{R}^1)$ function which *is not analytic at any point of an interval* $(-\rho, \rho)$, where ρ is a positive constant, and both of $a(y)$ and $b(y)$ be real-valued analytic functions in $\boldsymbol{R}^1$ which vanish only at $y=0$ of order$\geqq 2$. And let n be a non-negative integer.

Let us consider the following differential equation in two real variables x and y:

$$\mathcal{L}_1 u = \frac{\partial u}{\partial x} + i x^{4n+1} a(y) \frac{\partial u}{\partial y} - x^{2n} b(y) f(y) u = 0. \tag{4.1}$$

First, we find that $(1)_x \otimes \delta_y$ is a distribution solution of (4.1), where $(1)_x$ denotes the distribution which is constant 1 on the x-space. Therefore, *the operator* $\mathcal{L}_1 = \frac{\partial}{\partial x} + i x^{4n+1} a(y) \frac{\partial}{\partial y} - x^{2n} b(y) f(y)$ *is not hypoelliptic at the origin.* On the other hand, we get the following theorem:

Theorem 4.1. *Any C^1 solution u of* (4.1) *in a neighborhood of the origin is identically zero.*

Proof. Let $u=u(x, y)$ be an arbitrary C^1 solution of (4.1) in a neighborhood ω of the origin. We can assume that $\omega = \{(x, y); |x| < \varepsilon, |y| < \varepsilon\}$, where ε is a positive constant $< \rho$.

First of all, we shall prove the following fact:

$$u(0, y) = 0 \quad \textit{for every} \quad y \in \mathcal{I}_\varepsilon = \{y; |y| < \varepsilon\}.$$

On the contrary, suppose that there is some $y_0 \neq 0 (y_0 \in \mathcal{I}_\varepsilon)$ such that $u(0, y_0) \neq 0$. Then, we can define the single-valued function $v=v(x, y)$ by $\log u(x, y)$ in a neighborhood $\mathcal{V}$ of the point $P(0, y_0)$. From (4.1), we have

$$\frac{\partial v}{\partial x} + i x^{4n+1} a(y) \frac{\partial v}{\partial y} = x^{2n} b(y) f(y). \tag{4.2}$$

Consider the change of variables $(x, y) \to (X, Y)$ such that $X = x^{2n+1}/(2n+1)$ and $Y = \int_{y_0}^{y} 1/a(\xi) d\xi$. This defines a homeomorphism from a neighborhood $\mathcal{V}_1 (\subseteq \mathcal{V})$ of P onto a neighborhood of $\mathcal{V}_1^*$ of the origin. Then we set $v^* = v^*(X, Y) = v(x, y)$ and $f^*(Y) = b(y) f(y)$. v^* is continuous in V_1^* and continuously differentiable in $\mathcal{V}_{1\pm}^* = \mathcal{V}_1^* \setminus \{X=0\}$. (4.2) becomes the *Mizohata equation*

$$\frac{\partial v^*}{\partial X} + i X \frac{\partial v^*}{\partial Y} = f^*(Y)$$

in $\mathcal{V}_{1\pm}^*$. Therefore, from Lemma 3.1, it follows that $f^*(Y)$ is analytic at $Y=0$. Since $Y \equiv Y(y) = \int_{y_0}^{y} 1/a(\xi) d\xi$ is analytic with respect to the variable y at $y=y_0$, $b(y) f(y)$ is also analytic at $y=y_0$. Hence, this implies that $f(y)$ must be analytic at $y=y_0$, which contradicts our assumption on $f(y)$. Therefore, we see that $u(0, y)=0$ in $\mathcal{I}_\varepsilon \setminus \{0\}$. Therefore, by the continuity of $u(x, y)$, we conclude that $u(0, y)=0$ in $\mathcal{I}_\varepsilon$.

Now, let t be an another independent real variable. We can consider the function $u(x, y)$ that of the variables x, y, and t. Then, from (4.1), since $\frac{\partial u}{\partial t} = 0$, we have the following equation in $\omega \times \boldsymbol{R}_t^1$:

$$\frac{\partial u}{\partial x} + i x^{4n+1} \left(a(y) \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \right) - x^{2n} b(y) f(y) u = 0. \tag{4.3}$$

Let $U=U(t, y)$ be a real-valued analytic solution of

$$\frac{\partial U}{\partial x}+a(y)\frac{\partial U}{\partial y}=0 \tag{4.4}$$

in a neighborhood of the origin such that $U(0, y)\equiv y$. Let us consider the change of variables $(x, y, t)\rightarrow(x, Y, t)$ such that $x=x$, $Y=U(t, y)$, and $t=t$. Since $\frac{\partial}{\partial y}U(0, y)=1$, by the implicit function theorem, y is expressed by $y=y(Y, t)(y(Y, 0)=Y)$, where $y(Y, t)$ is analytic in a neighborhood $\{(Y, t); |Y|<\varepsilon_1, |t|<\varepsilon_1\}(0<\varepsilon_1\leqq\varepsilon)$ of the origin. We denote by $u^*=u^*(x, Y, t)$ the function $u(x, y(Y, t))$, and $F(Y, t)$ the function $b(y(Y, t))f(y(Y, t))$. Then, $u^*\in C^1(\mathscr{D}_{\varepsilon_1})$ and $F(Y, t)\in C^\infty(\mathscr{D}_{\varepsilon_1})$, where $\mathscr{D}_{\varepsilon_1}=\{(x, Y, t); |x|<\varepsilon, |Y|<\varepsilon_1, |t|<\varepsilon_1\}$. From (4.3) and (4.4), it follows that

$$\begin{cases} \dfrac{\partial u^*}{\partial x}+ix^{4n+1}\dfrac{\partial u^*}{\partial t}-x^{2n}F(Y, t)u^*=0 \quad \text{in} \quad \mathscr{D}_{\varepsilon_1} \\ u^*(0, Y, t)=0 \quad \text{in} \quad \{(Y, t); |Y|<\varepsilon_1, |t|<\varepsilon_1\} \end{cases}$$

Therefore, by virtue of the *uniqueness theorem* ([6], [12]), for every Y such that $|Y|<\varepsilon_1$, there is a constant $\varepsilon(Y)(0<\varepsilon(Y)\leqq\varepsilon_1)$ such that $u^*\equiv0$ in $\{(x, t); |x|<\varepsilon(Y), |t|<\varepsilon(Y)\}$. On the other hand, $\mathscr{L}_1$ is *elliptic for* $xy\neq0$. Therefore, in view of the fact that $u(x, Y)=u(x, y(Y, 0))=u^*(x, Y, 0)$, by virtue of *unique continuation property* of elliptic operator, we conclude that $u\equiv0$ in ω.

Remark. We refer to H. Saltzmann-K. Zeller [10] concerning on the existence of function having the property such that it is not analytic at any point of an interval.

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