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Differentiable vectors and analytic vectors in completions of certain representation spaces of a Kac-Moody algebra

By

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Introduction.

Let $\mathfrak{g}_{\boldsymbol{R}}$ be a Kac-Moody algebra over the real number field $\boldsymbol{R}$ with a symmetrizable generalized Cartan matrix (GCM), and $\mathfrak{h}_{\boldsymbol{R}}$ the Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{R}}$. Then, the Kac-Moody algebra $\mathfrak{g}$ over $\boldsymbol{C}$ corresponding to the same GCM and its Cartan subalgebra $\mathfrak{h}$ are given by

$$\mathfrak{g} = \boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}_{\boldsymbol{R}} \quad \text{and} \quad \mathfrak{h} = C\otimes_{\boldsymbol{R}}\mathfrak{h}_{\boldsymbol{R}},$$

respectively. We denote by $\mathfrak{k}$ the unitary form of $\mathfrak{g}$, and put $\mathfrak{k}_{\boldsymbol{R}}=\mathfrak{k}\cap\mathfrak{g}_{\boldsymbol{R}}$ (for the precise definition, see [8] and [7]).

In [8] and [7], we constructed and studied groups K^{Λ} and $K^{\Lambda}_{\boldsymbol{R}}$ consisting of unitary operators on a Hilbert space $H(\Lambda)$ which is a completion of the integrable highest weight module $L(\Lambda)$ for $\mathfrak{g}$ with dominant integral highest weight $\Lambda\in\mathfrak{h}^*_{\boldsymbol{R}}$. These groups are generated by naturally defined exponentials of elements in $\mathfrak{k}$ and $\mathfrak{k}_{\boldsymbol{R}}$ respectively. In this paper, we show that the exponential map exp: $\mathfrak{k}\rightarrow\boldsymbol{U}(H(\Lambda))$ can be extended to a certain completion $H^{u}_{1}(\text{ad})$ of $\mathfrak{k}$. We show, in prallel, that taking the adjoint representation of $\mathfrak{g}$ on itself in place of the highest weight representation on $L(\Lambda)$, and completing the representation space $\mathfrak{g}$ to a Hilbert space $H(\text{ad})$, the exaponintial map exp: $H^{u}_{1}(\text{ad})\mapsto\boldsymbol{B}(H(\text{ad}))$ can be defined naturally. Here $\boldsymbol{U}(H)$ is the group of unitary operators and $\boldsymbol{B}(H)$ is the algebra of bounded operators on a Hilbert space H. Note that the adjoint representation is quite different from $L(\Lambda)$ at the point that the set of its weights is unbounded both in positive and negative directions when $\mathfrak{g}$ is of infinite-dimension. For these exponentials, we define the differentiable vectors and the analytic vectors, and prove some properties of them, which we expect to utilize for studying fine structures of K^{Λ} and $K^{\Lambda}_{\boldsymbol{R}}$.

Let us explain in more detail. We denote by $\underline{\mathfrak{g}}$ the infinite direct products of $\mathfrak{g}^0=\mathfrak{h}$ and the root spaces $\mathfrak{g}^{\alpha}$ over α, and by $\underline{L}(\Lambda)$ that of all the weight spaces $L(\Lambda)_{\mu}$ over μ, respectively. $\mathfrak{g}$ acts on $\underline{\mathfrak{g}}$ and $\underline{L}(\Lambda)$ naturally. Let $H(\text{ad})$ and $H(\Lambda)$ be the

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completions of $(\mathfrak{g}, (\cdot|\cdot)_1)$ and $(L(\Lambda), (\cdot|\cdot)_\Lambda)$, respectively, where $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_\Lambda$ are standard inner products on $(\mathfrak{g}, \mathrm{ad})$ and $L(\Lambda)$. We may consider $H(\mathrm{ad})$ as a subspace of $\underline{\mathfrak{g}}$ and $H(\Lambda)$ of $\underline{L}(\Lambda)$.

The spaces of vectors of class C^m for $\mathfrak{g}$-action ($m=0, 1, 2, \cdots$) are defined naturally by

$$
\begin{aligned}
&H_0(\mathrm{ad}) = H(\mathrm{ad}),\\
&H_m(\mathrm{ad}) = \{y \in H_{m-1}(\mathrm{ad});\ [x, y] \in H_{m-1}(\mathrm{ad}) \quad \text{for any} \quad x \in \mathfrak{g}\};\\
&H_0(\Lambda) = H(\Lambda),\\
&H_m(\Lambda) = \{v \in H_{m-1}(\Lambda);\ xv \in H_{m-1}(\Lambda) \quad \text{for any} \quad x \in \mathfrak{g}\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad 0<m<+\infty.
\end{aligned}
$$

As for spaces of infinitely differentiable vectors, we put

$$H_\infty(\mathrm{ad}) = \cap_{k\geq 0} H_k(\mathrm{ad}), \quad H_\infty(\Lambda) = \cap_{k\geq 0} H_k(\Lambda),$$

and for spaces of analytic vectors,

$$
\begin{aligned}
H_\omega(\mathrm{ad}) = \Big\{ y \in H_\infty(\mathrm{ad});\ &\text{for any } x \in \mathfrak{g}, \text{ there exists } \varepsilon>0\\
&\text{such that } \textstyle\sum_{m\geq 0} \frac{1}{m!} \varepsilon^m \|(\mathrm{ad}\, x)^m y\|_1 < +\infty \Big\},\\
H_\omega(\Lambda) = \Big\{ v \in H_\infty(\Lambda);\ &\text{for any } x \in \mathfrak{g}, \text{ there exists } \varepsilon>0\\
&\text{such that } \textstyle\sum_{m\geq 0} \frac{1}{m!} \varepsilon^m \|x^m v\|_\Lambda < +\infty \Big\}.
\end{aligned}
$$

We call an element $h_0 \in \mathfrak{h}_{\boldsymbol{R}}$ *strictly dominant* if

$$\alpha(h_0)>0 \qquad \text{for any positive root } \alpha,$$

and fix such an element in the follwing.

As the first main result, we show that the differentiability and the analyticity are characterized by means of one element h_0. In other words, we may replace $x \in \mathfrak{g}$ in the above definitions of spaces with only one element h_0, any strictly dominant element in $\mathfrak{h}_{\boldsymbol{R}}$ (Theorem 2.6). Based on this result, we can define certain topologies on the above spaces. Then with such topologies, $H_\infty(\mathrm{ad})$ and $H_\omega(\mathrm{ad})$ become topological Lie algebras, which we denote by $\mathfrak{g}_\infty$ and $\mathfrak{g}_\omega$ (see § 3).

Let $H_1^u(\mathrm{ad})$ be the closure of the unitary form $\mathfrak{k} \subset \mathfrak{g}$ in the Hilbert space $H_1(\mathrm{ad})$ of class C^1 vectors. We see that each element in $H_1^u(\mathrm{ad})$ acts on $H(\mathrm{ad})$ and $H(\Lambda)$ as a closable operator (cf. § 4). The second goal is to prove the exponentiability of elements in $H_1^u(\mathrm{ad})$ as operators on the Hilbert spaces $H(\mathrm{ad})$ or $H(\Lambda)$. On $H(\Lambda)$, the exponentiability is considered also in [3]. However, on $H(\mathrm{ad})$, there exists much more difficulty firstly because the inner product on $H(\mathrm{ad})$ is not contravariant and so the action of elements of $H_1^u(\mathrm{ad})$ on $H(\mathrm{ad})$ is not anti-symmetric, and secondly because the weights of $H(\mathrm{ad})$ are unbounded in both positive and negative directions.

Let $\mathfrak{k}_\omega$ be the closure of $\mathfrak{k}$ in $\mathfrak{g}_\omega$. The last goal is to show the invariance of the spaces $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ of vectors of class C^m ($m=0, 1, 2, \cdots, \infty, \omega$) under the exponentials of elements in $\mathfrak{k}_\omega$. This result will be a useful tool for the study of the fine structure of the group K^A.

This paper is organized as follows. In § 1, we prepare notations and basic results about Kac-Moody algebras. In § 2, we define the notion of differentiable vectors and analytic vectors. And then, it is proved that the differentiability and the analyticity are characterized by only one element h_0. In § 3, topologies on the spaces of differentiable vectors and of analytic vectors are studied. In § 4, we extend the exponential map on $\mathfrak{k}$ to its closure $H_1^u(\mathrm{ad})$ in $H_1(\mathrm{ad})$. And then, in § 5, we prove invariance of the spaces of differentiable vectors and of analytic vectors under exponentials of elements of the closure $\mathfrak{k}_\omega$ of $\mathfrak{k}$ in $\mathfrak{g}_\omega$. In § 6, we investigate convergence of Campbell-Hausdorff formula for exponentials defined in § 4.

Notations. We denote by $\boldsymbol{C}$ the complex number field, $\boldsymbol{R}$ the real number field, $\boldsymbol{Z}$ the ring of rational integers. For an ordered set $(S, \leqq)$ and $s \in S$, we define subsets $S_{>s}$ and $S_{\geqq s}$ of S by

$$S_{>s} = \{t \in S;\ t>s\},$$
$$S_{\geqq s} = \{t \in S;\ t \geqq s\}.$$

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§ 1. Some basic results for Kac-Moody algebras.

In this section, we prepare notations and fundamental results about Kac-Moody algebras which will be needed in the succeeding sections. For detailed accounts, see [4] for example.

1.1. Kac-Moody algebras. Let $n \in \boldsymbol{Z}_{>0}$ and $A=(a_{ij})_{i,j=1}^n$ be an integral matrix which satisfies

i) $a_{ii}=2$ for all $i=1, \cdots, n$,
ii) $a_{ij} \leqq 0$ if $i \neq j$,
iii) $a_{ij}=0$ if and only if $a_{ji}=0$

Such a matrix is called a generalized Cartan matrix (GCM).

For a field k with characteristic 0, we denote by $\mathfrak{g}_k=\mathfrak{g}_k(A)$ the Kac-Moody algebra over k associated with the GCM A, and $\mathfrak{h}_k=\mathfrak{h}_k(A)$ its Cartan subalgebra (cf. [4, Chap. I]).

Let $\Delta=\Delta(A)$ be the root system of $(\mathfrak{g}_k, \mathfrak{h}_k)$, $\Delta_+=\Delta_+(A)$ the set of positive roots, Π the simple roots, $\Pi^\vee$ the simple coroots, and $\mathfrak{g}_k=\mathfrak{h}_k+\sum_{\alpha\in\Delta}\mathfrak{g}_k^\alpha$ the root space decomposition.

Put $\mathfrak{n}_{\pm,k}=\mathfrak{n}_{\pm,k}(A)=\sum_{\alpha\in\Delta_+}\mathfrak{g}_k^{\pm\alpha}$. Then $\mathfrak{n}_{\pm,k}$ are both subalgebras of $\mathfrak{g}_k$ and $\mathfrak{g}_k=\mathfrak{n}_{-,k}+\mathfrak{h}_k+\mathfrak{n}_{+,k}$. Let $P_{\pm,k}$ and $P_{0,k}$ be the projections from $\mathfrak{g}_k$ onto $\mathfrak{n}_{\pm,k}$ and $\mathfrak{h}_k$ respectively with respect to this decomposition.

If the GCM A is symmetrizable, that is, there is a non-degenerate diagonal matrix D such that DA is symmetric, then there exists a symmetric, non-degenerate, invariant bilinear form $(\cdot\,|\,\cdot)_k$ on $\mathfrak{g}_k$ called the standard invariant form, which plays an important role in the theory of Kac-Moody algebras (loc. cit. Chap. II).

By the invariance of $(|\cdot|)_k$, we have

$$(1.2)\qquad (\mathfrak{g}_k^{\alpha}\,|\,\mathfrak{g}_k^{\beta})_k=0 \qquad \text{if}\ \ \alpha+\beta\neq 0\,,$$

$$(1.3)\qquad (\mathfrak{h}_k\,|\,\mathfrak{g}_k^{\alpha})_k=0 \qquad (\alpha\in\Delta)\,.$$

In paticular, the restriction of $(\cdot\,|\,\cdot)_k$ to $\mathfrak{h}_k$ is non-degenerate. Hence, there exists a linear bijection ν_k from $\mathfrak{h}_k$ onto its dual $\mathfrak{h}_k^*$ such that

$$(h_1\,|\,h_2)_k=\nu_k(h_1)(h_2) \qquad (h_1,\ h_2\in\mathfrak{h}_k)\,,$$

and it holds that

$$(1.4)\qquad [x,\ y]=(x\,|\,y)_k\nu_k^{-1}(\alpha) \qquad (\alpha\in\Delta,\ x\in\mathfrak{g}_k^{\alpha},\ y\in\mathfrak{g}_k^{-\alpha})\,.$$

Throughout this paper, we assume A to be a symmetrizable GCM, and hence we have the standard invariant form $(\cdot\,|\,\cdot)_k$ on $\mathfrak{g}_k$.

1.2. Unitary form. In the following, we concentrate on the case where $k=\boldsymbol{R}$ or $\boldsymbol{C}$. If $k=\boldsymbol{C}$, then the subscript $\boldsymbol{C}$ should be ommitted. Clearly $\mathfrak{g}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{h}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{h}_{\boldsymbol{R}}$.

There exists a canonically defined anti-linear anti-automorphism $\mathfrak{g}\in x\mapsto x^*\in\mathfrak{g}$, such that

$$(1.5)\qquad (\mathfrak{g}^{\alpha})=\mathfrak{g}^{-\alpha} \qquad (\alpha\in\Delta)\,,$$

$$(1.6)\qquad h^*=h \qquad (h\in\mathfrak{h}_{\boldsymbol{R}})\,.$$

We define a real subalgebra $\mathfrak{k}$, called the unitary form, of $\mathfrak{g}$ by

$$(1.7)\qquad \mathfrak{k}=\{x\in\mathfrak{g};\ x+x^*=0\}\,.$$

If $\mathfrak{g}$ is finite-dimensional, $\mathfrak{k}$ is nothing but a compact real form of $\mathfrak{g}$.

Define a sesquilinear form $(\cdot\,|\,\cdot)_0$ on $\mathfrak{g}$ by

$$(1.8)\qquad (x\,|\,y)_0=(x\,|\,y^*) \qquad (x,\ y\in\mathfrak{g}))\,.$$

$(\cdot\,|\,\cdot)_0$ is Hermitian and positive definite on each root space $\mathfrak{g}^{\alpha}$ (cf. [5, Th.1]). By invariance of $(\cdot\,|\,\cdot)$ and orthgonality (1.2) and (1.3), it holds that

$$(1.9)\qquad ([x,\ y]\,|\,z)_0=(y\,|\,[x^*,\ z])_0 \qquad (x,\ y,\ z\in\mathfrak{g})\,,$$

$$(1.10)\qquad (\mathfrak{g}^{\alpha}\,|\,\mathfrak{g}^{\beta})_0=0 \qquad \text{for}\ \alpha\neq\beta;\ (\mathfrak{h}\,|\,\mathfrak{g}^{\alpha})_0=0\ \text{for}\ \alpha\in\Delta.$$

In paticular, $(\cdot\,|\,\cdot)_0$ is $\mathfrak{k}$-invariant and positive definite on the space $\mathfrak{n}_-+\mathfrak{n}_+$. We call the property (1.9) the contravariance.

1.3. Irreducible highest weight modules. Let $\lambda \in \mathfrak{h}^*$. We denote by $L(\lambda)$ the irreducible highest weight module for $\mathfrak{g}$ with highest weight λ, $P(\lambda)$ the set of weights of $L(\lambda)$, and $L(\lambda)=\sum_{\mu\in P(\lambda)} L(\lambda)_\mu$ the weight space decomposition of $L(\lambda)$.

If λ is in $\mathfrak{h}_R^*$, then there exists a non-degenerate Hermitian form $(\cdot\,|\,\cdot)_\lambda$ (see [8, §2] for example), unique up to scalar multiples, which is contravariant:

$$(1.11) \qquad (xv\,|\,u)_\lambda = (v\,|\,x^*u)_\lambda \qquad (x\in\mathfrak{g},\quad v,\, u\in L(\lambda))\,.$$

Moreover, if λ is dominant integral, that is, $\lambda(\alpha^\vee)$ is a non-negative integer for any simple coroot $\alpha^\vee$, then $(\cdot\,|\,\cdot)_\lambda$ is definite ([5, Th.1]).

In the following, we denote by Λ a dominant integral element of $\mathfrak{h}_R^*$ and fix a contravariant inner product $(\cdot\,|\,\cdot)_\Lambda$ on $L(\Lambda)$.

1.4. Inner product on $\mathfrak{g}$. Take a basis $h_1, \cdots, h_{2n-l}$, $l=\text{rank } A$, of $\mathfrak{h}_R$ such that

$$((h_i\,|\,h_j)_0)_{i,j} = (\varepsilon_i\delta_{ij})_{i,j}$$

where ε_i is ± 1 for each $1\leqq i\leqq 2n-1$.

We define an inner product $(\cdot\,|\,\cdot)_1$ on $\mathfrak{h}$ by

$$(h\,|\,h')_1 = \textstyle\sum_i c_i\bar{c}'_i \qquad \text{for} \quad h=\sum_i c_ih_i,\ h'=\sum_i c'_ih_i\in\mathfrak{h}\,,$$

and extend it to $\mathfrak{g}$ by

$$(x\,|\,y)_1 = (P_-(x)\,|\,P_-(y))_0+(P_0(x)\,|\,P_0(y))_1+(P_+(x)\,|\,P_+(y))_0 \qquad (x,\, y\in\mathfrak{g})\,.$$

where $P_\pm$ and P_0 are projections from $\mathfrak{g}$ onto $\mathfrak{n}_\pm$ and $\mathfrak{h}$ respectively defined in 1.1.

Let T be an operator on $\mathfrak{g}$ given as follows. On $\mathfrak{h}\subset\mathfrak{g}$,

$$(Th_1, \cdots, Th_{2n-l}) = (h_1, \cdots, h_{2n-l})(\varepsilon_i\delta_{ij})_{i,j}$$

and on $\mathfrak{n}_-+\mathfrak{n}_+\subset\mathfrak{g}$, it is equal to the identity. T is a unitary and selfadjoint operator with respect to $(\cdot\,|\,\cdot)_1$. In particular, T is involutive: $T^2=id$. Moreover, there hold that

$$(1.12) \qquad (x\,|\,y)_0 = (x\,|\,Ty)_1 \qquad (x,\, y\in\mathfrak{g})\,,$$

$$(1.13) \qquad \begin{aligned} id-T &= 2\,(\text{the orthogonal projection onto} \\ &\qquad \text{the } -1\text{-eigenspace of } T) \\ &\leqq 2P_0. \end{aligned}$$

1.5. Completions of $\mathfrak{g}$ and $L(\Lambda)$. Put $\mathfrak{g}^0=\mathfrak{h}$, and define infinite products $\underline{\mathfrak{g}}$ and $\underline{L}(\Lambda)$ by

$$(1.14) \qquad \underline{\mathfrak{g}} = \textstyle\prod_{\alpha\in\Delta\cup\{0\}}\mathfrak{g}^\alpha\,, \quad \underline{L}(\Lambda) = \prod_{\alpha\in P(\Lambda)} L(\Lambda)_\mu$$

Each element x of $\mathfrak{g}$ acts on $\underline{\mathfrak{g}}\supset\mathfrak{g}$ and on $\underline{L}(\Lambda)\supset L(\Lambda)$ naturally. And $(\cdot\,|\,\cdot)_0$ and $(\cdot\,|\,\cdot)_\Lambda$ are exended to the pairings of $\underline{\mathfrak{g}}\times\mathfrak{g}$ and of $\underline{L}(\Lambda)\times L(\Lambda)$ respectively so that (1.9) and (1.11) hold with $x,\, z\in\mathfrak{g}$, $u\in L(\Lambda)$, $y\in\underline{\mathfrak{g}}$, $v\in\underline{L}(\Lambda)$.

Let $H(\text{ad})$ and $H(\Lambda)$ be the completions of $(\mathfrak{g},\, (\cdot\,|\,\cdot)_1)$ and $(L(\Lambda),\, (\cdot\,|\,\cdot)_\Lambda)$

respectively. We may regard them as subspaces of $\underline{\mathfrak{g}}$ and $\underline{L}(\Lambda)$ respectively:

$$H(\mathrm{ad}) = \{(x^{\alpha})_{\alpha}\in\underline{\mathfrak{g}};\ \sum_{\alpha}||x_{\alpha}||_1^2<+\infty\}\ ,$$
$$H(\Lambda) = \{(v_{\mu})_{\mu}\in\underline{L}(\Lambda);\ \sum_{\mu}||v_{\mu}||_{\Lambda}^2<+\infty\}\ ,$$

Since T is identical on each root space, it is extended to a linear bijection from $\underline{\mathfrak{g}}$ onto $\underline{\mathfrak{g}}$ itself, which is denoted again by the same symbol. Similarly, *-operation is extended to $\underline{\mathfrak{g}}$, and again we use the same notation for the extension.

§ 2. Differentiable vectors and analytic vectors.

In this section, we define differentiable vectors and analytic vectors for $\mathfrak{g}$-action on the Hilbert spaces $H(\mathrm{ad})$ and $H(\Lambda)$, which are obtained by completing $\mathfrak{g}$-modules $(\mathfrak{g}, \mathrm{ad})$ and $L(\Lambda)$ respectively. Then, we shall prove that the differentiability and the analyticity are charcterized by means of a strictly dominant element of $\mathfrak{h}_{\boldsymbol{R}}$.

2.1. Estimate of norms of $\mathfrak{g}$-action. Take and fix an element h_0 of $\mathfrak{h}_{\boldsymbol{R}}$ such that

$$\alpha(h_0)>0 \qquad \text{for all} \quad \alpha\in\Delta_+ . \tag{2.1}$$

We call such an element *strictly dominant.*

For our purpose described above, we need to improve the estimates of $\mathfrak{g}$-action on $\mathfrak{g}$-modules $(\mathfrak{g}, \mathrm{ad})$ and $L(\Lambda)$ in [8]. And, modifying slightly the method in [5, Prop. 3.1], we obtain the following evaluations.

Proposition 2.1. *There exist positive numbers C_0, C_1, $C_{0,\Lambda}$ and $C_{1,\Lambda}$ such that*

i) *for* $(\mathfrak{g}, \mathrm{ad})$,

$$||P_-([x, y])||_1\leqq C_0||x||_1||[h_0, P_-(y)]||_1 \qquad (x\in\mathfrak{n}_+,\ y\in\mathfrak{g})\ ,$$
$$||[x, y]||_1\leqq C_1(||x||_1||[h_0, y]||_1+||[h_0, x]||_1||y||_1) \qquad (x,\ y\in\mathfrak{g})\ ,$$

ii) *for* $L(\Lambda)$,

$$||xv||_{\Lambda}\leqq C_{0,\Lambda}||x||_1(\Lambda(h_0)||v||_{\Lambda}+||h_0 v||_{\Lambda}) \qquad (x\in\mathfrak{n}_+,\ v\in L(\Lambda))\ ,$$
$$||xv||_{\Lambda}\leqq C_{1,\Lambda}(||x||_1||v||_{\Lambda}+||[h_0, x]||_1||v||_{\Lambda}+||x||_1||h_0 v||_{\Lambda}) \qquad (x\in\mathfrak{g},\ v\in L(\Lambda))\ .$$

Here, P_- is the projection from $\mathfrak{g}$ onto $\mathfrak{n}_-$ defined in 1.1, *and $||\cdot||_1$, $||\cdot||_{\Lambda}$ are norms defined by the inner products $(\cdot\,|\,\cdot)_1$ and $(\cdot\,|\,\cdot)_{\Lambda}$ respectively.*

Proof is essentially the same as in [5, Prop. 3.1].

From these estimates, we have the following proposition, which gives rather exact estimates of norms of iterations of $\mathfrak{g}$-actions. The order of increasing of these norms according to m are very important in the following, for instance, when we wish to define exponential maps.

Proposition 2.2. *For any $x_1, x_2, \cdots, x_m\in\mathfrak{g}$ and $v\in L(\Lambda)$, we have the following evaluations.*

i) *In case of* $(\mathfrak{g}, \mathrm{ad})$,

$$\|[x_1, [x_2, \cdots, [x_{m-1}, x_m]\cdots]]\|_1 \leqq$$
$$\leqq C_1^{m-1} \sum C^{(m)}_{p_1,\cdots,p_m} \prod_{j=1}^m \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1 ,$$

where the sum is taken over non-negative integers $p_1, \cdots, p_m$ *such that* $p_1+\cdots+p_m= m-1$, *and* $C^{(m)}_{p_1,\cdots,p_m}$ *are defined inductively as follows*:

$$C^{(m)}_{p_1,\cdots,p_{m-2},p_{m-1},p_m} = \qquad (\textit{for } m>1)$$
$$\begin{cases} = C^{(m-1)}_{p_1,\cdots,p_{m-2},p_{m-1}+p_m-1} \dfrac{(p_{m-1}+p_m)!}{p_{m-1}!p_m!} & \textit{if } p_{m-1}+p_->0 , \\ = 0 & \textit{if } p_{m-1}+p_m = 0 , \end{cases}$$
$$C^{(1)}_0 = 1 , \qquad (\textit{for } m = 1) .$$

ii) *In case of* $L(\Lambda)$

$$\|x_1 x_2 \cdots x_m v\|_\Lambda \leqq$$
$$\leqq C^m_{1,\Lambda} \sum C^{(m)}_{p_1,\cdots,p_m;q} (\prod_{j=1}^m \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1) \|h_0^q v\|_\Lambda$$

where the sum is taken over non-negative integers $p_1, \cdots, p_m$, q *such that* $p_1+\cdots+p_m+ q\leqq m$ *and* $C^{(m)}_{p_1,\cdots,p_m;q}$ *is given by*

$$C^{(m)}_{p_1,\cdots,p_{m-1},p_m;q} = \qquad (\textit{for } m>1)$$
$$\begin{cases} = (C^{(m-1)}_{p_1,\cdots,p_{m-1};p_m+q} + C^{(m-1)}_{p_1,\cdots,p_{m-1};p_m+q-1}) \dfrac{(p_m+q)!}{p_m!q!} \\ \qquad \textit{if } p_1+\cdots+p_{m-1}+p_m+q<m \textit{ and } p_m+q>0 , \\ = C^{(m-1)}_{p_1,\cdots,p_{m-1};p_m+q-1} \dfrac{(p_m+q)!}{p_m!q!} \\ \qquad \textit{if } p_1+\cdots+p_{m-1}+p_m+q = m \textit{ and } p_m+q>0 , \\ = C^{(m-1)}_{p_1,\cdots,p_{m-1};0} & \textit{if } p_1+\cdots+p_{m-1}<m , \\ = 0 & \textit{otherwise,} \end{cases}$$
$$C^{(1)}_{0;0} = C^{(1)}_{0;1} = C^{(1)}_{1;0} = 1 , \qquad (\textit{for } m = 1) .$$

Proof is carried out by induction on m.

i) If $m=1$, then the assertion is clear.

Assume that the assertion is valid until m. Then, we have

$$\|[x_1, \cdots, [x_{m-1}, [x_m, x_{m+1}]]\cdots]]\|_1 \leqq$$
$$\leqq C_1^{m-1} \sum_{p_1,\cdots,p_m} C^{(m)}_{p_1,\cdots,p_m} (\prod_{j=1}^{m-1} \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1) \|(\mathrm{ad}\ h_0)^{p_m}[x_m, x_{m+1}]\|_1$$
$$\leqq C_1^{m-1} \sum_{p_1,\cdots,p_m} C^{(m)}_{p_1,\cdots,p_m} (\prod_{j=1}^{m-1} \|(\mathrm{ad}\ h_0)^{p_j}{}_j\|_1) \times$$
$$\times \sum_{k=0}^{p_m} \binom{p_m}{k} \|[(\mathrm{ad}\ h_0)^{p_m-k} x, (\mathrm{ad}\ h_0)^k x_{m+1}]\|_1 .$$

By Proposition 2.1,

$$
\begin{aligned}
&\leqq C_1^{m-1} \sum_{p_1,\cdots,p_m} C^{(m)}_{p_1,\dots,p_m} \left(\prod_{j=1}^{m-1} \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1\right) \times \\
&\qquad \times \sum_{k=0}^{p_m} \binom{p_m}{k} C_1 \left(\|(\mathrm{ad}\ h_0)^{p_m-k+1} x_m\|_1 \|(\mathrm{ad}\ h_0^k) x_{m+1}\|_1 + \right. \\
&\qquad \left. + \|(\mathrm{ad}\ h_0)^{p_m-k} x_m\|_1 \|(\mathrm{ad}\ h_0)^{k+1} x_{m+1}\|_1\right) \\
&= C_1^m \sum_{p_1,\cdots,p_m} C^{(m)}_{p_1,\dots,p_m} \left(\prod_{j=1}^{m-1} \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1\right) \times \\
&\qquad \times \sum_{k=0}^{p_m+1} \binom{p_m+1}{k} \|(\mathrm{ad}\ h_0)^{p_m+1-k} x_m\|_1 \|(\mathrm{ad}\ h_0)^k x_{m+1}\|_1 \\
&= C_1^m \sum_{\substack{p_1,\cdots,p_m,p_{m+1} \\ p_m+p_{m+1}}} C^{(m)}_{p_1,\dots,p_m p_{m=1}-1} \binom{p_m+p_{m+1}}{p_m} \left(\prod_{j=1}^{m-1} \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1\right).
\end{aligned}
$$

Hence, the assertion is valid also for $m+1$.

ii) is proved in a similar way. Q.E.D.

Corollary 2.3. *For any $x_1, x_2, \cdots, x_m \in \mathfrak{g}$ and $v \in L(\Lambda)$, we have*

!) *in case of* ($\mathfrak{g}$, ad),

$$
\begin{aligned}
&\|[x_1, \cdots, [x_{m-1}, x_m]\cdots]\|_1 \leqq \\
&\leqq (m-1)!\, C_1^{m-1} \sum_{p_1,\cdots,p_m} \prod_j \frac{1}{p_j!} \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1,
\end{aligned}
$$

ii) *in case of $L(\Lambda)$*,

$$
\begin{aligned}
&\|x_1 \cdots x_m v\|_\Lambda \leqq \\
&\leqq (m+1)!\, C_{1,\Lambda}^m \sum_{p_1,\cdots,p_m,q} \left\{\prod_j \frac{1}{p_j!} \|(\mathrm{ad}\ h_0)^{p_j} x_j\|_1\right\} \frac{1}{q!} \|h_0^2 v\|_\Lambda.
\end{aligned}
$$

Proof. By induction on m, we see that

$$
C^{(m)}_{p_1,\dots,p_m} \leqq \frac{m!}{p_1! \cdots p_m!}, \quad C^{(m)}_{p_1,\dots,p_m;q} \leqq \frac{(m+1)!}{p_1! \cdots p_m!\, q!}.
$$

Then, the assertions are obvious. Q.E.D.

2.2. Differentiability and analyticity of elements of H(ad) and $H(\Lambda)$

Now, we introduce the notion of differentiable vectors and of analytic vectors, from the point of view that an element of a Lie algebra can be regaraded as a differentiable operator of degree 1 in the finite dimensional case. In this sense, our definition is quite natural, or rather necessary in studying Kac-Moody algebras.

Definition 2.4. We define the subspaces H_m(ad) and $H_m(\Lambda)$ of H(ad) and $H(\Lambda)$ inductively as follows:

$$
\begin{aligned}
&H_0(\mathrm{ad}) = H(\mathrm{ad}), \\
&H_m(\mathrm{ad}) = \{y \in H_{m-1}(\mathrm{ad});\ [x, y] \in H_{m-1}(\mathrm{ad}) \quad \text{for all } x \in \mathfrak{g}\}; \\
&H_0(\Lambda) = H(\Lambda),
\end{aligned}
$$

$$H_m(\Lambda) = \{v \in H_{m-1}(\Lambda);\ xv \in H_{m-1}(\Lambda) \quad \text{for all} \quad x \in \mathfrak{g}\};$$
$$H_\infty(\mathrm{ad}) = \cap_{m=0}^{\infty} H_m(\mathrm{ad}), \quad H_\infty(\Lambda) = \cap_{m=0}^{\infty} H_m(\Lambda),$$

We call elements of $H_\infty(\mathrm{ad})$ and $H_\infty(\Lambda)$ differentiable vectors in $H(\mathrm{ad})$ and $H(\Lambda)$ respectively.

Definition 2.5. We define the spaces of analytic vectors $H_\omega(\mathrm{ad})$ and $H_\omega(\Lambda)$ as follows:

$$H_\omega(\mathrm{ad}) = \Big\{y \in H_\infty(\mathrm{ad});\ \text{for any } x \in \mathfrak{g}, \text{ there exists } \varepsilon > 0$$
$$\text{such that } \sum_{m=0}^{\infty} \frac{1}{m!} \varepsilon^m \|(\mathrm{ad}\, x)^m y\|_1 < +\infty\Big\},$$
$$H_\omega(\Lambda) = \Big\{v \in H_\infty(\Lambda);\ \text{for any } x \in \mathfrak{g}, \text{ there exists } \varepsilon > 0$$
$$\text{such that } \sum_{m=0}^{\infty} \frac{1}{m!} \varepsilon^m \|x^m v\|_\Lambda < +\infty\Big\}.$$

We call elements of $H_\omega(\mathrm{ad})$ and $H_\omega(\Lambda)$ analytic vectors in $H(\mathrm{ad})$ and in $H(\Lambda)$ respectively.

Roughly speaking, using the estimates of norms of $\mathfrak{g}$-actions on $H(\mathrm{ad})$ and $H(\Lambda)$ given in Proposition 2.2, we can prove the following theorem, one of our main results. Thanks to this theorem, one may concentrate on only one element h_0, instead of all the elements in $\mathfrak{g}$, for the implication of the differentiability or the analyticity of an element of $H(\mathrm{ad})$ or $H(\Lambda)$.

Theorem 2.6. *Let $h_0 \in \mathfrak{h}_{\boldsymbol{R}}$ strictly dominant. The spaces $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ ($m=0, 1, 2, \cdots, \infty, \omega$) are characterized by means of one element h_0 as follows:*

i) *in case of* $(\mathfrak{g}, \mathrm{ad})$,

$$H_m(\mathrm{ad}) = \{y \in \mathfrak{g};\ (\mathrm{ad}\, h_0)^m y \in H(\mathrm{ad})\} \qquad (m \in \boldsymbol{Z}_{\geqq 0}), \tag{1}$$

$$H_\omega(\mathrm{ad}) = \{y \in H_\infty(\mathrm{ad});\ \textit{there exist } \varepsilon > 0 \textit{ such that} \tag{2}$$
$$\sum_{m=0}^{\infty} \frac{1}{m!} \varepsilon^m \|(\mathrm{ad}\, h_0)^m y\|_1 < +\infty\};$$

ii) *in case of* $L(\Lambda)$,

$$H_m(\Lambda) = \{v \in L(\Lambda);\ h_0^m v \in H(\Lambda)\} \qquad (m \in \boldsymbol{Z}_{\geqq 0}), \tag{3}$$

$$H_\omega(\Lambda) = \{v \in H_\infty(\Lambda);\ \textit{there exist } \varepsilon > 0 \textit{ such that} \tag{4}$$
$$\sum_{m=0}^{\infty} \frac{1}{m!} \varepsilon^m \|h_0^m v\|_\Lambda < +\infty\}.$$

Proof. It is clear that

$$\min_{\alpha \in \Delta} |\alpha(h_0)| = \min_{\alpha \in \Pi} |\alpha(h_0)| > 0,$$
$$\min_{\mu \in P(\Lambda), \mu(h_0) \neq 0} |\mu(h_0)| > 0,$$
$$\#\{\mu \in P(\Lambda);\ \mu(h_0) = 0\} < +\infty,$$

Then, we have, for any $y \in \mathfrak{g}$ and any $v \in L(\Lambda)$,

$$[h_0, y] \in H(\mathrm{ad}) \Rightarrow y \in H(\mathrm{ad}),$$
$$h_0 v \in H(\Lambda) \Rightarrow v \in H(\Lambda).$$

These implications together with Proposition 2.2 imply (1) and (3) respectively.

Now, let y be an arbitrary element of the right-hand side of (2). By definition, there is a positive number ε such that

$$\sum_{m \geqq 0} \frac{1}{m!} \varepsilon^m \|(\mathrm{ad}\, h_0)^m y\|_1 < +\infty .$$

By Corollary 2.3, we see that for any $x \in \mathfrak{g}$ and $\delta > 0$,

$$\frac{1}{m!} \delta^m \|(\mathrm{ad}\, x)^m y\|_1 \leqq$$
$$\leqq C_1^m \Big(\sum_{l \geqq 0} \frac{1}{l!} \delta^l \|(\mathrm{ad}\, h_0)^l x\|_1 \Big)^m \sum_{l \geqq 0} \frac{1}{l!} \delta^l \|(\mathrm{ad}\, h_0)^l y\|_1 .$$

Take δ, $\delta' > 0$ so that

$$\delta' \Big(\sum_{l \geqq 0} \frac{1}{l!} \delta^l \|(\mathrm{ad}\, h_0)^l x\|_1 \Big) < C_1^{-1} \quad \text{and} \quad \delta \leqq \varepsilon .$$

Then, it holds that

$$\sum_{m \geqq 0} \frac{1}{m!} (\delta' \delta)^m \|(\mathrm{ad}\, x)^m y\|_1 \leqq$$
$$\leqq \frac{1}{1 - C_1 \delta' \sum_{m \geqq 0} \frac{1}{m!} \delta^m \|(\mathrm{ad}\, h_0)^m x\|_1} \sum_{l \geqq 0} \frac{1}{l!} \delta^l \|(\mathrm{ad}\, h_0)^l y\|_1 < +\infty .$$

Hence, $y \in H_\omega(\mathrm{ad})$ and (2) is proved.

The equality (4) is proved in a similar way as (2). Q.E.D.

§ 3. Topologies on the spaces of differentiable vectors or of analytic vectors.

Now, we consider natural topologies on the spaces of differentiable vectors $H_\infty(\mathrm{ad})$, $H_\infty(\Lambda)$ and those of analytic vectors $H_\omega(\mathrm{ad})$, $H_\omega(\Lambda)$. With respect to these topologies, the spaces $H_\infty(\mathrm{ad})$ and $H_\omega(\mathrm{ad})$ become topological Lie algebras. Particularly, we obtain the Lie algebra $\mathfrak{g}_\omega = H_\omega(\mathrm{ad})$ on which the adjoint action can be exponentiated locally as shown in Corollary 3.8.

3.1. Topologies on the spaces of differentiable vectors.

Definition 3.1. For $m \in \boldsymbol{Z}_{\geqq 0}$, x, $y \in H_m(\mathrm{ad})$, u, $v \in H_m(\Lambda)$, define inner products on $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ respectively as

$$(x \mid y)_{\mathrm{ad},m} = \sum_{l=0}^{m} ((\mathrm{ad}\, h_0)^l x \mid (\mathrm{ad}\, h)^l y)_1 ,$$

$$(u\,|\,v)_{\Lambda,m}=\sum_{l=0}^{m}(h_0^l u\,|\,h_0^l v)_\Lambda\,.$$

Clearly, $(H_m(\mathrm{ad}), (\cdot\,|\,\cdot)_{\mathrm{ad},m})$ and $(H_m(\Lambda), (\cdot\,|\,\cdot)_{\Lambda,m})$ are both Hilbert spaces. On $H_\infty(\mathrm{ad})$ and $H_\infty(\Lambda)$, consider projective limit topologies of these Hilbert spaces.

Using Proposition 2.2, we can prove the following proposition, which gives a topological Lie algebra structure to $H_\infty(\mathrm{ad})$ and its continuous representation on $H_\infty(\Lambda)$.

Proposition 3.2. i) *The bracket product of* $\mathfrak{g}$: $\mathfrak{g}\times\mathfrak{g}\ni(x, y)\mapsto[x, y]\in\mathfrak{g}$, *is extended to a continuous bilinear map from* $H_m(\mathrm{ad})\times H_m(\mathrm{ad})$ *into* $H_{m-1}(\mathrm{ad})$ *for* $m>0$. *In particular,* $H_\infty(\mathrm{ad})$ *is a Lie algebra and its bracket product is continuous.*

ii) *The map* $\mathfrak{g}\times L(\Lambda)\ni(x, v)\mapsto xv\in L(\Lambda)$ *is extended to a continuous bilinear map from* $H_m(\mathrm{ad})\times H_m(\Lambda)$ *into* $H_{m-1}(\Lambda)$ *for* $m>0$. *In paticular,* $H_\infty(\mathrm{ad})\supset\mathfrak{g}$ *acts continuously on* $H_\infty(\Lambda)$.

We denote by $\mathfrak{g}_\infty$ the topological Lie algebra $H_\infty(\mathrm{ad})$.

Proof. i) By Proposition 2.2, we have, for any $x, y\in H_m(\mathrm{ad})$,

$$\|(\mathrm{ad}\,h_0)^{m-1}[x, y]\|_1\leqq C_1^m\sum_{p+q=m}C^{(m)}_{0,\dots,0,p,q}\|(\mathrm{ad}\,h_0)^p x\|_1\cdot\|(\mathrm{ad}\,h_0)^q y\|_1\,.$$

This implies i).

ii) is similarly proved. Q.E.D.

3.2. Topologies on the spaces of analytic vectors.

Definition 3.3. For $\delta>0$, $x\in H_\omega(\mathrm{ad})$, and $v\in H_\omega(\Lambda)$, we define norms on $H_\omega(\mathrm{ad})$ and on $H_\omega(\Lambda)$ which may take the value $+\infty$ as

$$\|x\|_{\mathrm{ad},\omega,\delta}=\sum_{m=0}^{\infty}\frac{1}{m!}\delta^m\|(\mathrm{ad}\,h_0)^m x\|_1\,,$$

$$\|v\|_{\Lambda,\omega,\delta}=\sum_{m=0}^{\infty}\frac{1}{m!}\delta^m\|h_0^m v\|_\Lambda\,.$$

Definition 3.4. We define the series of subspaces of $H_\omega(\mathrm{ad})$ and $H_\omega(\Lambda)$ which are parametrized in $0<\varepsilon\leqq+\infty$, using norms defined above, by

$$H_\omega(\mathrm{ad};\,\varepsilon)=\{x\in H_\omega(\mathrm{ad})\colon \|x\|_{\mathrm{ad},\omega,\delta}<+\infty \text{ for all } 0<\delta<\varepsilon\}\,,$$
$$H_\omega(\Lambda;\,\varepsilon)=\{v\in H_\omega(\Lambda)\colon \|v\|_{\Lambda,\omega,\delta}<+\infty \text{ for all } 0<\delta<\varepsilon\}\,.$$

The subspaces $H_\omega(\mathrm{ad};\,\varepsilon)$ with a family of norms $\{\|\cdot\|_{\mathrm{ad},\omega,\delta}(0<\delta<\varepsilon)\}$ are Fréchet spaces. Similarly, $\{H_\omega(\Lambda;\,\varepsilon), \|\cdot\|_{\Lambda,\omega,\delta}\,(0<\delta<\varepsilon)\}$ are Fréchet spaces.

We see that

$$H_\omega(\mathrm{ad})=\cup_{\varepsilon>0}H_\omega(\mathrm{ad};\,\varepsilon) \quad\text{and}\quad H_\omega(\Lambda)=\cup_{\varepsilon>0}H_\omega(\Lambda;\,\varepsilon)\,.$$

So, we consider inductive limit topologies on $H_\omega(\mathrm{ad})$ and on $H_\omega(\Lambda)$ associated with these family of Fréchet spaces.

Then, we get the following proposition on the continuity of $\mathfrak{g}$-action.

Proposition 3.5. *Fix* $0<\varepsilon\leqq+\infty$, *and let* $0<\delta<\delta'<\varepsilon$.

i) *For* $x, y\in H_\omega(\mathrm{ad}; \varepsilon)$, *it holds that*

$$\|[x, y]\|_{\mathrm{ad},\omega,\delta}\leqq C_1\frac{\delta'}{(\delta'-\delta)^2}\|x\|_{\mathrm{ad},\omega,\delta'}\|y\|_{\mathrm{ad},\omega,\delta'}\,.$$

In paticular, $H_\omega(\mathrm{ad}; \varepsilon)\supset\mathfrak{g}$ *is a subalgebra of* $H_\infty(\mathrm{ad})$ *and its bracket product is continuous.*

ii) *For* $x\in H_\omega(\mathrm{ad}; \varepsilon)$ *and* $v\in H_\omega(\Lambda; \varepsilon)$, *it holds that*

$$\|xv\|_{\Lambda,\omega,\delta}\leqq C_{1,\Lambda}(\|x\|_{\mathrm{ad},\omega,\delta}\|v\|_{\Lambda,\omega,\delta}+\frac{\delta'}{(\delta'-\delta)^2}\|x\|_{\mathrm{ad},\omega,\delta'}\|v\|_{\Lambda,\omega,\delta'})\,.$$

In paticular, $H_\omega(\mathrm{ad}; \varepsilon)\supset\mathfrak{g}$ *acts on* $H_\omega(\Lambda; \varepsilon)$ *continuously.*

We denote by $\mathfrak{g}_{\omega,\varepsilon}$ the topological Lie algebra $H_\omega(\mathrm{ad}; \varepsilon)$.

Proof. By Corollary 2.3, it holds that

$$\begin{aligned}\|(\mathrm{ad}\,h_0)^m[x, y]\|_1&\leqq\textstyle\sum_{l=0}^{m}\binom{m}{l}\|[(\mathrm{ad}\,h_0)^l x, (\mathrm{ad}\,h_0)^{m-l}y]\|_1\\&\leqq C_1\textstyle\sum_{l=0}^{m}\binom{m}{l}(\|(\mathrm{ad}\,h_0)^{l+1}x\|_1\|(\mathrm{ad}\,h_0)^{m-l}y\|_1+\\&\qquad+\|(\mathrm{ad}\,h_0)^l x\|_1\|(\mathrm{ad}\,h_0)^{m-l+1}y\|_1)\\&\leqq C_1\textstyle\sum_{l=0}^{m+1}\binom{m+1}{l}\|(\mathrm{ad}\,h_0)^l x\|_1\|(\mathrm{ad}\,h_0)^{m+1-l}y\|_1\,.\end{aligned}$$

Summing up this estimation over m, we get

$$\begin{aligned}\|[x, y]\|_{\mathrm{ad},\omega,\delta}&=\textstyle\sum_{m=0}^{\infty}\frac{1}{m!}\delta^m\|(\mathrm{ad}\,h_0)^m[x, y]\|_1\\&\leqq C_1\delta^{-1}\textstyle\sum_{m=0}^{\infty}(m+1)\left(\frac{\delta}{\delta'}\right)^{m+1}\sum_{l=0}^{m+1}\frac{1}{l!}\delta'^l\|(\mathrm{ad}\,h_0)^l x\|_1\times\\&\qquad\times\frac{1}{(m+1-l)!}\delta'^{m+1-l}\|(\mathrm{ad}\,h_0)^{m+1-l}y\|_1\\&\leqq C_1\delta^{-1}\|x\|_{\mathrm{ad},\omega,\delta'}\|y\|_{\mathrm{ad},\omega,\delta'}\textstyle\sum_{m\geq1}m\left(\frac{\delta}{\delta'}\right)^m\,.\end{aligned}$$

The proof for $H_\omega(\Lambda; \varepsilon)$ is similar as for $H_\omega(\mathrm{ad}; \varepsilon)$. Q.E.D.

Corollary 3.6. i) $\mathrm{H}_\omega(\mathrm{ad})\supset\mathfrak{g}$, *with the inductive limit topology, is a subalgebra of* $H_\infty(\mathrm{ad})$, *with the projective limit topology. Moreover, Lie algebra operation on* $H_\omega(\mathrm{ad})$ *is continuous. Thus,* $H_\omega(\mathrm{ad})$ *is a topological Lie algebra.*

ii) *The topological Lie algebra* $H_\omega(\mathrm{ad})$ *acts continuously on* $H_\omega(\Lambda)$ *with the inductive limit topology.*

We denote by $\mathfrak{g}_\omega$ the topological Lie algebra $H_\omega(\mathrm{ad})$.

3.3. Exponentiation of $\mathfrak{g}_\omega$-action. The next lemma and its corollary are concerned with the exponentiation of $\mathfrak{g}_\omega$-action, and will be needed in the next section.

Lemma 3.7. *Fix* $0<\varepsilon\leqq+\infty$, *and let* δ, $\delta'>0$, $\delta+\delta'<\varepsilon$.

i) *Let* $x\in\mathfrak{g}_{\omega,\varepsilon}$, $y\in H_\omega(\mathrm{ad};\varepsilon)\supset\mathfrak{g}$. *If* $C_1\delta'^{-1}\|x\|_{\mathrm{ad},\omega,\delta+\delta'}<1$, *then it holds that*

$$\sum_{k,m\geqq 0}\frac{1}{k!m!}\|\delta^k(\mathrm{ad}\,h_0)^k(\mathrm{ad}\,x)^m y\|_1\leqq$$
$$\leqq(1-C_1\delta'^{-1}\|x\|_{\mathrm{ad},\omega,\delta+\delta'})^{-1}\|y\|_{\mathrm{ad},\omega,\delta+\delta'}\,.$$

ii) *Let* $x\in\mathfrak{g}_{\omega,\varepsilon}$ *and* $v\in H_\omega(\Lambda;\varepsilon)$. *If* $C_{1,\Lambda}(1+\delta'^{-1})\|x\|_{\mathrm{ad},\omega,\delta+\delta'}<1$, *then it holds that*

$$\sum_{k,m\geqq 0}\frac{1}{k!m!}\delta^k\|h_0^k x^m v\|_\Lambda\leqq$$
$$\leqq(1-C_{1,\Lambda}(1+\delta'^{-1})\|x\|_{\mathrm{ad},\omega,\delta+\delta'})^{-2}\|v\|_{\Lambda,\omega,\delta+\delta'}\,.$$

Proof. i) For any k, $m\in\boldsymbol{Z}_{\geqq 0}$ and $x\in\mathfrak{g}_{\omega,\varepsilon}$, $y\in H_\omega(\mathrm{ad};\varepsilon)$, we have, by Corollary 2.3,

$$\|(\mathrm{ad}\,h_0)^k(\mathrm{ad}\,x)^m y\|_1\leqq$$
$$\leqq\sum'\frac{k!}{q_1!\cdots q_m!q_0!}\|[(\mathrm{ad}\,h_0)^{q_1}x,\cdots,[(\mathrm{ad}\,h_0)^{q_m}x,(\mathrm{ad}\,h_0)^{q_0}y]\cdots]\|_1$$
$$\leqq C_1^m\sum''\sum'\frac{k!m!}{p_1!\cdots p_m!p_0!q_1!\cdots q_m!q_0!}\times$$
$$\times\{\Pi_{j=1}^m\|(\mathrm{ad}\,h_0)^{p_j+q_j}x\|_1\}\cdot\|(\mathrm{ad}\,h_0)^{p_0+q_0}y\|_1\,,$$

where $\sum'$ means the sum running over all the non-negative integers $q_1,\cdots,q_m,q_0$ such that $q_1+\cdots+q_m+q_0=k$ and $\sum''$ the sum running over all the non-negative integers $p_1,\cdots,p_m,p_0$ such that $p_1+\cdots+p_m+p_0=m$.

Hence, it holds that

$$\sum_{k,m\geqq 0}\frac{1}{k!m!}\delta^k\delta'^m\|(\mathrm{ad}\,h_0)^k(\mathrm{ad}\,x)^m y\|_1\leqq\sum_{m\geqq 0}C_1^m\delta'^m\sum\frac{1}{p_1!\cdots p_m!p_0!}\times$$
$$\times\left\{\Pi_{j=1}^m\sum_{q\geqq 0}\frac{1}{q!}\delta^q\|(\mathrm{ad}\,h_0)^{p_j+q}x\|_1\right\}\left(\sum_{q\geqq 0}\frac{1}{q!}\delta^q\|(\mathrm{ad}\,h_0)^{p_0+q}y\|_1\right)$$
$$\leqq\sum_{m\geqq 0}\left\{C_1\sum_{p,q\geqq 0}\frac{1}{p!q!}\delta'^p\delta^q\|(\mathrm{ad}\,h_0)^{p+q}x\|_1\right\}^m\times$$
$$\times\left(\sum_{p,q\geqq 0}\frac{1}{p!q!}\delta'^p\delta^q\|(\mathrm{ad}\,h_0)^{p+q}y\|_1\right)=$$
$$=(1-C_1\|x\|_{\mathrm{ad},\omega,\delta+\delta'})^{-1}\|y\|_{\mathrm{ad},\omega,\delta+\delta'}\,.$$

Replacing x in this inequality with $\delta'^{-1}x$, we get the assertion i).

ii) is proved similarly as i). Q.E.D.

Corollary 3.8. *Fix* $0<\varepsilon'<\varepsilon\leqq+\infty$, *and let* $0<\delta<\varepsilon-\varepsilon'$. *Let* $x\in\mathfrak{g}_{\omega,\varepsilon}$, *and* $\zeta\in\boldsymbol{C}$.

i) *If* $|\zeta|<\delta(C_1\|x\|_{\mathrm{ad},\omega,\varepsilon'+\delta})^{-1}$, *then*

(1) *the series* $\sum_{m\geqq0}\dfrac{1}{m!}(\mathrm{ad}\,\zeta x)^m y$ *is absolutely convergent in* $H(\mathrm{ad})$ *and the sum* $(\exp\mathrm{ad}\,\zeta x)y=\sum_{m\geqq0}\dfrac{1}{m!}(\mathrm{ad}\,\zeta x)^m y$ *belongs to* $H_\omega(\mathrm{ad};\varepsilon')$ *for any* $y\in H_\omega(\mathrm{ad};\varepsilon)$,

(2) *the map*

$$\exp(\mathrm{ad}\,\zeta x)\colon H_\omega(\mathrm{ad};\varepsilon)\ni y\mapsto\sum_{m\geqq0}\frac{1}{m!}(\mathrm{ad}\,\zeta x)^m y\in H_\omega(\mathrm{ad};\varepsilon')$$

is continuous.

ii) *If* $|\zeta|<(1+\delta^{-1})^{-1}(C_{1\Lambda}\|x\|_{\mathrm{ad},\omega,\varepsilon'+\delta})^{-1}$, *then*

(1) *the series* $\sum_{m\geqq0}\dfrac{1}{m!}(\zeta x)^m v$ *is absolutely convergent in* $H(\Lambda)$ *and the sum* $(\exp\zeta x)v=\sum_{m\geqq0}\dfrac{1}{m!}(\zeta x)^m v$ *belongs to* $H_\omega(\Lambda;\varepsilon')$ *for any* $v\in H_\omega(\Lambda;\varepsilon)$,

(2) *the map*

$$\exp\zeta x\colon H_\omega(\Lambda;\varepsilon)\ni v\mapsto\sum_{m\geqq0}\frac{1}{m!}(\zeta x)^m v\in H_\omega(\Lambda;\varepsilon')$$

is continuous.

Remark 3.9. Let $x\in\mathfrak{g}_\omega$. In Corollary 3.8, the range of $\zeta\in\boldsymbol{C}$, in which the exponentials $e^{\mathrm{ad}\,\zeta x}y$ ($y\in H_\omega(\mathrm{ad})$) or $e^{\zeta x}v$ ($v\in H_\omega(\Lambda)$) can be defined, depends heavily on y or v. In this sense, the definition of exponentials on $\mathfrak{g}_\omega$ by means of absolutely convergent series is local in $y\in H_\omega(\mathrm{ad})$ or $v\in H_\omega(\Lambda)$. But, restricting ourselves to the closure $H_1^u(\mathrm{ad})$ of the unitary form $\mathfrak{k}$ in $H_1(\mathrm{ad})$, we get globally defined exponentials as shown in the next section.

3.4. Relations between the spaces defined above. Here, for convinience of readers, we illustrate the relations between the spaces introduced in § 2.

3.4.1. Continuous inclusions.

$$\begin{array}{l}
\underline{\mathfrak{g}}=\prod_{\alpha\in\Delta\cup(0)}\mathfrak{g}^\alpha\\
\cup\\
H(\mathrm{ad})=H_0(\mathrm{ad})\supset H_1(\mathrm{ad})\supset H_2(\mathrm{ad})\supset\cdots\supset\mathfrak{g}_\infty=H_\infty(\mathrm{ad})=\bigcap_{m\geqq0}H_m(\mathrm{ad})\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\cup\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\mathfrak{g}_\omega=H_\omega(\mathrm{ad})=\bigcup_{0<\varepsilon\leqq+\infty}H_\omega(\mathrm{ad};\varepsilon)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\cup\\
\mathfrak{g}\subset\mathfrak{g}_{\omega,+\infty}\subset\cdots\subset\mathfrak{g}_{\omega,\varepsilon}=H_\omega(\mathrm{ad};\varepsilon)\subset\cdots\subset\mathfrak{g}_{\omega,\delta}=H_\omega(\mathrm{ad};\delta)\quad(0<\delta<\varepsilon\leqq+\infty).
\end{array}$$

$$\begin{array}{l}
\underline{L}(\Lambda)=\prod_{\mu\in P(\Lambda)}L(\Lambda)_\mu\\
\cup\\
H(\Lambda)=H_0(\Lambda)\supset H_1(\Lambda)\supset H_2(\Lambda)\supset\cdots\supset H_\infty(\Lambda)=\bigcap_{m\leqq0}H_m(\Lambda)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\cup\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad H_\omega(\Lambda)=\bigcup_{0>\varepsilon\leqq+\infty}H_\omega(\Lambda;\varepsilon)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\cup\\
L(\Lambda)\subset H_\omega(\Lambda;+\infty)\subset\cdots\subset H_\omega(\Lambda;\varepsilon)\subset\cdots\subset H_\omega(\Lambda;\delta)\quad(0<\delta<\varepsilon\leqq+\infty).
\end{array}$$

3.4.2. Continuous actions of subspaces of $\boldsymbol{H}$(ad).

$$
\begin{array}{ll}
[H_m(\mathrm{ad}),\, H_m(\mathrm{ad})]\subset H_{m-1}(\mathrm{ad}), & H_m(\mathrm{ad})\cdot H_m(\Lambda)\subset H_{m-1}(\Lambda) \quad (0<m<+\infty)\,. \\
[\mathfrak{g}_\infty,\, \mathfrak{g}_\infty]\subset \mathfrak{g}_\infty & ,\ \mathfrak{g}_\infty\cdot H_\infty(\Lambda)\subset H_\infty(\Lambda)\,. \\
[\mathfrak{g}_\omega,\, \mathfrak{g}_\omega]\subset \mathfrak{g}_\omega & ,\ \mathfrak{g}_\omega\cdot H_\omega(\Lambda)\subset H_\omega(\Lambda)\,. \\
[\mathfrak{g}_{\omega,\varepsilon},\, \mathfrak{g}_{\omega,\varepsilon}]\subset \mathfrak{g}_{\omega,\varepsilon} & ,\ \mathfrak{g}_{\omega,\varepsilon}\cdot H_\omega(\Lambda;\,\varepsilon)\subset H_\omega(\Lambda;\,\varepsilon) \quad (0<\varepsilon\leqq+\infty)\,.
\end{array}
$$

§ 4. 1-parameter groups exp t(ad x) and exp tx.

In this section, we construct the exponentials of each element x in the closure $H_1^u(\mathrm{ad})$ of unitary form $\mathfrak{k}$ in $H_1(\mathrm{ad})$ as an operator on $H(\mathrm{ad})$ and on $H(\Lambda)$.

4.1. Completions of the unitary form. Since the $*$-operation on $\mathfrak{g}$ is isometric with respect to $(\cdot\,|\,\cdot)_1$ and h_0 is $*$-invariant, all the spaces $H_m(\mathrm{ad})\subset H(\mathrm{ad})$ defined in § 2 are $*$-invariant and $*$-operation is isometric for $m\in \boldsymbol{Z}_{\geqq 0}$, and bicontinuous for $m=\infty$ or ω. And so, we can define the completions of unitary form as follows.

Definitions 4.1. We define real subspaces of $H(\mathrm{ad})$ as

$$
\begin{aligned}
H_m^u(\mathrm{ad}) &= \{x\in H_m(\mathrm{ad});\ x+x^* = 0\} \qquad (m = 0,\, 1,\, \cdots,\, \infty,\, \omega)\,, \\
H_\omega^u(\mathrm{ad};\,\varepsilon) &= \{x\in H_\omega(\mathrm{ad};\,\varepsilon);\ x+x^* = 0\} \qquad (0<\varepsilon\leqq+\infty);
\end{aligned}
$$

and define real Lie subalgebras $\mathfrak{k}_\infty$, $\mathfrak{k}_\omega$ of $\mathfrak{g}_\infty=H_\infty(\mathrm{ad})$, $\mathfrak{g}_\omega=H_\omega(\mathrm{ad})$ respectively as

$$
\mathfrak{k}_\infty = H_\infty^u(\mathrm{ad})\,, \quad \mathfrak{k}_\omega = H_\omega^u(\mathrm{ad})\,.
$$

Our goal of this section is to prove the following theorem. We denote by $\boldsymbol{B}(H)$ the set of all bounded linear operators on a Hilbert space H and by $\boldsymbol{U}(H)$ the group of all the unitary operators on H.

Theorem 4.2. *Take an arbitrary element* $x=\sum_{\alpha\in\Delta\cup\{0\}} x_\alpha\in H_1^u(\mathrm{ad})$.

i) *There exists a unique* 1*-parameter group of operators* $\exp t(\mathrm{ad}\, x)=e^{t\,\mathrm{ad}\,x}$ *in* $\boldsymbol{B}(H(\mathrm{ad}))$ *whose infinitesimal generator contains* ad x:

$$
\frac{d}{dt}\{(\exp t(\mathrm{ad}\, x))y\} = (\exp t(\mathrm{ad}\, x))[x,\, y] \qquad \textit{for all} \quad y\in H_1(\mathrm{ad})\,.
$$

Moreover, the operator norm of $\exp t(\mathrm{ad}\, x)$ *is evaluated as*

$$
\|(\exp t(\mathrm{ad}\, x)\|_{op}\leqq \exp(2|t|(\textstyle\sum_{\alpha\in\Delta}\|\alpha\|_1^2\|x_\alpha\|_1^2)^{1/2}) \qquad \textit{for all} \quad t\in\boldsymbol{R}\,.
$$

ii) *There exists a unique* 1*-parameter group* $\exp tx=e^{tx}$ *in* $\boldsymbol{U}(H(\Lambda))=\boldsymbol{U}(\Lambda)$ *whose infinitesimal generator contains* x:

$$
\frac{d}{dt}\{(\exp tx)v\} = (\exp tx)xv \qquad \text{for all} \quad v\in H_1(\Lambda)\,.
$$

4.2. Resolvents of the closures of operators ad x **on** H(ad) **with** x **in** H_1^u(ad). For the proof of Theorem 4.2, we need several lemmas.

Recall the operator T on $\underline{\mathfrak{g}}$ defined in 1.4. For any $x, y, z \in H_1(\mathrm{ad}) \subset \underline{\mathfrak{g}}$, we have

$$((T\circ(\mathrm{ad}\ x))y\,|\,z)_1 = ([x, y]\,|\,z)_0 = (y\,|\,[x^*, z_0]) = (y\,|\,(T\circ(\mathrm{ad}\ x^*))z)_1 .$$

Hence,

$$(T\circ(\mathrm{ad}\ x))^+ = T\circ(\mathrm{ad}\ x^*) \qquad \text{for all} \quad x \in H_1(\mathrm{ad}) . \tag{4.1}$$

where $^+$ means the adjoint with respect to $(\cdot\,|\,\cdot)_1$. Therefore, $T\circ(\mathrm{ad}\ x)$ has the densely defined adjoint, and so $T\circ(\mathrm{ad}\ x)$ is closable. Hence, ad x is also closable for any $x \in H_1(\mathrm{ad})$ because T is unitary. We denote the closure of ad x again by the same symbol ad x and its domain by $H(\mathrm{ad}; x)$.

Let $x \in H_1^u(\mathrm{ad})$. Because of (4.1) and $x+x^*=0$, $T\circ(\mathrm{ad}\ x)$ is anti-symmetric. Hence, it holds that for $y \in H(\mathrm{ad}; x)$

$$\begin{aligned} \|(1-\mathrm{ad}\ x)y\|_1^2 &= \|y\|_1^2 + \|[x, y]\|_1^2 - 2\,\mathrm{Re}\,(y\,|\,[x, y])_1 \\ &= \|y\|_1^2 + \|[x, y]\|_1^2 - 2\,\mathrm{Re}\,(y\,|\,(1-T)[x, y])_1 . \end{aligned}$$

By (1.13), we see that

$$\mathrm{Re}\,(y\,|\,(1-T)[x, y])_1 \leqq 2\|y\|_1 \|P_0([x, y])\|_1 ,$$

where P_0 is the projection from $\mathfrak{g}$ onto $\mathfrak{h}$ defined in 1.1. Further, using the equality (1.4), we have

$$\begin{aligned} \|P_0([x, y])\|_1 &\leqq \textstyle\sum_{\alpha\in\Delta} \|[x_\alpha, y_{-\alpha}]\|_1 \\ &= \textstyle\sum_{\alpha\in\Delta} |(x_\alpha\,|\,y_{-\alpha})|\,\|\nu^{-1}(\alpha)\|_1 \\ &\leqq \textstyle\sum_{\alpha\in\Delta} \|x_\alpha\|_1 \|y_{-\alpha}\|_1 \|\alpha\|_1 \\ &\leqq (\textstyle\sum_{\alpha\in\Delta} \|\alpha\|_1^2 \|x_\alpha\|_1^2)^{1/2} (\textstyle\sum_{\alpha\in\Delta} \|y_\alpha\|_1^2)^{1/2} . \\ &\leqq (\textstyle\sum_{\alpha\in\Delta} \|\alpha\|_1^2 \|x_\alpha\|_1^2)^{1/2} \cdot \|y\|_1 . \end{aligned}$$

Thus, we obtain the following estimate.

Lemma 4.3. *For any* $x \in H_1^u(\mathrm{ad})$ *and any* $y \in H(\mathrm{ad}; x)$, *it holds that*

$$\|(1-\mathrm{ad}\ x)y\|_1^2 \geqq \{1 - 4\,(\textstyle\sum_{\alpha\in\Delta} \|\alpha\|_1^2 \|x_\alpha\|_1^2)^{1/2}\} \|y\|_1^2 + \|[x, y]\|_1^2 .$$

Remark 4.4. By the same argument as in the proof of Theorem 2.6, we see that $x \in \mathfrak{g}$ belongs to $H_1(\mathrm{ad})$ if and only if

$$\textstyle\sum_{\alpha\in\Delta} \|\alpha\|_1^2 \|x_\alpha\|_1^2 < +\infty ,$$

whence the above inequality has a sense for $x \in H_1^u(\mathrm{ad})$.

We see that if $x \in H_2^u(\mathrm{ad})$, and $\varepsilon \in \boldsymbol{R}$ is sufficiently small, then $(1-\mathrm{ad}\ \varepsilon x)H_1(\mathrm{ad})$ is dense in $H(\mathrm{ad})$, which is proved in another paper [10, § 4], and hence we have the following.

Lemma 4.5. *Let* $x \in H_2^u(\mathrm{ad})$. *If* $\varepsilon \in \boldsymbol{R}$ *is sufficiently small, then there holds that*

$$H(\mathrm{ad}) = (1-\mathrm{ad}\ \varepsilon x)H(\mathrm{ad};\ x)\ .$$

Corollary 4.6. *Let* $x \in H_2^u(\mathrm{ad})$. *For any sufficiently small* $\varepsilon \in \boldsymbol{R}$, *the inverse* $(1-\mathrm{ad}\ \varepsilon x)^{-1}$ *exists and belongs to* $B(H(\mathrm{ad}))$. *Further, we have*

$$\|(1-\mathrm{ad}\ \varepsilon x)^{-1}\|_{op} \leqq \{1-4|\varepsilon|(\textstyle\sum_{\alpha\in\Delta}\|\alpha\|_1^2\|x_\alpha\|_1^2)^{1/2}\}^{-1/2}\ .$$

4.3. Proof of Theorem 4.2. We apply to the closed operator ad x on $H(\mathrm{ad})$ the criterions in [9, Chap. IX] for exponentiability of a closed operator on a Banach space.

i) For $0 \leqq s < 1$, it holds that

$$(1-s)^{1/2} \geqq 1-\frac{1}{2}s\left(1+\frac{s}{4(1-s)}\right)$$

Take $0<\delta<1$. If $4|\varepsilon|(\sum_{\alpha\in\Delta}\|\alpha\|_1^2\|x_\alpha\|_1^2)^{1/2}<\delta$, then, by the above formula, we have

$$\begin{aligned}&\{1-4|\varepsilon|(\textstyle\sum_{\alpha\in\Delta}\|\alpha\|_1^2\|x_\alpha\|_1^2)^{1/2}\}^{-1/2} \leqq \\ &\leqq \left\{1-2|\varepsilon|(\textstyle\sum_{\alpha\in\Delta}\|\alpha\|_1^2\|x_\alpha\|_1^2)^{1/2}\left(1+\frac{\delta}{4(1-\delta)}\right)\right\}^{-1}.\end{aligned}$$

Hence, by [9, Chap. IX §9, Cor. 1] and Corollary 4.6, there exists a unique continuous 1-parameter group $\exp t(\mathrm{ad}\ x)$ in $\mathbf{B}(H(\mathrm{ad}))$ such that its infinitesimal generator is equal to ad x, the closed one, and

$$\|\exp t(\mathrm{ad}\ x)\|_{op} \leqq \exp\left\{2|t|(\textstyle\sum_{\alpha\in\Delta}\|\alpha\|_1^2\|x_\alpha\|_1^2)^{1/2}\left(1+\frac{\delta}{4(1-\delta)}\right)\right\}.$$

Since δ may be arbitrarily small, i) is proved.

ii) is proved in a similar way but much more easily than i), because any element x in $H_1^u(\mathrm{ad})$ acts as an anti-symmetric operator on $H(\Lambda)$. Q.E.D.

4.4. A consequence of Theorem 4.2. By Theorem 4.2, we see that differentiable vectors in $H(\mathrm{ad})$ or $H(\Lambda)$ defined in 2.1 are really defferentiable as expected.

Proposition 4.7. *For* $m \in \boldsymbol{Z}_{\geqq 0}$, *every vector* $y \in H_m(\mathrm{ad})$ *or* $v \in H_m(\Lambda)$ *is m-times differentiable in the sense that for any* $x \in H_m^u(\mathrm{ad})$, *it holds that*

$$\frac{d^m}{dt^m}\{(\exp t(\mathrm{ad}\ x))y\} = (\exp t(\mathrm{ad}\ x))(\mathrm{ad}\ x)^m y\ ,$$

$$\frac{d^m}{dt^m}\{(\exp tx)v\} = (\exp tx)x^m v\ .$$

§ 5. Properties of exponentials exp ad $\mathfrak{k}_\omega$ and exp $\mathfrak{k}_\omega$.

In this section, we prove some properties of the exponentials exp ad $\mathfrak{k}_\omega$ and exp $\mathfrak{k}_\omega$. The first one, one of our main results, says that the spaces of differentiable vectors and of analytic vectors are invariant under the action of the exponentials. The others concern the natural relations of actions of the exponentials.

5.1. Invariance of the spaces of differentiable vectors and of analytic vectors under exp ad $\mathfrak{k}_\omega$ or exp $\mathfrak{k}_\omega$. At first, we show that the analytic vectors $y \in H_\omega(\mathrm{ad})$ and $v \in H_\omega(\Lambda)$ are really analytic, similarly as differentiable vectors in Proposition 4.7.

Proposition 5.1. *Let* $x \in \mathfrak{k}_\omega$.

i) *For any* $y \in H_\omega(\mathrm{ad})$, *there exists* $\varepsilon > 0$ *such that if* $|t| < \varepsilon$, *the series* $\sum_{m \geqq 0} \frac{1}{m!} t^m (\mathrm{ad}\, x)^m y$ *is absolutely convergent in* $H(\mathrm{ad})$ *and equals to* $(\exp t(\mathrm{ad}\, x))y$.

ii) *For any* $v \in H_\omega(\Lambda)$, *there exists* $\varepsilon > 0$ *such that if* $|t| < \varepsilon$, *the series* $\sum_{m \geqq 0} \frac{1}{m!} t^m x^m v$ *is absolutely convergent in* $H(\Lambda)$ *and is equal to* $(\exp tx)v$.

Proof. i) Absolutely convergence is already proved in Corollary 3.8.

According to [9, Chap. IX], the 1-parameter group $\exp t(\mathrm{ad}\, x)$ is defined by

$$(*) \qquad (\exp t(\mathrm{ad}\, x))z = \lim_{\delta \to 0} \sum_{m \geqq 0} \frac{1}{m!} t^m (\mathrm{ad}\, x)^m (1 - \delta(\mathrm{ad}\, x))^{-m} z\,, \qquad (z \in H(\mathrm{ad}))\,.$$

Here, we have

$$(1 - \delta(\mathrm{ad}\, x))^{-m} - 1 = \delta(\mathrm{ad}\, x) \sum_{j=1}^{m} (1 - \delta(\mathrm{ad}\, x))^{-j}\,.$$

Let ε_1, ε_2 be positive numbers such that $||x||_{\mathrm{ad},\omega,\varepsilon_1}$, $||y||_{\mathrm{ad},\omega,\varepsilon_1} < +\infty$ and that

$$\{1 - 4\varepsilon_2 (\sum_{\alpha \in \Delta} ||\alpha||_1^2 \, ||x_\alpha||_1^2)^{1/2}\}^{-1/2} \leqq 2\,.$$

For this ε_1, take $\varepsilon > 0$ so that

$$\varepsilon < \varepsilon_1 / (2C_1 ||x||_{\mathrm{ad},\omega,\delta_1})\,.$$

Then, by Corollaries 3.8 and 4.6, for any $0 < \delta \leqq \varepsilon_2$ and $|t| \leqq \varepsilon$, it holds that,

$$\begin{aligned} &\sum_{m \geqq 0} \frac{1}{m!} ||t^m (\mathrm{ad}\, x)^m (1 - (\mathrm{ad}\, x))\delta^{-m} y - t^m (\mathrm{ad}\, x)^m y||_1 \\ &\leqq \sum_{m \geqq 1} \frac{1}{m!} ||\delta(\mathrm{ad}\, x)(\sum_{j=1}^{m} (1 - \delta(\mathrm{ad}\, x))^{-j}) t^m (\mathrm{ad}\, x)^m y||_1 \\ &\leqq \delta \sum_{m \geqq 1} \frac{1}{(m-1)!} ||(\mathrm{ad}\, x)(2t(\mathrm{ad}\, x))^m y||_1\,. \end{aligned}$$

Since δ may be arbitrarily small, the sum $\sum_{m \geqq 0} \frac{1}{m!} t^m (\mathrm{ad}\, x)^m y$ is equal to $e^{t(\mathrm{ad}\, x)}$ by $(*)$.

Proof of ii) is similar as for i). Q.E.D.

Lemma 5.2. *Let* $0 < \delta < \varepsilon \leqq +\infty$, $x \in H_\omega^u(\mathrm{ad}; \varepsilon)$ *and* $y \in \mathfrak{g}$. *Let* C_1 *and* $C_{1,\Lambda}$ *be the constants in Proposition* 2.1.

i) *If* $C_1\|x\|_{\mathrm{ad},\omega,\delta}<\delta$, *then there holds the equality*

$$[y, e^{\mathrm{ad}\,x}z] = e^{\mathrm{ad}\,x}[e^{-\mathrm{ad}\,x}y, z] \qquad \textit{for all} \quad z\in H_1(\mathrm{ad})\,.$$

ii) *If* $\max(C_1, C_{1,\Lambda})\|x\|_{\mathrm{ad},\omega,\delta}<(1+\delta^{-1})^{-1}$, *then there holds the equality*

$$ye^x v = e^x(e^{-\mathrm{ad}\,x}y)v \qquad \textit{for all} \quad v\in H_1(\Lambda)\,.$$

Proof. We begin with ii). For any $v\in H_1(\Lambda)$, there exists a sequence $\{v_j\}$ in $L(\Lambda)$ which converges to v in $H_1(\Lambda)$.

Let $u\in L(\Lambda)$. By Proposition 5.1 and Corollary 3.8, it holds that

$$(e^x(e^{-\mathrm{ad}\,x}y)v\,|\,u)_\Lambda = \Big(\Big(\sum_{m\geq 0}\frac{1}{m!}(-\mathrm{ad}\,x)^m y\Big)(\lim_j v_j)\,|\,\sum_{l\geq 0}\frac{1}{l!}(-x)^l u\Big)_\Lambda.$$

Since $e^{-\mathrm{ad}\,x}y=\sum_{m\geq 0}\frac{1}{m!}(-\mathrm{ad}\,x)^m y\in H_\omega(\mathrm{ad})\subset H_1(\mathrm{ad})$ by Corollary 3.8, and each element in $H_1(\mathrm{ad})$ defines a continuous linear map from $H_1(\Lambda)$ into $H(\Lambda)$ by Proposition 3.2, we have

$$\begin{aligned}(e^x(e^{-\mathrm{ad}\,x}y)v\,|\,u)_\Lambda &= \lim_j \sum_{m,l\geq 0}\frac{1}{m!\,l!}(x^l((-\mathrm{ad}\,x)^m y)v_j\,|\,u)_\Lambda \\ &= \lim_j \sum_{k\geq 0}\frac{1}{k!}(yx^k v_j)\,|\,u)_\Lambda \\ &= \lim_j \sum_{k\geq 0}\frac{1}{k!}(v_j\,|\,(-x)^k y^* u)_\Lambda \\ &= \lim_j (v_j\,|\,e^{-x}y^* u)_\Lambda \\ &= (ye^x v\,|\,u)_\Lambda\,.\end{aligned}$$

This implies ii), because $(\cdot\,|\,\cdot)_\Lambda$ gives a non-degenerate pairing of $\underline{L}(\Lambda)\times L(\Lambda)$ as noted in 1.5.

i) is proved quite similarly as ii), using $(\cdot\,|\,\cdot)_0$ (not $(\cdot\,|\,\cdot)_1$). Q.E.D.

Now, we prove another main result which advances our study on fine structures of our group K^Λ in [8, § 3] associated with the unitary form $\mathfrak{k}$.

Theorem 5.3. *Let* $x\in\mathfrak{k}_\omega$ *and* $m=0, 1, 2, \cdots, \infty, \omega$.

i) $e^{\mathrm{ad}\,x}$ *leaves invariant each* $H_m(\mathrm{ad})$, *and the restriction of* $e^{\mathrm{ad}\,x}$ *to* $H_m(\mathrm{ad})$ *is continuous with respect to its topology defined in* § 3.

ii) e^x *leaves invariant each* $H_m(\Lambda)$, *and the restriction of* e^x *to* $H_m(\Lambda)$ *is continous with respect to its topology defined in* § 3.

Proof. i) Take $\delta>0$ so that

$$\|x\|_{\mathrm{ad},\omega,\delta}<+\infty\,.$$

Clearly, we may assume that

$$C_1\delta^{-1}\|x\|_{\mathrm{ad},\omega,\delta}<1\,.$$

(I) The Case $0 \leqq m < +\infty$. For any $y \in H_m(\mathrm{ad})$, we have by Lemma 5.2

$$\begin{aligned} \|h_0^m e^{\mathrm{ad}\, x} y\|_1 &= \|e^{\mathrm{ad}\, x}(e^{-\mathrm{ad}\, x} h_0)^m y\|_1 \\ &\leqq \|e^{\mathrm{ad}\, x}\|_{op} \|(e^{-\mathrm{ad}\, x} h_0)^m y\|_1 , \end{aligned}$$

and $e^{-\mathrm{ad}\, x} h_0 \in H_m(\mathrm{ad})$ by Corollary 3.8. Hence, the assertion is clear from Proposition 2.2 i).

(II) The case $m = \infty$ is immediate from the definition of $H_\infty(\mathrm{ad})$ and the Case (I).

(III) The Case $m = \omega$. Since $e^{-\mathrm{ad}\, x} h_0 \in H_\omega(\mathrm{ad})$, by Corollary 3.8, there exists, for any $y \in H_\omega(\mathrm{ad})$, $\varepsilon > 0$ such that

$$\sum_{m \geqq 0} \frac{1}{m!} \varepsilon^m \|(\mathrm{da}\, e^{-\mathrm{ad}\, x} h_0)^m y\|_1 < +\infty .$$

Hence, by Lemma 5.2, we have the evaluation

$$\sum_{m \geqq 0} \frac{1}{m!} \varepsilon^m \|(\mathrm{ad}\, h_0)^m e^{\mathrm{ad}\, x} y\|_1 \leqq \|e^{\mathrm{ad}\, x}\|_{op} \cdot \sum_{m \geqq 0} \frac{1}{m!} \varepsilon^m \|(\mathrm{ad}\, e^{-\mathrm{ad}\, x} h_0)^m y\|_1 < +\infty .$$

Here, $\|e^{\mathrm{ad}\, x}\|_{op}$ denotes the operator norm of $e^{\mathrm{ad}\, x}$ as an operator on $H(\mathrm{ad})$. Therefore, $e^{\mathrm{ad}\, x} y \in H_\omega(\mathrm{ad})$, whence $H_\omega(\mathrm{ad})$ is left invariant under $e^{\mathrm{ad}\, x}$.

The continuity follows from the above inequality and Proposition 3.7.

ii) is proved in a quite similar way as i). Q.E.D.

5.2. Mutual relations of actions of exponentials. Finally, we complete the assertions in Lemma 5.2, taking off many restrictive assumptions in it, as follows.

Proposition 5.4. *Let* $x \in \mathfrak{k}_\omega$ *and* $y \in H_1(\mathrm{ad})$.

i) *For any* $z \in H_1(\mathrm{ad})$, *there holds the equality*

$$[y, e^{\mathrm{ad}\, x} z] = e^{\mathrm{ad}\, x}[e^{-\mathrm{ad}\, x} y, z] .$$

ii) *For any* $v \in H_1(A)$, *there holds the equality*

$$y e^x v = e^x (e^{-\mathrm{ad}\, x} y) v .$$

Proof is immediate from continuity of $e^{\mathrm{ad}\, x}$ and e^x as operators on $H_1(\mathrm{ad})$ and $H_1(A)$ respectively.

Proposition 5.5. *For any* $x \in \mathfrak{k}_\omega$ *and* $y \in H_1^u(\mathrm{ad})$, *it holds that*

i) $$e^{\mathrm{ad}\, x} e^{\mathrm{ad}\, y} e^{-\mathrm{ad}\, x} = e^{\mathrm{ad}\, (e^{\mathrm{ad}\, x} y)} ,$$

ii) $$e^x e^y e^{-x} = e^{(e^{\mathrm{ad}\, x} y)} .$$

Proof. Both sides of i) are continuous 1-parameter groups in $\boldsymbol{B}(H(\mathrm{ad}))$ with the same infinitesimal generator $\mathrm{ad}(e^{\mathrm{ad}\, x} y)$, by Proposition 5.4. Hence, they are the same thing.

ii) is valid by the same reason as i). Q.E.D.

Let K^{Λ}_{ω} be the subgroup of $\boldsymbol{U}(\Lambda)$, the unirary group on $H(\Lambda)$, generated by $\exp \mathfrak{k}_{\omega}$, and let K^{ad}_{ω} be the subgroup of $\boldsymbol{B}(H(\mathrm{ad}))^{\times}$, the group of all the invertible elements in $\boldsymbol{B}(H(\mathrm{ad}))$, generated by $\exp \mathrm{ad}\, \mathfrak{k}_{\omega}$. If Λ is strictly dominant, the last two propositions enable us to define a group homomorphism Ad from K^{Λ}_{ω} into K^{ad}_{ω} such that

$$g\cdot x\cdot g^{-1}\cdot v = ((\mathrm{ad}\; g)x)v \qquad \text{for any} \quad g\in K^{\Lambda}_{\omega},\, x\in H_1(\mathrm{ad}),\, v\in H(\Lambda_1)\,.$$

Thus we get the adjoint representation of K^{Λ}_{ω} through the homomorphism Ad. Furthermore, Proposition 5.4 connects the group structure of K^{Λ}_{ω} and of K^{ad}_{ω} with the Lie algebra structure of $\mathfrak{k}_{\omega}$.

§ 6. Remarks about the Campbell-Hausdorff formula.

This section is devoted to analyse the posibility to apply the Campbell-Hausdorff formula for our study of groups. In that case, the central problem is the convergence of Campbell-Hausdorff formula for the Lie algebra $\mathfrak{g}_{\omega}$. Since $\mathfrak{g}_{\omega}$ is not a normed Lie algebra, this is a very delicate and hard problem.

6.1. Dynkin-Cartier's formula. Let $\mathfrak{a}$ be a Lie algebra freely generated by two elements x and y. In [1], it was proved the following equality in the algebra of non-comutative formal power series:

$$\begin{aligned} (6.1)\qquad & \log((\exp x)(\exp y)) \\ &= \sum \frac{(-1)^{k-1}}{k\cdot(p_1+q_1+\cdots+p_k+q_k)\cdot p_1!\,q_1!\cdots p_k!q_k!}\times \\ &\qquad \times(\mathrm{ad}\; x)^{p_1}(\mathrm{ad}\; y)^{q_1}\cdots(\mathrm{ad}\; x)^{p_k}(\mathrm{ad}\; y)^{q_k-1}y\,, \end{aligned}$$

where log and exp are defined by means of formal power series, and the sum is taken over the set $\mathfrak{Q}$ of sequences $Q=(k;\, p_1,\, q_1,\, \cdots,\, p_k,\, q_k)$ for which

$$(6.2)\qquad \begin{cases} k\in \boldsymbol{Z}_{>0};\; p_1,\, q_1,\, \cdots,\, p_k,\, q_k\in \boldsymbol{Z}_{\geq 0} \\ p_j+q_j>0 \qquad \text{for all} \quad j=1,\, \cdots,\, k\,. \end{cases}$$

Here, if $q_k=0$, we understand as

$$\begin{aligned} &(\mathrm{ad}\; x)^{p_1}(\mathrm{ad}\; y)^{q_1}\cdots(\mathrm{ad}\; x)^{p_k}(\mathrm{ad}\; y)^{q_k-1}y = \\ &= (\mathrm{ad}\; x)^{p_1}(\mathrm{ad}\; y)^{q_1}\cdots(\mathrm{ad}\; x)^{p_k-1}x\,. \end{aligned}$$

Note that in cace $q_k>1$ or in case $q_k=0$ and $p_k>1$, the corresponding term is understood as 0. We put $|Q|=p_1+q_1+\cdots+p_k+q_k$ for $Q\in\mathfrak{Q}$.

Notice that in the right-hand side of (6.1), there appear plural terms which differ only in their coefficients. For example, there appear three terms

$$\frac{(-1)^2}{2\cdot(1+0+1+1)\cdot 1!0!1!1!}[x,\,[x,\,y]] \qquad (k=2,\, p_1=p_2=q_2=1,\, q_1=0)\,,$$

$$\frac{(-1)}{1\cdot(2+1)\cdot 2!1!}[x,[x,y]] \qquad (k=1, p_1=2, q_1=1),$$

$$\frac{(-1)^2}{2\cdot(1+1+1+0)\cdot 1!1!1!0!}[x,[y,x]] \qquad (k=2, p_1=q_1=p_2=1, q_2=0).$$

The absolute convergence proved in [1] concerns the reduced series whose terms are the sums of such terms in the original series (6.1), with x, y in a Banach Lie algebra. However, in Banach Lie algebra case, we can prove that the right-hand side of (6.1) is already term-by-term absolutely convergent as shown in Corollary 6.2 below.

We denote by $c_m(x, y)$ the sum of terms of degree m in the right-hand side of (6.1):

(6.2) $$c_m(x,y)= \\ =\sum_{Q\in\mathfrak{Q},|Q|=m}\frac{(-1)^{k-1}}{k\cdot m\cdot p_1!q_1!\cdots p_k!q_k!}\times \\ \times(\mathrm{ad}\,x)^{p_1}(\mathrm{ad}\,y)^{q_1}\cdots(\mathrm{ad}\,x)^{p_k}(\mathrm{ad}\,y)^{q_k-1}y$$

The following lemma gives the key evaluation for the convergence of the Campbell-Hausdorff formula in its original fom in (6.1).

Lemma 6.1. *Let* $0\leqq a\leqq b$. *We denote by* $f_m(a, b)$ *the sum of terms of degree* m *in the series*

$$\sum_{Q\in\mathfrak{Q}}\frac{1}{k\cdot(p_1+q_1+\cdots+p_k+q_k)\cdot p_1!q_1!\cdots p_k!q_k!}a^{p_1+\cdots+p_k}\cdot b^{q_1+\cdots+q_k},$$

that is,

$$f_m(a,b)=\sum_{Q\in\mathfrak{Q},|Q|=m}\frac{1}{k\cdot m\cdot p_1!q_1!\cdots p_k!q_k!}a^{p_1+\cdots+p_k}b^{q_1+\cdots+q_k}.$$

Then, we have the evaluations

$$\frac{2m^{m-2}\cdot a^m}{m!}\leqq f_m(a,b)\leqq\frac{2^m m^{m-2}\cdot b^m}{m!}.$$

Proof. Clearly, it holds that

$$f_m(a,b)\geqq\sum_{Q\in\mathfrak{Q},|Q|=m}\frac{1}{k\cdot m\cdot p_1!q_1!\cdots p_k!q_k!}a^m \\ =\frac{1}{m}a^m\sum_{\substack{k\geqq1,r_1,\dots,r_k>0\\ r_1+\cdots+r_k=m}}\frac{1}{k}\prod_{i=1}^{k}\sum_{p_i+q_i+r_i}\frac{1}{p_i!q_i!} \\ =\frac{1}{m}a^m\sum\frac{1}{k\cdot r_1!\cdots r_k!}2^m.$$

Put $c_k=\sum_{\substack{r_1,\dots,r_k>0\\ r_1+\cdots+r_k=m}}\frac{1}{k\cdot r_1!\cdots r_k!}$. Then, there holds that

$$\sum_{k=1}^{m} k\binom{m}{k} c_k = \sum_{r_1,\cdots,r_m \geqq 0, r_1+\cdots+r_m=m} \frac{1}{r_1!\cdots r_m!} = \frac{m^m}{m!}$$

and that

$$k\binom{m}{k} = k\cdot\frac{m!}{k!(m-k)!} = m\binom{m-1}{k-1} \leqq m\cdot 2^{m-1}.$$

Hence, we have

$$f_m(a, b) \geqq \frac{1}{m}(2a)^m \sum_{k=2}^{m} c_k \geqq \frac{1}{m}(2a)^m \cdot \frac{1}{m} 2^{-m+1} \sum_{k=1}^{m} k\binom{m}{k} c_k = \frac{2m^{m-2}\cdot a^m}{m!}.$$

By a similar calculation as above, it holds the inequality

$$f_m(a, b) \leqq \frac{1}{m}(2b)^m \sum_{k=1}^{m} c_k .$$

Since $k\binom{m}{k} \geqq m$, we have

$$f_m(a, b) \leqq \frac{1}{m}(2b)^m \cdot \frac{1}{m} \sum_{k=1}^{m} k\binom{m}{k} c_k = \frac{2^m m^{m-2}\cdot b^m}{m!}. \qquad \text{Q.E.D.}$$

Corollary 6.2. *Take a normalized norm* $\|\cdot\|$ *on a Banach Lie algebra such that*

$$\|[x, y]\| \leqq \|x\|\,\|y\| \qquad \textit{for any} \quad x, y .$$

If $\max(\|x\|, \|y\|) \leqq (2e)^{-1}$, *then we have*

$$\sum_{Q\in\mathfrak{Q}} \frac{1}{k\cdot(p_1+q_1+\cdots+p_k+q_k)p_1!q_1!\cdots p_k!q_k!} \times \|(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1}\cdots(\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k-1}y\| < +\infty .$$

In other words, the right-hand side of (6.1) *is term-by-term absolutely convergent for sufficiently small* x, y, *in the case of a Banach Lie algebra.*

6.2. Analytic scale of Goodman-Wallach [3]. Now, let us consider the case of an analytic scale introduced in [3, §4]. By definition an analytic scale is a filtration $\{\mathfrak{g}_t, \|\cdot\|_t\}_{t\geqq 0}$ of Banach spaces, which are subspaces of a Lie algebra, parametrized by non-negative numbers $t\in\boldsymbol{R}_{\geqq 0}$, and has the following properties: let $0<s<t$, then $\mathfrak{g}_t$ is a dense subspace of $\mathfrak{g}_s$ and it holds that

$$(6.3) \qquad \begin{cases} \|x\|_s \leqq \|x\|_t & \text{for any} \quad x\in\mathfrak{g}_t , \\ [\mathfrak{g}_t, \mathfrak{g}_t] \subset \mathfrak{g}_s , \\ \|[x, y]\|_s \leqq \dfrac{1}{t-s}\|x\|_t \|y\|_t & \text{for any} \quad x, y\in\mathfrak{g}_t . \end{cases}$$

As seen in Proposition 3.5, the series of Lie algebras $g_{\omega,\varepsilon}=H_\omega(\operatorname{ad};\varepsilon)\supset\mathfrak{g}$, $\varepsilon>0$, are very like the analytic scale, where each $g_{\omega,\varepsilon}$, is not a Banach space but a Fréchet space in general.

Let us return to the Goodman-Wallach's case. We see from [loc. cit. §4] that for any $\delta>0$ and $t>0$, there holds

$$(6.4)\qquad \|[x_1, [x_2, \cdots, [x_{m-1}, x_m]\cdots]]\|_t \leqq \left(\frac{m}{\delta}\right)^m \|x_1\|_{t+\delta}\|x_2\|_{t+\delta}\cdots\|x_m\|_{t+\delta},$$

for any $x_1, x_2, \cdots, x_{m-1}, x_m \in \mathfrak{g}_{t+\delta}$.

Hence, for $x, y \in \mathfrak{g}_{t+\delta}$, when we estimate the norm of each term of the right hand side of (6.1) by using the evaluation (6.4), we have to multiply $f_m(a, b)$ in Lemma 6.1 by the above factor $\left(\frac{m}{\delta}\right)^m$ in (6.4). Therefore, we arrived the series

$$(6.5)\qquad \sum_{Q\in\mathfrak{Q}} \frac{1}{k\cdot(p_1+q_1+\cdots+p_k+q_k)\cdot p_1!q_1!\cdots p_k!q_k!}\times$$
$$\times\left(\frac{|Q|}{\delta}\right)^{|Q|}\cdot\|x\|_{t+\delta}^{p_1+\cdots+p_k}\|y\|_{t+\delta}^{q_1+\cdots+q_k}$$
$$=\sum_{m\geqq 1}\cdot\left(\frac{m}{\delta}\right)^m\cdot f_m(a, b)$$

with $a=\min(\|x\|_{t+\delta}, \|y\|_{t+\delta})$, and $b=\max(\|x\|_{t+\delta}, \|y\|_{t+\delta})$. Then, by the first inequality in Lemma 6.1, this series is evaluated from below as

$$\sum_{m\geqq 1}\frac{2^{m-2}}{m!}\left(\frac{m}{\delta}\right)^m\cdot a^m=+\infty \qquad \text{if}\quad a>0,$$

contrary to the proof of [3, Th. 4.2]. In fact, the key inequality in their proof of Theorem 4.2 in [3]

$$(\textstyle\sum_{i=1}^k r_i)^{\Sigma r_i}\leqq\prod_{i=1}^k r_i!e^{r_i}$$

does not hold, for instance when $r_1=\cdots=r_k=r\in\boldsymbol{Z}_{>0}$, whereas, as in [2, p. 234], there holds the inequality for any fixed $k>0$

$$(\textstyle\sum_{i=1}^k r_i)^{\Sigma r_i}\leqq D_k\cdot\prod_{i=1}^k r_i!e^{r_i}$$

with a positive constant D_k depending on k.

6.3. Naimark-Štern's induction formula. Let $c_m(x, y)$ be as before the sum of terms of degree m in the equality (6.1) for $m=1, 2, \cdots$.

For the case where x, y are taken from a finite dimensional Lie algebra $\mathfrak{b}$, Naimark-Štern [6, Chap. XI, §1] proved the following induction formula for $c_m(x, y)$, $m>0$:

$$(6.6)\qquad (m+1)c_{m+1}(x, y)=$$
$$=\frac{1}{2}[x-y, c_m(x, y)]+$$
$$+\textstyle\sum_{p\geqq 1, 2p\leqq m} k_{2p}\cdot\sum[c_{m_1}(x, y), [\cdots, [c_{m_{2p}}(x, y), x+y]\cdots]]$$

for $m \geqq 2$, and

$$c_1(x, y) = x+y\,,$$

where k_{2p} are Bernoulli numbers defined by

$$\frac{z}{1-e^{-z}} - \frac{z}{2} = 1+\sum_{p\geq 1} k_{2p} z^{2p}\,, \tag{6.7}$$

and the last sum in (6.6) is taken over the non-negative integers $m_1, \cdots, m_{2p}$ such that $m_1+\cdots+m_{2p}=m$. Note that the convergence radius of the right hand side in (6.7) is 2π.

They proved that the so called Campbell-Hausdorff series $F(x, y)=\sum_{m\geq 1} c_m(x, y)$ is absolutely convergent if x and y are in a small neighbourhood of 0 in the Lie algebra $\mathfrak{b}$, that is,

$$\sum_{m\geq 1}||c_m(x, y)||<+\infty\,.$$

Thus, they gave a formula

$$(\exp x)\cdot(\exp y) = \exp F(x, y)$$

in a small neighbourhood of the identity element, for any Lie group associated with $\mathfrak{b}$.

We explain their proof briefly. Consider the differentiable equation

$$\frac{dw}{dz} = \frac{w}{2}+q(w)\,, \tag{6.8}$$

where $q(w)=1+\sum_{p\geq 1} k_{2p}\cdot w^{2p}$. Since $q(w)$ has the convergence radius 2π by definition of k_{2p}, (6.8) has an analytic solution

$$w(z) = \sum_{k\geq 1} \rho_k\cdot z^k \tag{6.9}$$

in a neighbourhood of 0. Substituting (6.9) into (6.8), we have an induction formula for $\{\rho_m\}_{m\geq 1}$:

$$(m+1)\rho_{m+1} = \frac{\rho_m}{2}+\sum_{p\geq 1,2p\leq m} k_{2p} \sum_{m_1+\cdots+m_{2p}=m} \rho_{m_1}\cdots\rho_{m_{2p}} \tag{6.10}$$

for $m\geqq 1$, and $\rho_1=1$.

On the other hand, by (6.6), it holds that

$$\begin{aligned}(m+1)||c_{m+1}(x, y)||\leqq\\ \leqq r||c_m(x, y)||+2r\sum_{p\geq 1,2p\leq m} k_{2p} \sum_{m_1+\cdots+m_{2p}=m} \prod_{j=1}^{2p}||c_{m_j}(x, y)||\end{aligned} \tag{6.11}$$

where $r=\max(||x||, ||y||)$, and $||\cdot||$ is a norm on $\mathfrak{b}$ such that

$$||[x, y]||\leqq||x||\cdot||y|| \qquad (x, y\in\mathfrak{b})\,.$$

From (6.10) and (6.11), we have

$$\|c_m(x, y)\| \leqq (2r)^m \rho_m, \quad r = \max(\|x\|, \|y\|), \tag{6.12}$$

for any $m=1, 2, 3, \cdots$. Hence the series $\sum_{m\geq 1} c_m(x, y)$ is absolutely convergent if r is sufficiently small.

Let us return to the case of infinite-dimensional Lie algebra, and try to apply their method to the analytic scale $\{\mathfrak{g}_t\}$ together with the evaluation (6.4), coming from definition of analytic scale. We look for an evaluation such as

$$\|c_m(x, y)\|_t \leqq (2r)^m \sigma_m, \quad r = \max(\|x\|_{t+\delta}, \|y\|_{t+\delta}), \tag{6.13}$$

with a series $\{\sigma_m\}$ of positive numbers such that $\sum_{m\geq 1} \sigma_m \cdot z^m$ has a positive convergent radius. Then we have to find a differential equation which has an analytic solution $w(z)=\sum_{m\geq 1} \sigma_m \cdot z^m$ such that the following inequality follows directly from the power series expression of the equation on $w(z)$:

$$(m+1)\sigma_{m+1} \leqq \frac{\sigma_m}{2} + \sum_{p\geq 1, 2p\leq m} k_{2p} \left(\frac{2p}{\delta}\right)^{2p} \sum_{m_1+\cdots+m_{2p}=m} \sigma_{m_1} \cdots \sigma_{m_{2p}}. \tag{6.14}$$

However we see that such an equation does not exist because the convergent radius of (6.9) is finite, in which the factors $\left(\frac{2p}{\delta}\right)^{2p}$ do not appear.

Similarly as for the Goodman-Wallach's case, we can see also that the method of Naimark-Štern can not be applied to our system $\{\mathfrak{g}_{\omega,\varepsilon}\}_{\varepsilon>0}$, and so the posibility of applying the Campbell-Hausdorff formula to our study of groups still remains in doubt.

Thus, in reality, Lie algebras with analytic scales have not yet been able to be treated in their full generality, except the case where the Lie algebra in question is itself Banach from the beginning.

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